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Complementary tree nil domination number and Chromatic number of Graphs

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Abstract: *A set D of a graph G = (V, E) is a dominating set if every vertex in V(G) – D is adjacent to some vertex in D. The domination number γ (G) of G is the minimum cardinality of a dominating set. A dominating set D is called a complementary tree nil dominating set, if the induced subgraph < V(G) – D > is a tree and the set V(G) – D is not a dominating set. The minimum cardinality of a complementary tree nil dominating set is called the complementary tree nil domination number of G and is denoted by $\gamma_{ctnd}(G)$. The minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour is chromatic number χ (G). In this paper, an upper bound for the sum of the complementary tree nil domination number and chromatic number of a graph is found and the corresponding extremal graphs are characterized.*

Key words: *Domination number, Complementary tree nil domination number, chromatic number.*

1. Introduction

Graphs discussed in this paper are finite, undirected and simple connected graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. A graph with p vertices and q edges is denoted by G(p, q). The concept of domination in graphs was introduced by Ore[5]. A set D ⊆ V(G) is said to be a dominating set of G, if every vertex in V(G) − D is adjacent to some vertex in D. The cardinality of a minimum dominating set in G is called the domination number of G and is denoted by γ(G). Muthammai, Bhanumathi and Vidhya[4] introduced the concept of complementary tree dominating set. A dominating set D ⊆ V(G) is said to be a complementary tree dominating set (ctd-set), if the induced subgraph < V(G)−D > is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. The minimum number of colours required to colour all the vertices such that adjacent vertices do not receive the same colour is chromatic number χ (G). Any undefined terms in this paper may be found in Harary[1].

The concept of complementary tree nil dominating set is introduced in [3]. A dominating set D ⊆ V(G) is said to be a complementary tree nil dominating set (ctnd-set), if the induced subgraph < V(G) – D > is a tree and the set V(G) – D is not a

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dominating set. The minimum cardinality of a ctnd-set is called the complementary tree nil domination number of G and is denoted by $\gamma_{ctnd}(G)$.

In this paper, an upper bound for the sum of the complementary tree nil domination number and chromatic number of a graph is found and the corresponding extremal graphs are characterized.

2. Prior Results

Theorem 2.1: [1] For any connected graph G, $\chi(G) \leq \Delta(G) + 1$.

Theorem 2.2: [3] For any connected graph G with p vertices, $2 \leq \gamma_{ctnd}(G) \leq p$,where $p \geq 2$.

Theorem 2.3: [3] Let G be a connected graph with p vertices. Then $\gamma_{ctnd}(G) = 2$ if and only if G is a graph obtained by attaching a pendant edge at a vertex of degree p - 2 in $T + K_1$, where T is a tree on (p – 2) vertices.

Theorem 2.4: [3] For any connected graph G, $\gamma_{ctnd}(G) = p$ if and only if $G \cong K_p$, where $p \geq 2$.

Theorem 2.5: [3] Let G be a connected graph with $p \geq 3$ and $\delta(G) = 1$. Then $\gamma_{ctnd}(G) = p-1$ if and only if the subgraph of G induced by vertices of degree atleast 2 is K_2 or K_1.

That is, G is one of the graphs $K_{1,p-1}$ or $S_{m,n}$ ($m + n = p$, $m, n \geq 2$), where $S_{m,n}$ is a bistar which is obtained by attaching m-1 pendant edges at one vertex of K_2 and n-1 pendant edges at other vertex of K_2.

Theorem 2.6: [3] Let G be a connected noncomplete graph with $\delta(G) \geq 2$. Then $\gamma_{ctnd}(G) = p - 1$ if and only if each edge of G is a dominating edge.

Theorem 2.7: [3] Let T be a tree on p vertices such that $\gamma_{ctnd}(T) \leq p - 2$. Then $\gamma_{ctnd}(T) = p - 2$ if and only if T is one of the following graphs.

