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On the Complement of the Boolean Function Graph $B(\overline{G}, \overline{K_q}, INC)$ of a Graph

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Abstract: *For any simple graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph B(G, K_q, NINC) of G is a graph with vertex set V(G) ∪ E(G) and two vertices in* $B(\overline{G}, \overline{K_q}, INC)$ *are adjacent if and only if they correspond to two nonadjacent vertices of G or to a vertex and an edge incident to it in G. For simplicity, this graph is denoted by $BF_1(G)$. Two vertices in the complement $\overline{BF_1(G)}$ of $BF_1(G)$ are adjacent if and only if they correspond to two adjacent vertices of G, two adjacent edges of G, two nonadjacent edges of G or to a vertex and an edge not incident to it in G. In this paper, structural properties of the complement $\overline{BF_1(G)}$ of $BF_1(G)$ including traversability and eccentricity properties are studied. Also covering numbers and various domination numbers are determined.*

Keywords: *Boolean function graph, domination number.*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. For two vertices u and v in a connected graph G, the distance d(u,v) from u to v is the length of a shortest u – v path in G. A connected graph G is said to be geodetic, if a unique shortest path joins any two vertices. A closed trail connecting all vertices and edges is called an Eulerian trail. A graph having an Eulerian trail is called an Eulerian graph. A Spanning cycle in a graph is called Hamiltonian cycle. A graph having a Hamiltonian cycle is called Hamiltonian graph. For a connected graph G, the eccentricity $e_G(v)$ of a vertex v in G is the distance to a vertex farthest from v. Thus, $e_G(v) = \{d_G(u, v) : u \in V(G)\}$, where $d_G(u, v)$ is the distance between the vertices u and v. The minimum and maximum eccentricities are the radius and diameter of G, denoted r(G) and diam(G) respectively.

A covering of a graph G = (V, E) is a subset K of V such that every edge of G is incident with a vertex in K. A covering K is called a minimum covering if G has no covering K' with $|K'| < |K|$. The number of vertices in a minimum covering of G is called

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the covering number of G and is denoted by $\alpha_0(G)$ or α_0. An edge covering of G is a subset L of E such that every vertex is incident with an edge of L. The number of edges in a minimum edge covering of G is called the edge covering number of G and is denoted by $\alpha_1(G)$ or α_1. For graph theoretic terminology, Harary [1] is referred.

The concept of domination in graphs was introduced by Ore [10]. A set $D \subseteq V(G)$ is said to be a dominating set of G, if every vertex in V(G) – D is adjacent to some vertex in D. D is said to be a minimal dominating set if D – {u} is not a dominating set, for any $u \in D$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. We call a set of vertices a $\gamma-$ set, if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on dominating sets. A dominating set D is called a independent dominating set, if the induced subgraph < D > is independent and is called a perfect dominating set, if every vertex in V(G) – D is adjacent to exactly one vertex in D is called a restrained dominating set, if every vertex in V(G) – D is adjacent to another vertex in V(G) – D . By γ_i, γ_p, γ_r we mean the minimum cardinality of a independent dominating set respectively. Janakiraman et al., introduced the concepts of Boolean and Boolean function graphs [2 – 5].

Kulli and Janakiram introduced the concept of split [7] and nonsplit [8] domination in graphs. A dominating set D of a connected graph G is a split (nonsplit), if < V(G) – D > is disconnected (connected). Split (nonsplit) domination number $\gamma_s(G)$ ($\gamma_{ns}(G)$) of G is the minimum cardinality of a split (nonsplit) dominating set.

For any simple graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph B(G, K_q, NINC) of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(\overline{G}, \overline{K_q}, INC)$ are adjacent if and only if they correspond to two nonadjacent vertices of G or to a vertex and an edge incident to it in G. For simplicity, this graph is denoted by $BF_1(G)$. Two vertices in the complement $\overline{BF_1(G)}$ of $BF_1(G)$ are adjacent if and only if they correspond to two adjacent vertices of G, two adjacent edges of G, two non adjacent edges of G or to a vertex and an edge not incident to it in G. In this paper, structural properties of $\overline{BF_1(G)}$ including traversability and eccentricity properties are studied. Also covering numbers and various domination numbers are determined.

2. Main Results

In this section, the properties including traversability and eccentricity properties are studied. Also decomposition of $\overline{BF_1(G)}$ for some known graphs are given.

The following elementary properties of $\overline{BF_1(G)}$ are immediate.

Observation 2.1:

