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Clique Partition Numbers of Boolean Function Graphs $B(\overline{K_p}, \overline{L(G)}, INC, NINC)$ and $B(\overline{K_p}, L(G), INC, NINC)$

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Abstract: *A clique in a graph G is a complete subgraph of G. A clique partition of G is a collection C of cliques such that each edge of G occurs in exactly one clique in C. The clique partition number cp(G) is the minimum size of a clique partition of G. In this paper upper bounds for the clique partition number of the Boolean function graphs* $BF_2(G)$ *and* $BF_3(G)$ *for some standard graphs are obtained.*

Keyword: *Boolean Function Graph, clique, clique partition.*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. A clique partition of G is a collection C of cliques such that each edge of G occurs in exactly one clique in C. The clique partition number cp(G) is the minimum size of a clique partition of G. The Line graphs, Middle graphs, Total graphs and Quasi-total graphs are very much useful in computer networks.

Whitney[16] introduced the concept of the line graph L(G) of a given graph G in 1932. The first characterization of line graph is due to Krausz. The Middle graph M(G) of a graph G was introduced by Hamada and Yoshimura [5]. Characterizations were presented for middle graphs of any graph, tree and complete graphs in [1]. The concept of total graphs was introduced by Behzad [2] in 1966. Sastry and Raju [15] introduced the concept of quasi-total graphs and they solved the graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. This motivates us to define and study other graph operations.

The points and Lines of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. The total graph T(G) of G has vertex set V(G)∪E(G) and vertices of T(G) are adjacent whenever they are neighbors in G. The quasi-total graph [9] P(G) of G is a graph with vertex set as that of T(G) and two vertices are adjacent if and only if they correspond to two nonadjacent vertices of G or to two adjacent edges of G or to a vertex and an edge incident to it in G. The middle graph M(G) of G is the one whose vertex set is as that of T(G) and two vertices are adjacent in M(G) whenever either they are adjacent edges of G or one is a vertex of G and the other is an edges of G incident with it. Clearly, E(M(G)) = E(T(G)) – E(G).

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The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1 of order n and n copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2.

For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph $B(\overline{K_p}, \overline{L(G)}, INC, NINC)$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(\overline{K_p}, \overline{L(G)}, INC, NINC)$ are adjacent if and only if they correspond to two nonadjacent edges of G or to a vertex and an edge incident to it in G, or to a vertex and an edge not incident to it in G, where L(G) is the line graph of G. For brevity, this graph is denoted by $BF_2(G)$.

For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph $B(\overline{K_p}, L(G), INC, NINC)$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(\overline{K_p}, L(G), INC, NINC)$ are adjacent if and only if they correspond to two adjacent edges of G or to a vertex and an edge incident to it in G, or to a vertex and an edge not incident to it in G, where L(G) is the line graph of G. For brevity, this graph is denoted by $BF_3(G)$.

In this paper, upper bounds for the clique partition numbers of the Boolean function graph $BF_2(G)$ and $BF_3(G)$ for some standard graphs are obtained. For unexplained terminology and notations, [4] is referred.

2. Clique partition of $BF_2(G)$

In the following, clique partition number of path, cycle, star and wheel graphs are found.

Theorem 2.1:

For the path P_n on n vertices (n ≥ 6), $cp(BF_2(P_n)) = \begin{cases} \dfrac{3n^2+2n-1}{4} & \text{if } n \text{ is odd.} \\ \dfrac{3n^2+2n}{4} & \text{if } n \text{ is even.} \end{cases}$

Proof:

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of P_n, where $e_i = (v_i, v_{i+1})$, $(1 \le i \le n-1)$. Then $v_1, v_2, v_3, \ldots, v_n, e_1, e_2, \ldots, e_{n-1} \in V(BF_2(P_n))$, $|V(BF_2(P_n))| = 2n-1$ and $|E(BF_2(P_n))| = |E(\overline{L(P_n)})| + n(n-1)$

$$= \frac{(n-1)(n-2)}{2} - (n-2) + n(n-1)$$

$$= \frac{3n^2-7n+6}{2}.$$

The clique number of $BF_2(P_n)$ is $\frac{n}{2}$.

$E(BF_2(P_n)) = E(\overline{L(P_n)}) \bigcup F$, where $F = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n-1}\left(v_j, e_i\right)\right)$; $|F| = n(n-1)$.

Case1: n is odd

The edge set of $BF_2(P_n)$ is decomposed into $K_{\frac{n-1}{2}}$, K_3 and $K_2{'}$s. Vertex sets of $K_{\frac{n-1}{2}}$ are listed as elements of the sets A_1 and A_2, where $A_1 = \{e_1, e_3, \ldots, e_{n-2}\}$; $A_2 = \{e_2, e_4, \ldots, e_{n-1}\}$, $< A_1 > \cong < A_2 > \cong K_{\frac{n-1}{2}}$. Vertex sets of $K_3{'}$s are given by

$A_3 = \{\{v_i, e_1, e_{2i+2}\}$, for each i, $1 \le i \le \frac{n-3}{2}\}$, $< A_3 > \cong \frac{n-3}{2} K_3$.