1. T is obtained from a path P_n ($n \geq 4$ and $n < p$) by attaching pendant edges at atleast one of the end vertices of P_n.
2. T is obtained from P_3 by attaching pendant edges at either both the end vertices or all the vertices of P_3

Notation 2.8: [3] Let $\mathcal{G}$ be the class of connected graphs G with $\delta(G) = 1$ having one of the following properties.

(a) There exist two adjacent vertices u, v in G such that $deg_G(u) = 1$ and <V(G) - {u, v}> contains P_3 as an induced subgraph such that end vertices of P_3 have degree atleast 2 and the central vertex of P_3 has degree atleast 3.

(b) Let P be the set of all pendant vertices in G and let there exist a vertex $v \in V(G)$-P having minimum degree in V(G) - P and is not a support of G such that V(G) - ($N_{\langle V-P \rangle}$ [v] - P) contains P_3 as an induced subgraph such that the end vertices of P_3 have degree atleast 2 and the central vertex of P_3 has degree atleast 3.

Theorem 2.9: [3] Let G be a connected graph with δ(G) = 1. Assume $\gamma_{ctnd}(G) \neq p - 1$. Then γ_{ctnd} (G) = p -2 if and only if G does not belong to the class $\mathcal{G}$ of graphs.

Theorem 2.10: [3] Let G be a connected, non-complete graph with p vertices ($p \geq 4$) and $\delta(G) \geq 2$. Then $\gamma_{ctnd}(G) = p - 2$ if and only if G is one of the following graphs.

1. A cycle on atleast five vertices.
2. A wheel on six vertices.
3. G is the one point union of complete graphs.
4. G is obtained by joining two complete graphs by an edges.
5. G is a connected noncomplete graph such that there exists a vertex $v \in V(G)$ such that G-v is a complete graph on (p-1) vertices.
6. G is a graph such that there exists a vertex $v \in V(G)$ such that G - v is K_{p-1} - e, ($e \in E(K_{p-1})$) and N(v) contains atleast one vertex of degree (p - 3) in K_{p-1} - e.

3. Main Results

Theorem 3.1: For any connected graph G, $\gamma_{ctnd}(G) + \chi(G) \leq 2p$, ($p \geq 2$). Equality holds if and only if $G \cong K_p$.

Proof: $\gamma_{ctnd}(G) + \chi(G) \leq p + \Delta(G) + 1 \leq p + p - 1 + 1 = 2p$.

If $\gamma_{ctnd}(G) + \chi(G) = 2p$, then the only possible case is $\gamma_{ctnd}(G) = p$ and $\chi(G) = p$.

But $\gamma_{ctnd}(G) = \chi(G) = p$ if and only if $G \cong K_p$. Conversely, if $G \cong K_p$, then $\gamma_{ctnd}(G) + \chi(G) = 2p$.

Theorem 3.2: For any noncomplete graph G, $\gamma_{ctnd}(G) + \chi(G) \leq 2p - 2$.

Proof: Since G is not complete, $\gamma_{ctnd}(G) + \chi(G) \leq 2p - 1$.

Assume $\gamma_{ctnd}(G) + \chi(G) = 2p - 1$. Then either $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 1$ or $\gamma_{ctnd}(G) = p - 1$ and $\chi(G) = p$.

Case 1: $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 1$.

$\gamma_{ctnd}(G) = p$ if and only if $G \cong K_p$ on p vertices, But $\chi(K_p) = p$.

Case 2: $\gamma_{ctnd}(G) = p - 1$ and $\chi(G) = p$.

$\chi(G) = p$ implies $G \cong K_p$. But for K_p, $\gamma_{ctnd}(G) = p$. From Case 1 and Case 2, no graph exists with $\gamma_{ctnd}(G) + \chi(G) = 2p - 1$. Hence $\gamma_{ctnd}(G) + \chi(G) \leq 2p - 2$.

Theorem 3.3: For any connected graph G ($p \geq 3$), $\gamma_{ctnd}(G) + \chi(G) = 2p - 2$ if and only if $G \cong P_3$ or $K_p - e$.