Let G be a (p,q) graph

1. G and L(G) are induced subgraphs of $\overline{BF_1(G)}$ and the subgraph of $\overline{BF_1(G)}$ induced by q vertices in L(G) is a complete graph.
2. Number of vertices in $\overline{BF_1(G)}$ = p + q.
3. Number of edges in $\overline{BF_1(G)}$ = | E(G)) | + | E(K_q) | + q(p – 2)
$$= q + \frac{q(q-1)}{2} + q(p-2) = \frac{q}{2}(2p + q - 3).$$
4. Degree of vertex vof G in $\overline{BF_1(G)}$ = $deg_G^v + (q - deg_G^v)$ = q.
5. Degree of vertex e ∈V(L(G)) in $\overline{BF_1(G)}$ = (q – 1) + (p – 2) + p + q – 3.
6. $\overline{BF_1(G)}$ is bi-regular if p ≠ 3 and regular if p = 3.
7. $\overline{BF_1(G)}$ contains isolated vertices if and only if q = 0 (or) p + q – 3 = 0.
If q = 0, then G ≅ nK_1. If p + q – 3 = 0, then p = 3, q = 0 (or) p = 2, q = 1.
Therefore, G ≅ nK_1, n ≥ 1 (or) G ≅ K_2.
8. $\overline{BF_1(G)}$ is disconnected if and only if G ≅ nK_1 (or) G ≅ $K_2 \cup mK_1$,
where n ≥ 1, m ≥ 0, since if q(G) ≥ 2, then $\overline{BF_1(G)}$ is connected.
9. If p ≥ 4, then $\delta(\overline{BF_1(G)})$ = q and $\Delta(\overline{BF_1(G)})$ = p + q – 3 if p ≥ 4.
If p ≤ 3, then $\delta(\overline{BF_1(G)})$ = p + q – 3 and $\Delta(\overline{BF_1(G)})$ = q.
10. For any connected graph G with atleast 3 vertices, no vertex of $\overline{BF_1(G)}$ is a cut vertex.
11. $\overline{BF_1(G)}$ contains a cut vertex, if and only if G ≅ $K_2 \cup nK_1$, K_2, $2K_2$.

In the following, the girth of $\overline{BF_1(G)}$ is found.

Theorem 2.2: For any (p, q) graph G having atleast 3 vertices, the girth of $\overline{BF_1(G)}$ is either 3 or 5.

Proof: Since K_q is an induced subgraph of $\overline{BF_1(G)}$, $\overline{BF_1(G)}$ contains triangles, if q ≥ 3. Assume q ≤ 2.

If G ≅ $2K_2 \cup nK_1$, n ≥ 0 or $P_3 \cup mK_1$, m ≥ 1, then $\overline{BF_1(G)}$ contains triangles. Therefore, G ≅ P_3. Then $\overline{BF_1(G)}$ ≅ C_5. If G ≅ P_3, then girth of $\overline{BF_1(G)}$ is 5.

Theorem 2.3: Let G be a graph with atleast two edges. Then $\overline{BF_1(G)}$ is geodetic if and only if G is one of the following graphs.$P_3 \cup nK_1$, $2K_2 \cup nK_1$, n ≥ 0.

Proof: Assume $\overline{BF_1(G)}$ is geodetic.

Case 1. G contains triangles.

Let $V(C_3) = \{v_1, v_2, v_3\}$ and $E(C_3) = \{e_{12}, e_{23}, e_{31}\}$,where $e_{12} = (v_1, v_2)$, $e_{23} = (v_2, v_3)$, $e_{31} = (v_3, v_1)$. Then $v_1, v_2, v_3, e_{12}, e_{23}, e_{31} \in V(\overline{BF_1(G)})$ and $\langle\{v_1, v_3, e_{12}, e_{23}\}\rangle \cong C_4$ in $\overline{BF_1(G)}$. Therefore $\overline{BF_1(G)}$ is not geodetic.

Case 2. G is triangle free

Assume q ≥ 3. Since G is a triangle free, G contains $P_3 \cup K_2$, $3K_2$, $K_{1,3}$ (or) P_4 as a subgraph of G. Then $\overline{BF_1(G)}$ contains either $K_4 - e$, C_4 as an induced subgraph.Therefore, q ≤ 2.Since G contains atleast 2 edges, q = 2. Therefore, $G \cong P_3 \cup nK_1$ or $2K_2 \cup nK_1$,n ≥ 0. Conversely, if $G \cong P_3 \cup nK_1$ or $2K_2 \cup nK_1$, n ≥ 0. Then $\overline{BF_1(G)}$ is geodetic.

In the following, a necessary and sufficient condition for $\overline{BF_1(G)}$ to be Eulerian is given.

Theorem 2.4: Let G be a (p, q) graph with q ≥ 2 and p + q ≥ 5. Then $\overline{BF_1(G)}$ is Eulerian if and only if p is odd and q is even.

Proof: Assume p is odd and q is even. Let v ∈ V(G), e ∈ V(L(G). Then v, e∈V($\overline{BF_1(G)}$). $\deg^{v}_{\overline{BF_1(G)}}$ = q, is even. $\deg^{e}_{\overline{BF_1(G)}}$ = p + q − 3 is also even, since p is odd and p + q ≥ 5. Therefore $\overline{BF_1(G)}$ is Eulerian.

Conversely, if $\overline{BF_1(G)}$ is Eulerian, then q is even and p is odd.

In the following, a necessary condition that $\overline{BF_1(G)}$ to be Hamiltonian is given.

Theorem 2.5: Let G be a connected (p, q) graph such that q ≥ 3.Then $\overline{BF_1(G)}$ is Hamiltonian.