$A_4 = \bigcup_{i=1}^{\frac{(n-5)}{2}} B_i$, where $B_1 = \{\{v_1, e_i, e_{i+3}\}$, $i = 2, 3, \ldots, n-4\}$, $B_2 = \{\{v_2, e_i, e_{i+5}\}$, $i = 2, 3, \ldots, n-6\}$, $B_3 = \{\{v_3, e_i, e_{i+7}\}$, $i = 2, 3, \ldots, n-8\}, \ldots, B_{\frac{n-5}{2}} = \{\{v_{\frac{n-5}{2}}, e_i, e_{i+(n-4)}\}$, $i = 2, 3\}$ and hence $< A_4 > \cong \frac{n^2 - 8n + 15}{4} K_3$.

These cover all the edges of $\overline{L(P_n)}$ and $\frac{n^2 - 6n + 9}{2}$ edges of F. The remaining $\frac{n^2 + 4n - 9}{2}$ edges of F are covered by $K_2{'}$s.

Therefore, $BF_2(P_n) = 2K_{\frac{n-1}{2}} \bigcup \left(\frac{n^2 - 6n + 9}{4}\right) K_3 \bigcup \left(\frac{n^2 + 4n - 9}{2}\right) K_2$ and hence

$$cp(BF_2(P_n)) = 2 + \left(\frac{n^2 - 6n + 9}{4}\right) + \left(\frac{n^2 + 4n - 9}{2}\right) = \frac{3n^2 + 2n - 1}{4}.$$

Case2: n is even

The edge set of $BF_2(P_n)$ is decomposed into $K_{\frac{n}{2}}$, $K_{\frac{n-2}{2}}$, K_3 and $K_2{'}$s. Vertex sets of $K_{\frac{n}{2}}$, $K_{\frac{n-2}{2}}$ are listed as elements of the sets C_1 and C_2. $C_1 = \{e_1, e_3, \ldots, e_{n-1}\}$; $C_2 = \{e_2, e_4, \ldots, e_{n-2}\}$ and $< C_1 > \cong K_{\frac{n}{2}}$, $< C_2 > \cong K_{\frac{n-2}{2}}$. Vertex sets of K_3's are given by $C_3 = \bigcup_{i=1}^{\frac{(n-4)}{2}} D_i$ where $D_1 = \{\{v_1, e_i, e_{i+3}\}$, $i = 1, 2, 3, \ldots, n-4\}$, $D_2 = \{\{v_2, e_i, e_{i+5}\}$, $i = 1, 2, 3, \ldots, n-6\}$,

$D_3 = \{\{ v_3, e_i, e_{i+7} \}, i = 1, 2, 3, \ldots, n - 8\}, \ldots D_{\frac{n-4}{2}} = \{\{ v_{\frac{n-4}{2}}, e_i, e_{i+(n-3)} \}, i = 1, 2\}$ and hence

$< C_3 > \cong \left(\frac{n^2 - 6n + 8}{4}\right) K_3.$

These cover all the edges of $\overline{L(P_n)}$ and $\frac{n^2 - 6n + 8}{2}$ edges of F. The remaining $\frac{n^2 + 4n - 8}{2}$ edges of F are covered by $K_2{}'s$.

Therefore, $BF_2(P_n) = K_{\frac{n}{2}} \bigcup K_{\frac{n-2}{2}} \bigcup \left(\frac{n^2 - 6n + 8}{4}\right) K_3 \bigcup \left(\frac{n^2 + 4n - 8}{2}\right) K_2$ and hence

$cp(BF_2(P_n)) = 2 + \left(\frac{n^2 - 6n + 8}{4}\right) + \left(\frac{n^2 + 4n - 8}{2}\right) = \frac{3n^2 + 2n}{4}.$

Therefore, $cp(BF_2(P_n)) = \begin{cases} \frac{3n^2 + 2n - 1}{4} & \text{if } n \text{ is odd.} \\ \frac{3n^2 + 2n}{4} & \text{if } n \text{ is even.} \end{cases}$

Theorem 2.2:

For any cycle C_n on n vertices $(n \geq 6)$, $cp(BF_2(C_n)) = \begin{cases} \frac{3n^2 + 6n - 1}{4} & \text{if } n \text{ is odd.} \\ \frac{3n^2 + 4n + 8}{4} & \text{if } n \text{ is even.} \end{cases}$

Proof:

Let v_i $(1 \leq i \leq n)$ be the vertices of C_n. Let $e_i = (v_i, v_{i+1})$, $(1 \leq i \leq n - 1)$ and $e_n = (v_n, v_1)$. Then $V(BF_2(C_n)) = V(C_n) \bigcup E(C_n)$, $|V(BF_2(C_n)| = 2n$, $|E((BF_2(C_n)| = |E(\overline{L(C_n)})| + n^2 = \left(\frac{3n(n-1)}{2}\right)$. The clique number of $BF_2(C_n)$ is $\frac{n+1}{2}$.