Proof: If $G \cong P_3$, then $\gamma_{ctnd}(G) = 2$, $\chi(G) = 2$ and $\gamma_{ctnd}(G) + \chi(G) = 4 = 2p - 2$.

If $G \cong K_p - e$, then $\gamma_{ctnd}(G) = p - 1$ and $\chi(G) = p - 1$, and $\gamma_{ctnd}(G) + \chi(G) = 2p - 2$.

Conversely, assume $\gamma_{ctnd}(G) + \chi(G) = 2p - 2$.

Then there are three cases to consider

1. $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 2$
2. $\gamma_{ctnd}(G) = p - 1$ and $\chi(G) = p - 1$ and
3. γ_{ctnd} G) = p - 2 and $\chi(G) = p$.

Case 1: $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 2$.

$\gamma_{ctnd}(G) = p$ if and only if $G \cong K_p$ on p vertices, But $\chi(K_p) = p$.

Therefore no graph exists with $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 2$.

Case 2: $\gamma_{ctnd}(G) = p - 1$ and $\chi(G) = p - 1$.

$\gamma_{ctnd}(G) = p - 1$ if and only if G is one of the following graphs

1. $K_{1, p-1}$ or $S_{m,n}$ ($m + n = p$, $m, n \geq 1$), if $\delta(G) = 1$.
2. G is a graph in which each edge is a dominating edge, if $\delta(G) \geq 2$.

Subcase 2.1: Let $G \cong K_{1, p-1}$ or $S_{m,n}$, ($m + n = p$, $m, n \geq 2$).

If $G \cong K_{1, p-1}$, $\chi(K_{1, p-1}) = 2$ implies $p = 3$ and hence, $G \cong P_3$.

If $G \cong S_{m, n}$, $\chi(S_{m, n}) = 2$ is not possible, since $m, n \geq 2$.

Subcase 2.2: G is a graph in which each edge is a dominating edge. $\chi(G) = p - 1$ implies G contains a clique K_{p-1} on $p - 1$ vertices. Let $x \in V(G)$ such that $x \notin V(K_{p-1})$ and let u_1, u_2, …, u_{p-1} be the vertices of K_{p-1}. Since G is connected, x is adjacent to u_i for some i, i = 1, 2, …, p – 1. x is not adjacent to all the u_i, since otherwise G will contain a clique on p vertices. Let x be adjacent to atleast two vertices and atmost (p – 3) vertices of K_{p-1}, then $\gamma_{ctnd}(G) = p - 2$, which is a contradiction to $\gamma_{ctnd}(G) = p - 1$. Therefore, x is adjacent to exactly (p – 2) vertices of K_{p-1} and hence G is isomorphic to $K_p - e$.

Case 3: $\gamma_{ctnd}(G) = p - 2$ and $\chi(G) = p$.

But, $\chi(G) = p$ implies $G \cong K_p$ and $\gamma_{ctnd}(G) = p$. Hence, no graph exists for this case.

From Case 1, Case 2 and Case 3, $G \cong P_3$ or $K_p - e$.

Notation 3.4: We use the following notations in this paper

(i) G_1 is a graph such that V(G) can be partitioned into two sets $X = \{x, y\}$ and $V - X = \{u_1, u_2, \ldots, u_{p-2}\}$, $\langle V - X \rangle \cong K_{p-2}$, $\langle X \rangle \cong 2K_1$ and both x and y are adjacent to exactly p – 3 vertices of K_{p-2}.

(ii) G_2 is a graph such that V(G) can be partitioned into two sets X = { x, y } and V – X = { $u_1, u_2, \ldots, u_{p-2}$ }, < V – X > $\cong K_{p-2}$, <X> $\cong K_2$ and both x and y are adjacent to exactly p – 3 vertices of K_{p-2}.

(iii) G_3 is a graph such that V(G) can be partitioned into two sets X = { x } and V – X = { $u_1, u_2, \ldots, u_{p-1}$ }, < V – X > $\cong K_{p-1}$ and x is adjacent to atleast one vertex and atmost (p – 3) vertices of K_{p-1}.