Proof: This is proved by finding the closure of $\overline{BF_1(G)}$. Let v ∈ V(G) and e ∈ V(L(G)). Then $\deg^{v}_{\overline{BF_1(G)}}$ = q and $\deg^{e}_{\overline{BF_1(G)}}$ = p + q − 3. In $\overline{BF_1(G)}$, any two vertices of L(G) are adjacent and any vertex of L(G) in $\overline{BF_1(G)}$ is adjacent to (p − 2) vertices of G in $\overline{BF_1(G)}$. Let v ∈ V(G) and e ∈ V(L(G)) such that (v, e) ∉ E($\overline{BF_1(G)}$). Then $\deg^{v}_{\overline{BF_1(G)}} + \deg^{e}_{\overline{BF_1(G)}}$ = p + 2q − 3 ≥ p + q, since q ≥ 3. Therefore, a vertex of G which is not adjacent to a vertex of L(G) in $\overline{BF_1(G)}$ can be joined by an edge in $\overline{BF_1(G)}$. Now, deg (e) = p + q − 1 and deg (v) = q + $\deg^{v}_{G}$. Let v_1, v_2 be two non adjacent vertices of G in $\overline{BF_1(G)}$. $\deg^{v_1}_{\overline{BF_1(G)}}$ + $\deg^{v_2}_{\overline{BF_1(G)}}$ = 2q + $\deg^{v_1}_{G}$ + $\deg^{v_2}_{G}$ ≥ 2q + 2 > p + q, since q ≥ p−1. Therefore, any two nonadjacent vertices of G in $\overline{BF_1(G)}$ can be joined by an edge. Therefore closure of $\overline{BF_1(G)}$ is complete and $\overline{BF_1(G)}$ is Hamiltonian.

Note 2.1: If q ≤ 2, then $G \cong P_3$ (or) K_2. $\overline{BF_1(G)} \cong C_5$ and $\overline{BF_1(K_2)} \cong 3K_1$.

Theorem 2.6: If G is disconnected such that each component contains atleast two edges, then $\overline{BF_1(G)}$ is Hamiltonian.

Proof:

By Theorem 2.5., each component of G induces a Hamiltonian cycle in $\overline{BF_1(G)}$.

Let G_1, G_2, …,G_w be the components of G, where w ≥ 2. Through one end of the Hamiltonian path of $\overline{BF_1(G)}$, we can enter into the Hamiltonian path of $\overline{BF_1(G)}$ and in turn into that of $\overline{BF_1(G)}$, …, covering all the Hamiltonian paths of $\overline{BF_1(G)}$ and come back to $\overline{BF_1(G)}$ through a vertex of G (or) L(G). Therefore, there exists a Hamiltonian cycle in $\overline{BF_1(G)}$ and hence $\overline{BF_1(G)}$ is Hamiltonian.

Theorem 2.7: $\overline{BF_1(nK_2)}$, (n ≥ 3) is Hamiltonian.

Proof : Let $v_1,v_2,…,v_{2n}$ be vertices of G and $e_{i,\, i+1} = (v_i, v_{i+1})$, i = 1, 3, 5,…, (2n – 1). Then $v_1,…,v_{2n}$, $e_{i,\, i+1} \in V(\overline{BF_1(G)})$. $e_{2n-1,2n} = (v_{2n-1},v_{2n})$

Case 1. n is odd.

Then $v_1v_2e_{2n-1,2n}v_3v_4e_{2n-3,2n-2}v_5v_6,…e_{n+2,n+3}v_nv_{n+1}e_{12}v_{n+2}v_{n+3}e_{34},…v_{2n-1}v_{2n}e_{n,n+1}v_1$ is a Hamiltonian cycle in $\overline{BF_1(nK_2)}$.

Case 2. n is even.

Then $v_1v_2e_{2n-1,2n}v_3v_{4,…,}e_{n+3,n+4}v_{n-1}v_ne_{12}v_{n+1}v_{n+2}v_{34},…,e_{n+1,n+2}v_1$ is a Hamiltonian cycle in $\overline{BF_1(nK_2)}$. Therefore $\overline{BF_1(nK_2)}$ is Hamiltonian.

Remark 2.1:

1, If G isa connected graph with atleast two edges, then $\overline{BF_1(nK_2)}$ is hamiltonian and hence $\overline{BF_1(G)}$ is 2-connected. Therefore, $\overline{BF_1(G)}$ has no cut vertices, if G is connected.

2, If G is disconnected with atleast two edges, then $\overline{BF_1(G)}$ contains cut vertices if and only $G \cong 2K_2 \cup nK_1$, n ≥ 0.

In the following a necessary and sufficient condition that $\overline{BF_1(G)}$ contains C_n (n ≥ 4), as an induced subgraph is obtained, where G is any graph which is not totally disconnected.

Theorem 2.8: $\overline{BF_1(G)}$ contains C_4 as an induced subgraph if and only if one of the following holds.

(i) G contains triangles.

(ii) G contains either C_4 or $K_{1,3}$ as an induced subgraph.

(iii) G contains P_4 as a subgraph.

Proof: (i) Let G contains triangles and let $V(C_3) = \{v_1, v_2, v_3\}$ and $E(C_3)=\{e_{12}, e_{23}, e_{31}\}$, where $e_{i,\, i+1} = (v_i, v_{i+1})$ i = 1, 2 and $e_{31} = (v_3, v_1)$. Then $\overline{BF_1(G)}$ contains C_4, induced by the vertices v_1, v_3, e_{12}, e_{23}.

(ii) Since G is an induced subgraph of $\overline{BF_1(G)}$, if G contains C_4 then $\overline{BF_1(G)}$ also contains C_4.

(iii) Let G contain $K_{1,3}$ as induced subgraph.

Then $\overline{BF_1(G)}$ contains C_4 induced by the central vertex w, any two pendant vertices u, v of $K_{1,3}$ and the vertex in $\overline{BF_1(G)}$ corresponding to the edges not incident with u and v.

(iv) Let G contain P_4 as a subgraph. Then $\overline{BF_1(G)}$ contains C_4 induced by the central vertices of P_4 and the vertices in $\overline{BF_1(G)}$ corresponding to the pendant edge of P_4.