$E(BF_2(C_n)) = E(\overline{L(C_n)}) \bigcup F$, where $F = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n}(v_j, e_i)\right)$; $|F| = n^2$.

Case1: n is odd

The edge set of $BF_2(C_n)$ is decomposed into $K_{\frac{n-1}{2}}$, K_3 and $K_2{}'s$.

Vertex sets of $2K_{\frac{n-1}{2}}$ are listed as elements of the sets A_1 and A_2, where

$A_1 = \{e_1, e_3, \ldots, e_{n-2}\}$; $A_2 = \{e_2, e_4, \ldots, e_{n-1}\}$, $< A_1 > \cong < A_2 > \cong K_{\frac{n-1}{2}}$.

Vertex sets of K_3's are given by $A_3 = \bigcup_{i=1}^{\frac{(n-3)}{2}} B_i$ where $B_1 = \{\{ v_1, e_i, e_{i+3} \}, i = 1, 2, 3, \ldots, n - 3\}$, $B_2 = \{\{ v_2, e_i, e_{i+5} \}, i = 1, 2, 3, \ldots, n - 5\}$, $B_3 = \{\{ v_3, e_i, e_{i+7} \}, i = 1, 2, 3, \ldots, n - 7\}, \ldots$, $B_{\frac{n-3}{2}} = \{\{ v_{\frac{n-3}{2}}, e_i, e_{i+(n-2)} \}, i = 1, 2\}$ and hence $< A_3 > \cong \left(\frac{n^2 - 4n + 3}{4}\right) K_3$.

These cover $\left(\frac{n^2 - 4n + 3}{2}\right)$ edges of $\overline{L(C_n)}$ and $\left(\frac{n^2 - 4n + 3}{2}\right)$ edges of F. The remaining $\frac{n-3}{2}$ edges of $\overline{L(C_n)}$ and $\left(\frac{n^2 + 4n - 3}{2}\right)$ edges of F are covered by K_2's and in total there are $\left(\frac{n^2 + 5n - 6}{2}\right) K_2'$s.

Therefore, $BF_2(C_n) = 2K_{\frac{n-1}{2}} \bigcup \left(\frac{n^2 - 4n + 3}{4}\right) K_3 \bigcup \left(\frac{n^2 + 5n - 6}{2}\right) K_2$ and hence

$$cp(BF_2(C_n)) = 2 + \left(\frac{n^2 - 4n + 3}{4}\right) + \left(\frac{n^2 + 5n - 6}{2}\right) = \frac{3n^2 + 6n - 1}{4}$$

Case2: n is even

The edge set of $BF_2(C_n)$ is decomposed into $K_{\frac{n}{2}}$, K_3 and K_2's. Vertex sets of $2K_{\frac{n}{2}}$ are listed as elements of the sets D_1 and D_2.

$D_1 = \{e_1, e_3, \ldots, e_{n-1}\}$; $D_2 = \{e_2, e_4, \ldots, e_n\}$ and $< D_1 > \cong < D_2 > \cong K_{\frac{n}{2}}$.

Vertex sets of K_3's are given by

$D_3 = \{\{v_i, e_1, e_{2i+2}\}$, for each i, $1 \leq i \leq \frac{n-4}{2}\}$ and $< D_3 > \cong \frac{n-4}{2} K_3$.

$D_4 = \bigcup_{j=1}^{\frac{(n-4)}{2}} D_{j,i}$ where $D_{1,i} = \{\{ v_1, e_i, e_{i+3} \}, i = 2, 3, \ldots, n - 3\}$, $D_{2,i} = \{\{ v_2, e_i, e_{i+5} \}, i = 2, 3, \ldots, n - 5\}$, $D_{3,i} = \{\{ v_3, e_i, e_{i+7} \}, i = 2, 3, \ldots, n - 7\}, \ldots, D_{\frac{n-4}{2},i} = \{\{ v_{\frac{n-4}{2}}, e_i, e_{i+(n-3)} \}, i = 2, 3\}$ and $< D_4 > \cong \left(\frac{n^2 - 6n + 8}{4}\right)$.

These cover all the edges of $\overline{L(C_n)}$ and $\frac{n^2 - 4n}{2}$ edges of F. The remaining $\frac{n^2 - 4n}{2}$ edges of F are covered by K_2's.

Therefore, $BF_2(C_n) = 2K_{\frac{n}{2}} \cup \left(\frac{n^2-4n}{2}\right)K_3 \cup \left(\frac{n^2+4n}{2}\right)K_2$ and hence

$$cp(BF_2(C_n)) = 2 + \left(\frac{n^2-4n}{2}\right) + \left(\frac{n^2+4n}{2}\right) = \frac{3n^2+4n+8}{4}.$$

$$\text{Therefore, } cp(BF_2(C_n)) = \begin{cases} \dfrac{3n^2+6n-1}{4} & \text{if } n \text{ is odd.} \\ \dfrac{3n^2+4n+8}{4} & \text{if } n \text{ is even.} \end{cases}.$$

Theorem 2.3:

For the star $K_{1,n}$ on n vertices ($n \geq 3$), $cp(BF_2(K_{1,n})) = n(n+1)$.