Theorem 3.5: For any connected graph G, $\gamma_{ctnd}(G) + \chi(G) = 2p - 3$ if and only if G is one of the following graphs. $K_{1,3}$, P_4, G_1, G_2 and G_3.

Proof: If G $\cong K_{1,3}$, then $\gamma_{ctnd}(G) = 3$, $\chi(G) = 2$, and $\gamma_{ctnd}(G) + \chi(G) = 5 = 2p - 3$.

If G $\cong P_4$, then $\gamma_{ctnd}(G) = 3$, $\chi(G) = 2$ and $\gamma_{ctnd}(G) + \chi(G) = 5 = 2p - 3$.

If G is isomorphic to the graphs G_2 or G_3, then $\gamma_{ctnd}(G) = p - 1$, $\chi(G) = p - 2$ and $\gamma_{ctnd}(G) + \chi(G) = 2p - 3$. If G is isomorphic to the graph G_1, then $\gamma_{ctnd}(G) = p - 2$ and $\chi(G) = p - 1$ and $\gamma_{ctnd}(G) + \chi(G) = 2p - 3$.

Conversely, assume $\gamma_{ctnd}(G) + \chi(G) = 2p - 3$.

Then there are four cases to consider

1. $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 3$
2. $\gamma_{ctnd}(G) = p - 1$ and $\chi(G) = p - 2$
3. $\gamma_{ctnd}(G) = p - 2$ and $\chi(G) = p - 1$ and
4. $\gamma_{ctnd}(G) = p - 3$ and $\chi(G) = p$.

Case 1: $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 3$.

$\gamma_{ctnd}(G) = p$ if and only if G $\cong K_p$ on p vertices, But $\chi(K_p) = p$.

Therefore no graph exists with $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 3$.

Case 2: $\gamma_{ctnd}(G) = p - 1$ and $\chi(G) = p - 2$.

$\gamma_{ctnd}(G) = p - 1$ if and only if G is one of the following

1. $K_{1, p-1}$ or $S_{m, n}$ (m + n = p, m, n $\geq$ 2) if $\delta(G) = 1$.
2. G is a graph in which each edge is a dominating edge if $\delta(G) \geq 2$.

Subcase 2.1: Let G $\cong K_{p-1}$ or $S_{m, n}$ (m + n = p, m, n $\geq$ 2).

$\chi(K_{1, p-1}) = 2$ and $\chi(G) = p - 2$ implies p = 4 and hence G $\cong K_{1, 3}$.

$\chi(S_{m,n}) = 2$, $\chi(G) = p - 2$ implies p = 4 and hence, G $\cong S_{2, 2}$. But $S_{2,2} \cong P_4$.

Subcase 2.2: G is a graph in which each edge is a dominating edge.

$\chi(G) = p - 2$ implies G contains a clique K_{p-2} on p – 2 vertices.

Let x, y $\in$ V(G) - V(K_{p-2}) and let $u_1, u_2, \ldots, u_{p-2}$ be the vertices of K_{p-2}. Now, G is a graph such that V(G) can be partitioned into two sets X = { x, y } and V – X = {$u_1, u_2, \ldots, u_{p-2}$}, where x, y $\in$ V(G). Let x, y be adjacent to atleast two vertices and atmost (p – 4) vertices of K_{p-2}, then $\gamma_{ctnd}(G) \leq p - 2$, which is a contradiction to $\gamma_{ctnd}(G) = p - 1$.

Subcase 2.2.1: $< X > \cong \overline{K_2}$.

If x or y is adjacent to all the vertices of V – X, then χ (G) = p – 1. If x or y is adjacent to atmost p – 4 vertices of V – X, then $\gamma_{ctnd}(G) \leq$ p – 2. Therefore, both x and y are adjacent to exactly p – 3 vertices of K_{p-2}. Hence $G \cong G_1$.

Subcase 2.2.2: $< X > \cong K_2$.