Conversely, assume $\overline{BF_1(G)}$ contains C_4 as an induced subgraph.

If the vertices of C_4 in $\overline{BF_1(G)}$ are vertices of G, then G contains C_4 as an induced subgraph.

If three vertices of C_4 in $\overline{BF_1(G)}$ are vertices of G and the fourth vertex is a vertex of L(G), then G contains $K_{1,3}$ as an induced subgraph.

If two vertices of C_4 in $\overline{BF_1(G)}$ are vertices of G and the remaining two vertices are vertices of L(G), then G contains either C_3 (or) P_4 as a subgraph.

Since any two vertices of L(G) are adjacent in $\overline{BF_1(G)}$, the cases in which three (or) four vertices of C_4 in $\overline{BF_1(G)}$ are vertices of L(G) are not possible.

Theorem 2.9: $\overline{BF_1(G)}$ Contains C_5 as an induced subgraph if and only if G contains P_3 as an induced subgraph.

Proof: Let G contain P_3 as an induced subgraph. Since $\overline{BF_1(P_3)} \cong C_5$, $\overline{BF_1(G)}$ contains C_5 as an induced subgraph.

Conversely, assume $\overline{BF_1(G)}$ contains C_5 as an induced subgraph. Since any two vertices of L(G) are adjacent in $\overline{BF_1(G)}$, any cycle on 5 vertices in $\overline{BF_1(G)}$ contains atmost 2 vertices of L(G).

(i) If all the vertices of C_5 in $\overline{BF_1(G)}$ are vertices of G, then G also contains C_5 as an induced subgraph of $\overline{BF_1(G)}$.

(ii) If four vertices of C_5 in $\overline{BF_1(G)}$ are vertices of G and the fifth vertex is a vertex of L(G), then G contains P_4 as an induced subgraph.

(iii) If three vertices of C_5 is $\overline{BF_1(G)}$ are vertices of G and the remaining two vertices are vertices of L(G), then G contains P_3 as an induced subgraph.

Theorem 2.10: $\overline{BF_1(G)}$ contains C_n ($n \geq 6$) as an induced subgraph if and only if G contains C_n ($n \geq 6$) as an induced subgraph.

Proof: If G contains C_n as an induced subgraph, then so is $\overline{BF_1(G)}$.

Conversely assume $\overline{BF_1(G)}$ contains C_n ($n \geq 6$) as an induced subgraph. If atleast one vertex of C_n in $\overline{BF_1(G)}$ is a vertex of L(G), then a C_n in $\overline{BF_1(G)}$ contains a chord.

Therefore no vertex of C_n in $\overline{BF_1(G)}$ is a vertex of L(G) and hence G contains C_n ($n \geq 6$) as an induced subgraph.

In the following, the edge Partition of $\overline{BF_1(G)}$ for some known graphs are given.

Theorem 2.11:

1. Edges of $\overline{BF_1(G)}$ can be partitioned into G, complete graph on q vertices and $qK_{1, p-2}$, such that central vertex of $K_{1, p-2}$ is a vertex of K_q.

2. Edges of $\overline{BF_1(C_n)}$ can be partitioned in to

(i) C_n, K_n, $\frac{n-2}{2}$ C_{2n}, if n is even.

(ii) C_n, K_n, $\frac{n-3}{2}$ C_{2n}, nK_2, if n is odd.

3. Edges of $\overline{BF_1(K_{1,n})}$ can be partitioned in to

(i) $K_{1,n}$, K_n, $\frac{n-1}{2}$ C_{2n}, if n is odd. n

(ii) $K_{1,n}$, K_n, $\frac{n-2}{2}$ C_{2n}, if n is even.

4. Edges of $\overline{BF_1(K_n)}$ ($n \geq 3$) can be partitioned in to

(i) K_n, $K_{\binom{n}{2}}$ and $\frac{n-1}{2}$ H, where H is a (n – 2) regular graph on 2n vertices, if n is odd.

(ii) K_n, $K_{\binom{n}{2}}$, $\frac{n-1}{2}$ H and $\frac{n}{2}$ $K_{1, n-2}$ if n is even.

In the following, eccentricity properties of $\overline{BF_1(G)}$ are discussed.

Theorem 2.12: Eccentricity of a vertex in $\overline{BF_1(G)}$ corresponding to an edge in G is 2.

Proof: Let e $\in$ E(G) and let e' be vertex in $\overline{BF_1(G)}$ corresponding to e. Then $e' \in V(L(G))$

Since the subgraph of $\overline{BF_1(G)}$ induced by vertices of L(G) is complete, distance between any two vertices of L(G) in $\overline{BF_1(G)}$ is 1.

Let v, $e' \in V(\overline{BF_1(G)})$, where $v \in V(G)$. e' is the vertex corresponding to an edge $e \in E(G)$. If e $\in$ E(G) is not incident with $v \in V(G)$, then $d_{\overline{BF_1(G)}}(v, e') = 1$. Let e $\in$ E(G) be incident with v. Let there exist an edge x in G, not incident with v and let x' be the vertex in $\overline{BF_1(G)}$ corresponding to x. Then $vx'e'$ is a geodesic path in $\overline{BF_1(G)}$ and hence $d(v, e') = 2$.

If all the edges of G are incident with v, then there exists a vertex $u \in V(G)$ such that u is not incident with e. Then vue' is a geodesic path in $\overline{BF_1(G)}$ and hence $d(v, e') = 2$. Therefore, eccentricity of e' in $\overline{BF_1(G)}$ is 2.