Proof:

Let v be the central vertex and $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $e_1, e_2, \ldots, e_n$ be the edges of $K_{1,n}$, where $e_i = (v, v_i)$, $(1 \leq i \leq n)$.

Then $v, v_1, v_2, v_3, \ldots, v_n, e_1, e_2, \ldots, e_n \in V(BF_2(K_{1,n}))$ and $|V(BF_1(K_{1,n}))| = 2n + 1$ and $|E(BF_1(K_{1,n}))| = n(n+1)$ and the clique number is 2. Since $BF_2(K_{1,n})$ is C_3 free and edges of $BF_2(K_{1,n})$ can be decomposed into K_2's only. The edge sets of $n(n+1)K_2$ are denoted as A_1 and A_2 are given as

$A_1 = \{(v, e_i): 1 \leq i \leq n\}$ and $A_2 = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n}\left(v_j, e_i\right)\right)$, $|A_1| = n$; $|A_2| = n^2$. $|A_1| + |A_2| = n(n+1)$.

Therefore $cp(BF_2(K_{1,n})) = n(n+1)$.

Theorem 2.4:

For the wheel W_{n+1} on (n+1) vertices ($n \geq 6$),

$$cp(BF_2(W_{n+1})) = \begin{cases} \dfrac{11n^2-2n-1}{4} & \text{if } n \text{ is odd.} \\ \dfrac{11n^2-4n+8}{4} & \text{if } n \text{ is even.} \end{cases}.$$

Proof:

Let v be the central vertex of W_{n+1} and $v_1, v_2, v_3, \ldots, v_n$ be the vertices of cycle C_n. Let $e_i = (v_i, v_j)$, $1 \leq i \leq n$ and $j \equiv (i+1)(\text{mod } n)$ and $f_i = (v, v_i)$, $1 \leq i \leq n$. Then $V(BF_2(W_{n+1})) = V(W_{n+1}) \cup E(W_{n+1})$.

$|V(BF_2(W_{n+1}))| = (n+1) + (2n) = 3n + 1.$

$|E(BF_2(W_{n+1}))| = |E(L(W_{n+1}))| + 2n(n+1) = \frac{7n^2 - 3n}{2}$ and the clique number of is $\frac{n+1}{2}$. Then $E(BF_2(W_{n+1})) = E(L(W_{n+1})) \cup F \cup H$, where $F = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n}\{(v_j, e_i)(v_j, f_i)\}\right)$ and $H = \bigcup_{i=1}^{n}\{(v, e_i)(v, f_i)\}$; $|F| = 2n^2$, $|H| = 2n$.

Case1: n is odd

The edge set of $BF_2(W_{n+1})$ is decomposed into $K_{\frac{n-1}{2}}$, K_3 and K_2's.

Vertex sets of $2K_{\frac{n-1}{2}}$ are listed as elements of the sets A_1 and A_2, where

$A_1 = \{e_1, e_3, \ldots, e_{n-2}\}$; $A_2 = \{e_2, e_4, \ldots, e_{n-1}\}$, $\langle A_1 \rangle \cong \langle A_2 \rangle \cong K_{\frac{n-1}{2}}$.

Vertex sets of K_3's are given by

$A_3 = \bigcup_{i=1}^{\frac{(n-3)}{2}} B_i$ where

$B_1 = \{\{v_1, e_i, e_{i+3}\}, i = 1, 2, 3, \ldots, n-3\}$.

$B_2 = \{\{v_2, e_i, e_{i+5}\}, i = 1, 2, 3, \ldots, n-5\}$.

$B_3 = \{\{v_3, e_i, e_{i+7}\}, i = 1, 2, 3, \ldots, n-7\}$.

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$B_{\frac{n-3}{2}} = \{\{v_{\frac{n-3}{2}}, e_i, e_{i+(n-2)}\}, i = 1, 2\}$ and $\langle A_3 \rangle \cong \left(\frac{n^2 - 4n + 3}{4}\right) K_3$.

$A_4 = \{\{v, f_i, e_{i+1}\}; 1 \le i \le n, e_{n+1} = e_1\}$ and $\langle A_4 \rangle \cong nK_3$.

These cover all the edges of H, $\frac{n^2 - 2n + 3}{2}$ edges of $\overline{L(W_{n+1})}$ and $\frac{n^2 - 4n + 3}{2}$ edges of F. The remaining $\frac{5n^2 - n - 6}{2}$ edges are covered by K_2's.