If x or y is adjacent to all the vertices of V – X, then χ (G) = p – 1. If x or y is adjacent to atmost p – 4 vertices of V – X, then $\gamma_{ctnd}(G) \leq$ p – 2. Therefore, x and y are adjacent to atmost p – 3 vertices of V – X and hence $G \cong G_2$.

Case 3: $\gamma_{ctnd}(G)$ = p – 2 and χ (G) = p – 1.

χ (G) = p – 1 implies G contains a clique K_{p-1} on p – 1 vertices. Let $x \in V(G) - V(K_{p-1})$ and let $u_1, u_2, \ldots, u_{p-1}$ be the vertices of K_{p-1}. Since G is connected, x is adjacent to u_i for some i, i = 1, 2, …, p-1. x is not adjacent to all the u_i, since otherwise G will contain a clique on p vertices. Let x be adjacent to exactly p – 2 vertices in K_{p-1} and u_i be the nonadjacent vertex to x, then D = V(G) – { u_i } or D = V(G) – { x } is a minimum ctnd-set and hence $\gamma_{ctnd}(G)$ = p – 1, which is a contradiction to $\gamma_{ctnd}(G)$ = p – 1. Therefore x is not adjacent to exactly (p – 2) vertices of K_{p-1}. Therefore x is adjacent to atleast one vertex and atmost (p – 3) vertices of K_{p-1}. Then D = {x, $u_1, u_2, \ldots, u_{p-3}$} is a minimum complementary dominating set. V – D = {u_{p-2}, u_{p-1}} and $< V - D > \cong K_2$ and hence $\gamma_{ctnd}(G)$ = p – 2. Hence $G \cong G_3$

Case 4: $\gamma_{ctnd}(G)$ = p – 3 and χ (G) = p.

But χ (G) = p implies $G \cong K_p$ and $\gamma_{ctnd}(G)$ = p. Hence no graph exists .

From Case 1, Case 2, Case 3 and Case 4, G is one of the graphs $K_{1,3}$, P_4, G is $K_{1,3}$, P_4, G_1, G_2 and G_3.

Notation 3.6: We use the following notations in this paper

(i) G_4 is a graph such that V(G) can be partitioned into two sets X = { x, y, z} and V – X = { $u_1, u_2, \ldots, u_{p-3}$ }, $< V - X > \cong K_{p-3}$, <X> is complete and each vertex in X is adjacent to exactly p – 4 vertices of K_{p-3}.

(ii) G_5 is a graph obtained by attaching two pendent edges at exactly one vertex of K_{p-2}.

(iii) G_6 is a graph obtained by attaching two pendent edges at any two vertex of K_{p-2}.

(iv) G_7 is isomorphic to a graph obtained by attaching P_3 at exactly one vertex of K_{p-2}.

(v) G_8 is isomorphic to a graph obtained by attaching C_3 at exactly one vertex of K_{p-2}.

Theorem 3.7: For any connected graph G, $\gamma_{ctnd}(G) + \chi(G)$ = 2p – 4 if and only if G is one of the following graphs.

1. $K_{1,4}$, $S_{3,2}$, C_5, G_4, G_5, G_6, G_7, G_8

2. G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is $K_{p-1} - e$, ($e \in E(K_{p-1})$) and N(v) contains atleast one vertex of degree (p–3) in $K_{p-1} - e$.

Proof: If G is a graph stated in the theorem, then $\gamma_{ctnd}(G) + \chi(G) = 2p - 4$.

Conversely, assume $\gamma_{ctnd}(G) + \chi(G) = 2p - 4$. Then there are five cases to consider

1. $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 4$
2. $\gamma_{ctnd}(G) = p - 1$ and $\chi(G) = p - 3$
3. $\gamma_{ctnd}(G) = p - 2$ and $\chi(G) = p - 2$ and
4. $\gamma_{ctnd}(G) = p - 3$ and $\chi(G) = p - 1$.
5. $\gamma_{ctnd}(G) = p - 4$ and $\chi(G) = p$.