Theorem 2.13:

Eccentricity of a vertex of G in $\overline{BF_1(G)}$ is 1, 2 or 3.

Proof: Let u be a vertex of G in $\overline{BF_1(G)}$. By the previous theorem, in $\overline{BF_1(G)}$, d(u, e′) = 2, where e′ is a vertex of L(G) in $\overline{BF_1(G)}$.

Case 1. G is connected.

Let u be a vertex of G in $\overline{BF_1(G)}$ such that u ≠ v. Since G is an induced subgraph of G, if $d_G(u, v) \le 2$, then $d_{\overline{BF_1(G)}}(u,v) \le 2$. If $d_G(u, v) \ge 3$, then there exists atleast one edge e not incident with both u and v. Let e′ be the vertex of $\overline{BF_1(G)}$ corresponding to e. Then ue′v is a geodesic path in $\overline{BF_1(G)}$ and hence $d_{\overline{BF_1(G)}}(u,v) = 2$. Therefore, eccentricity of a vertex of a connected graph G in $\overline{BF_1(G)}$ is 2. Thus, the distance between any two point vertices in $\overline{BF_1(G)}$ is 1 or 2.

Case 2. G is disconnected.

Let u, v ∈ V($\overline{BF_1(G)}$) ∩ V(G). If u and v belong to the same component of G, then $d_{\overline{BF_1(G)}}(u,v) \le 2$. Let u and v belong to the different components of G. Let G_1 and G_2 be two components of G such that each have atleast one edge. Assume u ∈ G_1 and v ∈ G_2. If there exists an edge x ∈ E(G_1)not incident with u ((or) x ∈E(G_2) not incident with v), then ux′v is a geodesic path in $\overline{BF_1(G)}$ and hence d(u, v) = 2 in $\overline{BF_1(G)}$.

Let all the edges of G be incident with one of u and v and let e_1 and $e_2$be the edges in G incident with u and v respectively. Let e_1' and e_2' be the vertices in $\overline{BF_1(G)}$ corresponding to e_1 and e_2 respectively . Then u $e_2'e_1'$ v is a geodesic path in $\overline{BF_1(G)}$ and hence $d_{\overline{BF_1(G)}}$ (u, v) = 3. Let G ≅ K_1 and V(K_1) = {u}. If there exists an edge in G_2 not incident with vthen d(u,v) = 2. Let all the edges of G_2 be incident with v. Let e_1 and e_2 be the edges incident with v and $e_1 = (v, v_1)$, $e_2 = (v, v_2)$. Then u e_1' v_2v is a geodesic path in G and hence d(u,v) = 3.

Let G have more than 3 components. Let u,v ∈ V(G) ∩ V($\overline{BF_1(G)}$). If each component has atleast one edge, then d(u,v) = 2 in $\overline{BF_1(G)}$. If G ≅ $G_1 \cup G_2 \cup mK_1$, m ≥ 1 and if G_1, G_2 are stars, then d(u,v) = 3. Otherwise, d(u,v) = 2.

Remark 2.2:

1. If $\overline{BF_1(G)}$ is connected, then diameter of $\overline{BF_1(G)}$ is at most 3.

2. For any connected graph G with atleast 3 vertices, $\overline{BF_1(G)}$ is self-centered with radius 2.

3. Let G be a disconnected graph such that $\overline{BF_1(G)}$ is connected. Then $\overline{BF_1(G)}$ is self-centered with radius 2 if and only if one of the following holds.

(i). $G \ncong 2K_2 \cup mK_1$, where $m \geq 0$.

(ii). $G \ncong K_{1,n} \cup K_{1,m}$, where $n, m \geq 2$ and

(iii). $G \ncong K_{1,n} \cup mK_1$, where $n \geq 2$ and $m \geq 1$

Proposition 2.1: Let G be a disconnected graph such that $\overline{BF_1(G)}$ is connected. Then $\overline{BF_1(G)}$ is bieccentric with radius 2 if and only if one of the following holds.

(i) $G \cong K_{1,n} \cup K_{1,m} \cup tK_1$ for $n, m \geq 2$.

(ii) $G \cong K_{1,n} \cup mK_1$, for $n \geq 2$ and $m \geq 1$ and

(iii) $G \ncong 2K_2 \cup mK_1$ form ≥ 0 .

Theorem 2.14: For any (p, q) graph G, $\alpha_0(\overline{BF_1(G)}) \leq \max\{q + \alpha_0(G), p + q - 1\}$. .

Proof: Let S be a minimum point cover of G such that $|S| = \alpha_0(G)$. Let $K = S \cup V(L(G)) \subseteq V(\overline{BF_1(G)})$. The q vertices of L(G) in $\overline{BF_1(G)}$ covers all the edges of L(G) together with the edges of the form (v_i, e_{jk}), where $v_i \in V(G)$ is not incident with $e_{jk} \in E(G)$. Therefore S is a point cover for $\overline{BF_1(G)}$. On the other hand , since the subgraph of $\overline{BF_1(G)}$ induced by vertices of L(G) in $\overline{BF_1(G)}$ is complete. $V(G) \cup K$, where K is a point cover of L(G) is a point cover, is a point cover of $\overline{BF_1(G)}$.$|K| = \alpha_0(K_q) = q-1$. Therefore $\alpha_0(\overline{BF_1(G)}) \leq p + q - 1$.