Therefore, $BF_2(W_{n+1}) = 2K_{\frac{n-1}{2}} \cup \left(\frac{n^2 + 3}{4}\right) K_3 \cup \left(\frac{5n^2 - n - 6}{2}\right) K_2$ and hence

$$cp(BF_2(W_{n+1})) = 2 + \left(\frac{n^2 + 3}{4}\right) + \left(\frac{5n^2 - n - 6}{2}\right) = \frac{11n^2 - 2n - 1}{4}.$$

Case 2: n is even

The edge set of $BF_2(W_{n+1})$ is decomposed into $K_{\frac{n}{2}}$, K_3 and K_2's.

Vertex sets of $2K_{\frac{n}{2}}$ are listed as elements of the sets C_1 and C_2.

$C_1 = \{e_1, e_3, \ldots, e_{n-1}\}$; $C_2 = \{e_2, e_4, \ldots, e_n\}$ and $< C_1 > \cong < C_2 > \cong K_{\frac{n}{2}}$.

Vertex sets of K_3's are given by

$C_3 = \{\{v_i, e_1, e_{2i+2}\}$, for each i, $1 \le i \le \frac{n-4}{2}\}$and $< C_3 > \cong \frac{n-4}{2} K_3$.

$C_4 = \bigcup_{i=1}^{\frac{(n-4)}{2}} D_i$, where $D_1 = \{\{ v_1, e_i, e_{i+3} \}, i = 2, 3, \ldots, n-3\}$, $D_2 = \{\{ v_2, e_i, e_{i+5} \}, i = 2, 3, \ldots, n-5\}$, $D_3 = \{\{ v_3, e_i, e_{i+7} \}, i = 2, 3, \ldots, n-7\}, \ldots, D_{\frac{n-4}{2}} = \{\{ V_{\frac{n-4}{2}}, e_i, e_{i+(n-3)} \}, i = 2, 3\}$and

$< C_4 > \cong \left(\frac{n^2 - 6n + 8}{4}\right) K_3$.

$c_5 = \{\{v, f_i, e_{i+1}\}; 1 \le i \le n, e_{n+1} = e_1\}$ and $< C_5 > \cong nK_3$.

These cover all the edges of H, $\left(\frac{n^2 - n}{2}\right)$ edges of $\overline{L(C_n)}$ and edges $\left(\frac{n^2 - 4n}{2}\right)$ of F.

The remaining $\left(\frac{5n^2 - 2n}{2}\right)$ edges of F are covered by K_2's.

Therefore, $BF_2(W_{n+1}) = 2K_{\frac{n}{2}} \cup \left(\frac{n^2}{4}\right) K_3 \cup \left(\frac{5n^2 - 2n}{2}\right) K_2$ and hence

$cp(BF_2(W_{n+1})) = 2 + \left(\frac{n^2}{4}\right) + \left(\frac{5n^2 - 2n}{2}\right) = \frac{11n^2 - 4n + 8}{4}$.

Therefore, $cp(BF_2(W_{n+1})) = \begin{cases} \frac{11n^2 - 2n - 1}{4} & \text{if n is odd.} \\ \frac{11n^2 - 4n + 8}{4} & \text{if n is even.} \end{cases}$

In the following clique partition number of $P_n \circ K_1$ and $C_n \circ K_1$ are found.

Theorem 2.5:

For the graph $P_n \circ K_1$ $(n \ge 6)$, $cp(BF_2(P_n \circ K_1)) = \begin{cases} \frac{9n^2 - 6n + 3}{2} & \text{if n is odd.} \\ \frac{9n^2 - 7n + 8}{2} & \text{if n is even.} \end{cases}$

Proof:

Let v_i $(1 \le i \le n)$ be the vertices of P_n with v_1 and v_n as pendant vertices and let $e_i = (v_i, v_{i+1})$, $(1 \le i \le n-1)$ be the edges of P_n. Let u_i be the pendant vertex adjacent to v_i $(1 \le i \le n)$ and let $f_i = (v_i, u_i)$, $(1 \le i \le n)$.

$V(BF_2(P_n \circ K_1)) = V(P_n \circ K_1) \bigcup E(P_n \circ K_1)$.

Therefore $|V(BF_2(P_n \circ K_1))| = 2n + 2n - 1 = 4n - 1$.

Let $F = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n-1}\{(v_j, e_i), (u_j, e_i)\}\right)$ and $H = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n}\{(v_j, f_i), (u_j, f_i)\}\right)$

$|F| = 2n(n-1)$; $|H| = 2n^2$. Then $E(BF_2(P_n \circ K_1)) = E(L(P_n \circ K_1) \bigcup F \bigcup H$.

$|E(BF_2(P_n \circ K_1)| = 6n^2 - 8n + 5$. The clique number of $BF_2(P_n \circ K_1)$ is $\frac{n-1}{2}$

Case1: n is odd

Vertex sets of $K_{\frac{n-1}{2}}$ are listed as elements of the sets A_1, A_2, A_3 and A_4.

$A_1 = \{e_1, e_3, \ldots, e_{n-2}\}$; $A_2 = \{e_2, e_4, \ldots, e_{n-1}\}$.