Case 1: $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 4$.

$\gamma_{ctnd}(G) = p$ if and only if $G \cong K_p$ on p vertices, But $\chi(K_p) = p$. Therefore no graph exists with $\gamma_{ctnd}(G) = p$ and $\chi(G) = p - 4$.

Case 2: $\gamma_{ctnd}(G) = p - 1$ and $\chi(G) = p - 3$.

$\gamma_{ctnd}(G) = p - 1$ if and only if G is one of the following

1. $K_{1, p-1}$ or $S_{m,n}$ ($m + n = p$, $m, n \geq 2$) if $\delta(G) = 1$.
2. G is the graph in which each edge is a dominating edge if $\delta(G) \geq 2$.

Subcase 2.1: Let $G \cong K_{p-1}$ or $S_{m,n}$ ($m + n = p$, $m, n \geq 2$).

$\chi(K_{p-1}) = 2$ and $\chi(G) = p - 3$ implies $p = 5$ and hence, $G \cong K_{1,4}$.

$\chi(S_{m,n}) = 2$ and $\chi(G) = p - 3$ implies $p = 5$ and hence, $G \cong S_{3,2}$.

Subcase 2.2: G is a graph in which each edge is a dominating edge.

$\chi(G) = p - 3$ implies G contains a clique K_{p-3} on $p - 3$ vertices.

Let $x, y, z \in V(G) - V(K_{p-3})$ and let $u_1, u_2, \ldots, u_{p-3}$ be the vertices of K_{p-3}. G is a graph such that V(G) can be partitioned into two sets $X = \{x, y, z\}$ and $V - X = \{u_1, u_2, \ldots, u_{p-3}\}$ where $x, y, z \in V(G)$.

Subcase 2.2.1: $\langle X \rangle \cong \overline{K_3}$.

If one or two vertices of X is adjacent to all the vertices of V – X, then $\chi(G) \geq p - 2$. If x, y and z are adjacent to atmost p – 4 vertices of V – X, then $\gamma_{ctnd}(G) \leq p - 2$. Therefore no graph exists in this case.

Subcase 2.2.2: $\langle X \rangle \cong K_1 \cup K_2$. $\gamma_{ctnd}(G) \leq p - 2$, since K_2 is not a dominating edge.

Subcase 2.2.3: $\langle X \rangle \cong K_3$.

If one or two vertices of X is adjacent to all the vertices of V – X, then $\chi(G) \geq p - 2$. If x, y or z are adjacent to atmost p – 5 vertices of V – X, then $\gamma_{ctnd}(G) \leq p - 2$. Therefore each vertex in X is adjacent to exactly p – 4 vertices of K_{p-3} and $G \cong G_4$.

Case 3: $\gamma_{ctnd}(G) = p - 2$ and $\chi(G) = p - 2$.

By Theorem 2.7. Notation 2.8. Theorem 2.9. and Theorem 2.10. $\gamma_{ctnd}(G) = p - 2$ if and only if

1. $G \cong T$, where T is tree either obtained from a path P_n ($n \geq 4$ and $n < p$) by attaching pendant edges at atleast one of the end vertices of P_n, or obtained from P_3 by attaching pendant edges at either both the end vertices or all the vertices of P_3.
2. $G \notin \mathcal{G}$, if $\delta(G) = 1$
3. If $\delta(G) \geq 2$, then G is one of the following graphs.
   (i) A cycle on atleast five vertices.
   (ii) A wheel on six vertices.
   (iii) G is the one point union of complete graphs.
   (iv) G is obtained by joining two complete graphs by an edge.
   (v) G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is a complete graph on $(p - 1)$ vertices.
   (vi) G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is $K_{p-1} - e$, ($e \in E(K_{p-1})$) and N(v) contains atleast one vertex of degree $(p - 3)$ in $K_{p-1} - e$.

Case 3.1: $G \cong T$,

$\chi(T) = 2$ and $\chi(G) = p - 2$ implies $p = 4$. But this case is not possible, since $p \geq 5$.