Therefore $\alpha_0(\overline{BF_1(G)}) \leq \max\{q + \alpha_0(G), p + q - 1\}$.

Theorem 2.15: If $G \neq nK_1, K_{1,n}$, then $\alpha_1(\overline{BF_1(G)}) \leq p$.

Proof: Case 1. $p \geq q$.

The set of p edges (v_i, e_j) $i = 1, 2, \ldots, p$ of $\overline{BF_1(G)}$ is a line cover for $\overline{BF_1(G)}$, e_j is not incident with v_i.

Case 2. $p < q$.

Let $S = \{(v_i, e_j)\ i = 1, 2, \ldots, p\}$, where e_j is not incident with v_i. This set S covers all the p vertices of G in $\overline{BF_1(G)}$ and p vertices of L(G) in $\overline{BF_1(G)}$. Let H be the subgraph of $\overline{BF_1(G)}$ induced by the remaining (q − p) vertices of L(G). H is a complete graph on (q − p) vertices. Therefore $\alpha_1(\overline{BF_1(G)}) \leq p + \alpha_1(K_{q-p}) = p + \left\lceil \frac{(q-p+1)}{2} \right\rceil$, since

$$\alpha_1(K_q) = \left\lceil \frac{p+1}{2} \right\rceil. \quad \alpha_1(\overline{BF_1(G)}) \leq \begin{cases} p & \text{if } p \geq q. \\ p + \left\lceil \frac{q-p+1}{2} \right\rceil & \text{if } p < q. \end{cases}$$

Example 2.1:

1. $\alpha_1(\overline{BF_1(P_n)}) \le$ n, if n ≥ 3.
2. $\alpha_1(\overline{BF_1(C_n)}) \le$ n, if n ≥ 3.
3. $\alpha_1(\overline{BF_1(K_{1,n})}) \le$ n + 1, if n ≥ 3.
4. $\alpha_1(\overline{BF_1(K_n)}) \le \left\lceil \frac{n(n+1)}{4} \right\rceil$, if $n \ge 3$.

3. Domination Numbers and other parameters for the complement of $B(\overline{G}, \overline{K_q}, INC)$

In the following, the graphs G for which the domination number γ of $\overline{BF_1(G)}$ is 2 is found.

Theorem 3.1: For any graph G, the following 2 – element sets are not dominating sets in $\overline{BF_1(G)}$

(i) two adjacent vertices of G

(ii) Vertices in $\overline{BF_1(G)}$ corresponding to two adjacent edges of G.

(iii) Vertex in $\overline{BF_1(G)}$ corresponding to an edge of G and a vertex of G.

Proof: Let D = {u, v} be any 2 – element set in V($\overline{BF_1(G)}$).

If u and v are adjacent vertices of G, then the edge uv ∈ E(G) is adjacent to neither u nor v in $\overline{BF_1(G)}$.

If u and v are vertices corresponding to adjacent edges in G, then the edges in G are incident with a common vertex, say x. But x is adjacent to neither x nor y in $\overline{BF_1(G)}$.

If u = xy ∈ E(G) and w ∈ V(G), w ≠ x, y such that d(w, x) = d(w, y) ≥ 2 in G. Then x and y are not adjacent to both u and w.

Thus in all cases, D is not dominating set in $\overline{BF_1(G)}$.

Theorem 3.2: For any graph G, $\gamma(\overline{BF_1(G)})$ = 2 if and only if q ≥ 1.

Proof: Since there is no graph G with $\gamma(\overline{BF_1(G)})$ = 1, $\gamma(\overline{BF_1(G)}) \ge 2$.

Let e=(u, v) be an edge in G. Then D = {u, e′} ⊆ V($\overline{BF_1(G)}$). v is adjacent to u in $\overline{BF_1(G)}$. Since L(G) is an induced sub graph of $\overline{BF_1(G)}$, each vertex in V(L(G)) ∩ V($\overline{BF_1(G)}$) is adjacent to e′ and each vertex in V($\overline{BF_1(G)}$) ∩ (V(G) – {u, v}) is also adjacent to e′. Therefore D is adominating set of $\overline{BF_1(G)}$ and hence $\gamma(\overline{BF_1(G)}) \le 2$.

Remark 3.1:

1. For any graph G, $\gamma(\overline{BF_1(G)}) \leq \gamma(G)$, since G is an induced subgraph of $\overline{BF_1(G)}$.
2. If $\alpha_0(G) = 2$ and if D is an independent point cover of G with |D| = 2 then $\gamma(\overline{BF_1(G)}) \leq \alpha_0(G) = 2$.
3. For any graph G, $\gamma(\overline{BF_1(G)}) \leq \alpha_0(G)$, if $\alpha_0(G) \geq 3$.

Remark 3.2: D is also a nonsplit (restrained) dominating set of $\overline{BF_1(G)}$.

Therefore $\gamma_r(\overline{BF_1(G)}) \leq \gamma_{ns}(\overline{BF_1(G)}) = 2$.

Remark 3.3:

(i) Any nonsplit dominating set of $\overline{BF_1(G)}$ containing vertices of G only need not be a nonsplit dominating set of G.

For example in C_5, the set containing any two nonadjacent vertices of C_5 is a nonsplit dominating set of $\overline{BF_1(C_5)}$, but is not a nonsplit dominating set of C_5.