$A_3 = \{f_1, f_3, \ldots, f_{n-2}\}$; $A_4 = \{f_2, f_4, \ldots, f_{n-1}\}$.

Vertex sets of K_3's are given by

$A_5 = \bigcup_{i=1}^{\frac{(n-5)}{2}} B_i$ where

$B_1 = \{\{v_1, e_i, e_{i+3}\}, i = 2, 3, \ldots, n-4\}$.

$B_2 = \{\{v_2, e_i, e_{i+5}\}, i = 2, 3, \ldots, n-6\}$.

$B_3 = \{\{v_3, e_i, e_{i+7}\}, i = 2, 3, \ldots, n-8\}$.

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$B_{\frac{n-5}{2}} = \{\{v_{\frac{n-5}{2}}, e_i, e_{i+(n-4)}\}, i = 2, 3\}$ and $< A_5 > \cong \left(\frac{n^2 - 8n + 15}{4}\right) K_3$.

$A_6 = \bigcup_{i=1}^{\frac{(n-3)}{2}} C_i$ where

$C_1 = \{\{u_1, f_i, f_{i+3}\}, i = 1, 2, 3, \ldots, n-3\}$.

$C_2 = \{\{u_2, f_i, f_{i+5}\}, i = 1, 2, 3, \ldots, n-5\}$.

$C_3 = \{\{u_3, f_i, f_{i+7}\}, i = 1, 2, 3, \ldots, n-7\}$.

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$C_{\frac{n-3}{2}} = \{\{u_{\frac{n-3}{2}}, f_i, f_{i+(n-2)}\}, i = 1, 2\}$ and $< A_6 > \cong \left(\frac{n^2 - 4n + 3}{4}\right) K_3$.

$A_7 = \{\{v_i, f_i, f_{i+1}\}; 1 \le i \le n, f_{n+1} = f_1\}$ and $< A_7 > \cong nK_3$.

$A_8 = \{\{v_i, e_1, e_{2i+2}\}$, for each i, $1 \leq i \leq \frac{n-3}{2}\}$ and $< A_7 > \cong \frac{n-3}{2} K_3$.

These cover $\left(\frac{2n^2 - 7n + 9}{2}\right)$ edges of $\overline{L(P_n \circ K_1)}$ and $\left(\frac{n^2 - 6n + 9}{2}\right)$ edges of F and $\left(\frac{n^2 + 3}{2}\right)$ edges of H. The remaining $\left(\frac{8n^2 - 3n - 11}{2}\right)$ edges are covered by K_2's.

Therefore, $BF_2(P_n \circ K_1) = 4K_{\frac{n-1}{2}} \cup \left(\frac{n^2 - 3n + 6}{2}\right) K_3 \cup \left(\frac{8n^2 - 3n - 11}{2}\right) K_2$ and hence

$cp(BF_2(P_n \circ K_1)) = 4 + \left(\frac{n^2 - 3n + 6}{2}\right) + \left(\frac{8n^2 - 3n - 11}{2}\right) = \frac{9n^2 - 6n + 3}{2}$.

Case2: n is even

Vertex sets of $K_{\frac{n}{2}}$, $K_{\frac{n-2}{2}}$ are listed as elements of the sets D_1, D_2, D_3 and D_4, where

$D_1 = \{e_1, e_3, \ldots, e_{n-1}\}$.

$D_2 = \{e_2, e_4, \ldots, e_{n-2}\}$.

$D_3 = \{f_1, f_3, \ldots, f_{n-1}\}$.

$D_4 = \{f_2, f_4, \ldots, f_n\}$ and $< D_1 > \cong < D_3 > \cong < D_4 > \cong K_{\frac{n}{2}}$; $< D_2 > \cong K_{\frac{n-2}{2}}$.

Vertex sets of K_3's are given by

$D_5 = \bigcup_{i=1}^{\frac{(n-4)}{2}} E_i$ where

$E_1 = \{\{v_1, e_i, e_{i+3}\}, i = 1, 2, 3, \ldots, n-4\}$.

$E_2 = \{\{v_2, e_i, e_{i+5}\}, i = 1, 2, 3, \ldots, n-6\}$.

$E_3 = \{\{v_3, e_i, e_{i+7}\}, i = 1, 2, 3, \ldots, n-8\}$.

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$E_{\frac{n-4}{2}} = \{\{v_{\frac{n-4}{2}}, e_i, e_{i+(n-3)}\}, i = 1, 2\}$ and $< D_5 > \cong \left(\frac{n^2 - 6n + 8}{4}\right) K_3$.

$D_6 = \bigcup_{i=1}^{\frac{(n-3)}{2}} J_i$, where

$J_1 = \{\{u_1, f_i, f_{i+3}\}, i = 2, 3, \ldots, n-3\}$.

$J_2 = \{\{u_2, f_i, f_{i+5}\}, i = 2, 3, \ldots, n-5\}$.