Therefore no graph exists in this case.

Case 3:2: $G \notin \mathcal{G}$ and $\delta(G) = 1$

$\chi(G) = p - 2$ implies G contains a clique K_{p-2} on $p - 2$ vertices.

Let $x, y \in V(G) - V(K_{p-2})$ and let $u_1, u_2, \ldots, u_{p-2}$ be the vertices of K_{p-2}. Now, G is a graph such that V(G) can be partitioned into two sets $X = \{x, y\}$ and $V - X = \{u_1, u_2, \ldots, u_{p-2}\}$, where $x, y \in V(G)$ and $< V - X > \cong K_{p-2}$. If $< X > \cong \overline{K_2}$, then $G \cong G_5$ or G_6.

If $< X > \cong K_2$, then x or y is adjacent to exactly one vertex of $V - X$. Hence $G \cong G_7$.

Case 3.3: $\delta(G) \geq 2$.

Subcase 3.3.1: A cycle on atleast five vertices.

$$\chi(C_p) = \begin{cases} 2 \text{ if p is even} \\ 3 \text{ if p is odd} \end{cases} \text{ and } \chi(G) = p - 3 \text{ implies } p = 5. \text{ Hence } G \cong C_5.$$

Subcase 3.3.2: A wheel on six vertices.

$\chi(W_6) = 4$ and $\chi(G) = p - 2$. This case is not possible.

Subcase 3.3.3: G is the one point union of complete graphs.

From Case 3.2, V(G) can be partitioned into two sets $X = \{x, y\}$ and $V–X = \{u_1, u_2,\ldots, u_{p-2}\}$, where $x, y \in V(G)$, both x and y are adjacent to exactly one vertex of K_{p-2}. Therefore G is isomorphic to a graph obtained by attaching C_3 at exactly one vertex of K_{p-2} and hence $G \cong G_8$.

Subcase 3.3.4: G is obtained by joining two complete graphs by an edge.

This case is also not possible.

Subcase 3.3.5: G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is a complete graph on $(p - 1)$ vertices. In this case $\chi(G) = p - 1$. Therefore no graph exists in this case.

Subcase 3.3.6: G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is $K_{p-1} - e$, $(e \in E(K_{p-1}))$ and N(v) contains atleast one vertex of degree $(p - 3)$ in $K_{p-1} - e$.

From this graph $\gamma_{ctnd}(G) = p - 2$ and $\chi(G) = p - 2$.

Case 4: $\gamma_{ctnd}(G) = p - 3$ and $\chi(G) = p - 1$.

$\chi(G) = p - 1$ implies G contains a clique K_{p-1} on $p - 1$ vertices.

Let x be the vertex other than the vertices of K_{p-1} and let $u_1, u_2, \ldots, u_{p-1}$ be the vertices of K_{p-1}. Since G is connected, x is adjacent to u_i for some i, $i = 1, 2, \ldots, p - 1$, x is not adjacent to all u_i, since otherwise G will contain a clique on p vertices.

Let x be adjacent to atleast one vertex and atmost $(p - 2)$ vertices of K_{p-1}. Then $\gamma_{ctnd}(G) \geq p - 2$, which is a contradiction to $\gamma_{ctnd}(G) = p - 3$. Hence no graph exists in this case.

Case 5: $\gamma_{ctnd}(G) = p - 4$ and $\chi(G) = p$.

$\chi(G) = p$ implies $G \cong K_p$. But for $G \cong K_p$, $\gamma_{ctnd}(G) = p$.

From Case 1, Case 2, Case 3, Case 4 and Case 5. G is one of the graphs $K_{1,4}$, $S_{3,2}$, C_5, G_4, G_5, G_6, G_7, G_8, G is a graph such that there exists a vertex $v \in V(G)$ such that $G - v$ is $K_{p-1} - e$, $(e \in E(K_{p-1}))$ and N(v) contains atleast one vertex of degree $(p-3)$ in $K_{p-1} - e$.

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