(ii) $D = \{e_1', e_2'\} \subseteq V(\overline{BF_1(G)})$, where e_1' and e_2' are vertices in $\overline{BF_1(G)}$ corresponding to edges e_1 and e_2 in G is a restrained dominating set of $\overline{BF_1(G)}$ if and only if $G \neq K_{1,n}$.

(iii) Any proper subset D of V(G) is restrained dominating set of $\overline{BF_1(G)}$ if and only if D is a dominating set of G.

Example 3.1 :

(i) If $G \cong P_3 \cup mK_1$,for $m \geq 0$, then $\gamma_{ns}(\overline{BF_1(G)}) = 3$.

(ii) If $G \cong 2K_2 \cup mK_1$, for $m \geq 1$, then $\gamma_{ns}(\overline{BF_1(G)}) = 4$.

(iii) If $G \cong P_3 \cup mK_1$,for $m \geq 0$, then $\gamma_r(\overline{BF_1(G)}) = 3$.

(iv)If $G \cong 2K_2 \cup mK_1$, for $m \geq 1$, then $\gamma_r(\overline{BF_1(G)}) = 2$.

In the following, the graphs G for the perfect domination number γ_p of $\overline{BF_1(G)}$ is 2 or 3 are obtained.

Theorem 3.3: Let G be a graph other than a star. If the set D of vertices of L(G) in $\overline{BF_1(G)}$ such that atleast three of the corresponding edges are independent in G, then D is not a perfect dominating set of $\overline{BF_1(G)}$.

Proof: Let D be a subset of V(L(G)) having atleast three vertices, say e_1', e_2', e_3' such that the corresponding edges e_1, e_2, e_3 in G are independent in G. Let $v \in V(G)$ be an end vertex of e_3. Then $v \in V(\overline{BF_1(G)}) - D$ is adjacent to both e_1' and e_2' and hence D is not a perfect dominating set of $\overline{BF_1(G)}$.

Theorem 3.4: Let G be anygraph such that there exists a pair of independent edges say e_1, e_2 in G. Then $D = \{e_1', e_2'\} \subseteq V(\overline{BF_1(G)})$ is perfect dominating set for $\overline{BF_1(G)}$ if and only if $G \cong 2K_2$.

Proof: Let $D = \{e_1', e_2'\}$ be a perfect dominating set of $\overline{BF_1(G)}$. Then $D = \{e_1', e_2'\}$ is a dominating set of $\overline{BF_1(G)}$. If there exists an edge in G, then the corresponding vertex in $V(\overline{BF_1(G)}) - D$ is adjacent to both e_1' and e_2'. Similarly, if there exists a vertex $v \in V(G)$ such that e_1 and e_2 are not incident with v, then $v \in V(\overline{BF_1(G)}) - D$ is adjacent to both e_1 and e_2. Thus, D is not a perfect dominating set of $\overline{BF_1(G)}$ and hence $G \cong 2K_2$. Conversely assume $G \cong 2K_2$. Let e_1', e_2' be the vertices in $V(\overline{BF_1(G)})$ corresponding to the edges in $2K_2$. Then $D = \{e_1', e_2'\}$ is a perfect dominating set for $\overline{BF_1(G)}$.

Theorem 3.5: The set $\{u, v\} \subseteq V(G)$ is a perfect dominating set of $\overline{BF_1(G)}$ if and only if $G \cong K_{1,n} \cup K_{1,m}$, n, m ≥ 1 or $K_{1,n} \cup K_1$, n ≥ 1.

Proof: Let $D = \{u, v\} \subseteq V(\overline{BF_1(G)})$, where $u, v \in V(G)$ be a perfect dominating set of $\overline{BF_1(G)}$. If u, v are adjacent in G, then D is not a dominating set of $\overline{BF_1(G)}$. Therefore u and v are nonadjacent in G and hence in $\overline{BF_1(G)}$. Since D is a perfect dominating set of $\overline{BF_1(G)}$, each vertex in $V(\overline{BF_1(G)}) - D$ is adjacent to exactly one vertex in D. Each vertex of G in $V(\overline{BF_1(G)})$ is adjacent to exactly one of u and v. Therefore $G \cong K_{1,n} \cup K_{1,m}$, n, m ≥ 1 or $K_{1,n} \cup K_1$, n ≥ 1.

Conversely, if G is one of the graphs $G \cong K_{1,n} \cup K_{1,m}$, n, m ≥ 1 or $K_{1,n} \cup K_1$, then $D = \{u, v\} \subseteq V(G)$ is a perfect dominating set for $\overline{BF_1(G)}$.

Theorem 3.6: Let G be a graph having atleast four vertices. Any subset of V(G) having at least three vertices is not a perfect dominating set of $\overline{BF_1(G)}$.

Proof: Let D be a subset of V(G) such that $|D| \geq 3$ and $v \in V(G) - D$. There exists a vertex $u \in D$ such that $e = (u, v) \in E(G)$ and the vertex e' in $\overline{BF_1(G)}$ corresponding to e, $e' \in V(\overline{BF_1(G)}) - D$ is adjacent to atleast two vertices in D. Hence, D is not a perfect dominating set of $\overline{BF_1(G)}$.

Theorem 3.7: For any graph G having atleast one edge, $\overline{BF_1(G)}$ = 3. if and only if G is one of the graphs: C_3 and P_3.

Proof: Assume $\overline{BF_1(G)}$ =3. Then there exists a perfect dominating set D of $\overline{BF_1(G)}$ having three vertices

Case 1. All the vertices of D are vertices of G .