$J_3 = \{\{u_3, f_i, f_{i+7}\}, i = 2, 3, \ldots, n-7\}$.

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$J_{\frac{n-3}{2}} = \{\{u_{\frac{n-4}{2}}, f_i, f_{i+(n-3)}\}, i = 2, 3\}$ and $< D_6 > \cong \left(\frac{n^2 - 6n + 8}{4}\right)K_3$.

$D_7 = \{\{u_i, f_1, f_{2i+2}\}; 1 \le i \le \frac{n-4}{2}\}$ and $< D_7 > \cong \frac{n-4}{2}K_3$.

$D_8 = \{\{v_i, f_i, f_{i+1}\}; 1 \le i \le n, f_{n+1} = f_1\}$ and $< D_8 > \cong nK_3$.

These cover $n^2 - 3n + 3$ edges of $\overline{L(P_n \circ K_1)}$ and $\frac{n}{2}$ edges of F and $\left(\frac{n^2 - 2n + 8}{2}\right)$ edges H. The remaining $\left(\frac{8n^2 - 4n - 4}{2}\right)$ edges are covered by K_2's.

Therefore, $BF_2(P_n \circ K_1) = 3K_{\frac{n}{2}} \cup K_{\frac{n-2}{2}} \cup \left(\frac{n^2 - 3n + 4}{2}\right)K_3 \cup \left(\frac{8n^2 - 4n - 4}{2}\right)K_2$

and hence $cp(BF_2(P_n \circ K_1)) = 4 + \left(\frac{n^2 - 3n + 4}{2}\right) + \left(\frac{8n^2 - 4n - 4}{2}\right) = \frac{9n^2 - 7n + 8}{2}$.

Therefore, $cp(BF_2(P_n \circ K_1)) = \begin{cases} \frac{9n^2 - 6n + 3}{2} & \text{if n is odd.} \\ \frac{9n^2 - 7n + 8}{2} & \text{if n is even.} \end{cases}$.

Theorem 2.6:

For the graph $C_n \circ K_1$ $(n \ge 6)$, $cp(BF_2(C_n \circ K_1)) = \begin{cases} \frac{9n^2 - 6n + 3}{2} & \text{if n is odd.} \\ \frac{9n^2 - 7n + 8}{2} & \text{if n is even.} \end{cases}$.

Proof:

Let v_i $(1 \le i \le n)$ be the vertices of C_n and let u_i $(1 \le i \le n)$ be the pendant vertex adjacent to v_i. Let $e_i = (v_i, v_{i+1})$, $(1 \le i \le n - 1)$, $e_n = (v_n, v_1)$ and $f_i = (v_i, u_i)$; $(1 \le i \le n)$. $V(BF_2(C_n \circ K_1)) = V(C_n \circ K_1) \cup E(C_n \circ K_1)$. Therefore $|V(BF_2(C_n \circ K_1))| = 4n$.

Then $E(BF_2(C_n \circ K_1)) = L(\overline{(C_n \circ K_1)}) \cup F$,

where $F = \bigcup_{j=1}^{n}\left\{\bigcup_{i=1}^{n}(v_j, e_i), (v_j, f_i), (u_j, e_i), (u_j, f_i)\right\}$; $|F| = 4n^2$.

$|E(BF_2(C_n \circ K_1))| = 6n^2 - 4n$.

Case1: n is odd

The edge set of $BF_2(C_n \circ K_1)$ is decomposed into $K_{\frac{n-1}{2}}$, K_3 and K_2's.

Vertex set of $K_{\frac{n-1}{2}}$ are listed as elements of the sets A_1, A_2, A_3 and A_4, where

$A_1 = \{e_1, e_3, \ldots, e_{n-2}\}$; $A_2 = \{e_2, e_4, \ldots, e_{n-1}\}$.

$A_3 = \{f_1, f_3, \ldots, f_{n-2}\}$; $A_4 = \{f_2, f_4, \ldots, f_{n-1}\}$.

Vertex sets of K_3's are given by

$A_5 = \bigcup_{i=1}^{\left(\frac{n-3}{2}\right)} B_i$, where

$B_1 = \{\{ v_1, e_i, e_{i+3} \}, i = 1, 2, 3, \ldots, n - 3\}$.

$B_2 = \{\{ v_2, e_i, e_{i+5} \}, i = 1, 2, 3, \ldots, n - 5\}$.

$B_3 = \{\{ v_3, e_i, e_{i+7} \}, i = 2, 3, \ldots, n - 7\}$.

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$B_{\frac{n-3}{2}} = \{\{ v_{\frac{n-3}{2}}, e_i, e_{i+(n-2)} \}, i = 1, 2\}$ and $< A_5 > \cong \left(\frac{n^2 - 4n + 3}{4}\right) K_3$.

$A_6 = \bigcup_{i=1}^{\left(\frac{n-3}{2}\right)} C_i$ where

$C_1 = \{\{ u_1, f_i, f_{i+3} \}, i = 1, 2, 3, \ldots, n - 3\}$.