By Theorem 3.6, V($\overline{BF_1(G)}$) – D contains no vertices of G. Therefore, V($\overline{BF_1(G)}$) – D contains vertices corresponding to the edges in $\langle D \rangle$ and G $\cong K_2 \cup K_1$, P_3 (or) C_3.

If G $\cong K_2 \cup K_1$, then $\overline{BF_1(G)} \cong 2K_2$ and $\overline{BF_1(G)}$ = 2. Therefore G $\cong$ P_3 (or) C_3.

Case 2. D contains two vertices of G and one vertex of L(G).

Let v_1 and v_2 be nonadjacent and let e = (u, v) ∈ E(G), u, v ∈ V(G).

Then D= {v_1, v_2, e'}.Since u is not adjacent to exactly one of v_1 and v_2, say v_1.

Let e_1 = (u, v). Then e_1' is adjacent to both v_2 and e'. Therefore v_1 and v_2 are adjacent.

Case 3. D contains two vertices of L(G) and one vertex of G.

Let e_1', e_2'∈ L(G) and e_1, e_2 be the edges in G. Let e_1 = (u_1, v_1) , e_2 = (u_2, v_2) and v ∈V(G).Then D = {e_1', e_2', v} ⊆ V($\overline{BF_1(G)}$) .

Let e_1 and e_2 be adjacent and v_1 = u_2 . Since v_1 is not adjacent to $e_1'$and e_2', v_1 is adjacent to v. If x_1 = (v, v_1), then x_1' is adjacent to both e_1', e_2'.Therefore either v = v_1 (or) e_1 and e_2 are nonadjacent.

Let v = v_1. Then $u_1$is adjacent to both e_2' and v. Similarly v_2 is adjacent to e_1' and v. This is a contradiction. Therefore G $\cong$ P_3.Similarly is the case when v = u_1 (or) v = u_2.

Therefore e_1 and $e_2$are not adjacent. Since u_1, v_1 are adjacent to e_2', they are not adjacent to v. Similarly, if u_2, v_2 are adjacent to e_2', then they are not adjacent to v. But the corresponding vertex in L(G) is adjacent to both $e_1'$and e_2'. Therefore this case is not possible.

Case 4. All the vertices of D are vertices L(G).

If the subgraph of G induced by the edges corresponding to the vertices of L(G) in D is C_3, then G $\cong$ C_3.

Conversely, if G is one of the graphs given in the theorem, then $\overline{BF_1(G)}$ = 3 .

Remark 3.4: There exists no perfect dominating set containing atleast 4 vertices in $\overline{BF_1(G)}$.

In the following, the graphs G for the independent domination number γ_i of $\overline{BF_1(G)}$ is 2 are obtained.

Theorem 3.8: For any (p, q) graph G with $q \geq 2$, then $\overline{BF_1(G)} = 2$.

Proof: Let $v \in V(G)$ and since $q \geq 2$, there exists and edge e in G incident with v in G. Let e' be the vertex in $\overline{BF_1(G)}$ corresponding to e. Then $D = (v, e) \subseteq V(\overline{BF_1(G)})$. All the vertices in $V(\overline{BF_1(G)})$ - D corresponding to the edges in G and the vertices of V(G) in $\overline{BF_1(G)}$ not incident with e are adjacent to e and the vertex in V(G) incident with e is adjacent to u . Also $\langle D \rangle \cong 2K_1$. Therefore D is an independent dominating set $\overline{BF_1(G)}$ and hence $\overline{BF_1(G)} \leq 2$. Therefore $\overline{BF_1(G)} = 2$.

In the following, an upper bound for the split domination number γ_s of $\overline{BF_1(G)}$ is determined.

Theorem 3.9: Let G be any graph with $\delta(G) \geq 2$. Then $\overline{BF_1(G)} \leq q$.

Proof: Let $v \in V(G)$ be such that $deg_G(v) = \delta(G) \geq 2$ and let D' be the set of all vertices in $\overline{BF_1(G)}$ corresponding to the edges not incident with v in G.

Then $D = D' \cup N_G(v) \subseteq V(\overline{BF_1(G)})$ is a dominating set of $\overline{BF_1(G)}$. Also v is an isolated vertex in the induced subgraph $\langle V(\overline{BF_1(G)}) - D \rangle$ and hence this subgraph is disconnected. Hence, D is a split dominating set of $\overline{BF_1(G)}$. Thus $\overline{BF_1(G)} \leq |D| = q$.

This bound is attained if $G \cong C_n$, $n \geq 3$.

Theorem 3.10: For any connected graph G, $\gamma_S(\overline{BF_1(G)}) \leq p + q - \Delta(G) - 1$.

Proof: Let v be a vertex of maximum degree in G and $deg_G(v) = \Delta(G)$. Let S be the set of vertices in $\overline{BF_1(G)}$ corresponding to the edges incident with v in G and let $D = S \cup \{v\}$ $\subseteq V(\overline{BF_1(G)})$. If $D' = V(\overline{BF_1(G)}) - D$, then D' is a dominating set of $\overline{BF_1(G)}$. Also v is an isolated vertex in the induced subgraph $\langle V(\overline{BF_1(G)}) - D \rangle$ and is disconnected.Hence, D' is a split dominating of $\overline{BF_1(G)}$.

Thus $\overline{BF_1(G)} \leq |D| = |V(\overline{BF_1(G)})| - |D| = p + q - \Delta(G) - 1$.

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