$C_2 = \{\{ u_2, f_i, f_{i+5} \}, i = 1, 2, 3, \ldots, n - 5\}$.

$C_3 = \{\{ u_3, f_i, f_{i+7} \}, i = 1, 2, 3, \ldots, n - 7\}$.

.

.

$C_{\frac{n-3}{2}} = \{\{ u_{\frac{n-3}{2}}, f_i, f_{i+(n-2)} \}, i = 1, 2\}$and $< A_6 > \cong \left(\frac{n^2 - 4n + 3}{4}\right) K_3$

$A_7 = \{\{v_i, f_i, f_{i+1}\}; 1 \le i \le n, f_{n+1} = f_1\}$ and $< A_7 > \cong nK_3$

These cover $n^2 - 3n + 3$ edges of $\overline{L(C_n \circ K_1)}$ and $n^2 - 2n + 3$ edges of F. The remaining $4n^2 + n - 6$ edges are covered by K_2's.

Therefore, $BF_2(C_n \circ K_1) = 4K_{\frac{n-1}{2}} \bigcup \left(\frac{n^2 - 2n + 3}{2}\right) K_3 \bigcup (4n^2 + n - 6)K_2$ and hence

$$cp(BF_2(C_n \circ K_1)) = 4 + \left(\frac{n^2 - 2n + 3}{2}\right) + (4n^2 + n - 6) = \frac{9n^2 - 1}{2}.$$

Case2: n is even

The edge set of $BF_2(C_n \circ K_1)$ is decomposed into edges of $K_{\frac{n}{2}}$, K_3, K_2's.

Vertex sets of $4K_{\frac{n}{2}}$ are listed as element of the sets D_1, D_2, D_3 and D_4, where

$D_1 = \{e_1, e_3, \ldots, e_{n-1}\}$; $D_2 = \{e_2, e_4, \ldots, e_n\}$.

$D_3 = \{f_1, f_3, \ldots, f_{n-1}\}$; $D_4 = \{f_2, f_4, \ldots, f_n\}$.

Vertex sets of K_3's are given by

$D_5 = \{\{v_i, e_1, e_{2i+2}\}; 1 \le i \le \frac{n-4}{2}\}$ and $< D_5 > \cong \left(\frac{n-4}{2}\right) K_3$.

$D_6 = \bigcup_{i=1}^{\left(\frac{n-4}{2}\right)} E_i$ where

$E_1 = \{\{v_1, e_i, e_{i+3}\}, i = 2, 3, \ldots, n-3\}$.

$E_2 = \{\{v_2, e_i, e_{i+5}\}, i = 2, 3, \ldots, n-5\}$.

$E_3 = \{\{v_3, e_i, e_{i+7}\}, i = 2, 3, \ldots, n-7\}$.

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$E_{\frac{n-4}{2}} = \{\{v_{\frac{n-4}{2}}, e_i, e_{i+(n-3)}\}, i = 2, 3\}$ and $< D_6 > \cong \left(\frac{n^2 - 6n + 8}{4}\right) K_3$,

$D_7 = \bigcup_{i=1}^{\left(\frac{n-3}{2}\right)} J_i$ where

$J_1 = \{\{u_1, f_i, f_{i+3}\}, i = 2, 3, \ldots, n-3\}$.

$J_2 = \{\{u_2, f_i, f_{i+5}\}, i = 2, 3, \ldots, n-5\}$.

$J_3 = \{\{u_3, f_i, f_{i+7}\}, i = 2, 3, \ldots, n-7\}$.

.

.

$J_{\frac{n-4}{2}} = \{\{u_{\frac{n-4}{2}}, e_i, e_{i+(n-3)}\}, i = 2, 3\}$and , $< D_7 > \cong \left(\frac{n^2 - 6n + 8}{4}\right) K_3$,

$D_8 = \{\{u_i, f_1, f_{2i+2}\}; 1 \le i \le \frac{n-4}{2}\}$ and $< D_8 > \cong \frac{n-4}{2} K_3$,

$D_9 = \{\{v_i, f_i, f_{i+1}\}; 1 \le i \le n, f_{n+1} = f_1\}$ and $< D_9 > \cong nK_3$.

These cover $n^2 - 2n$ edges of $\overline{L(C_n \circ K_1)}$ and $n^2 - 2n$ edges of F.

The remaining $4n^2$ edges are covered by K_2's.

Therefore, $BF_2(C_n \circ K_1) = 4K_{\frac{n}{2}} \cup \left(\frac{n^2 - 2n}{2}\right) K_3 \cup (4n^2)\, K_2$ and hence

$$cp(BF_2(C_n \circ K_1)) = 4 + \left(\frac{n^2 - 2n}{2}\right) + 4n^2 = \frac{9n^2 - 2n + 4}{2}.$$

$$\text{Therefore, } cp(BF_2(C_n \circ K_1)) = \begin{cases} \dfrac{9n^2 - 1}{2} & \text{if n is odd.} \\ \dfrac{9n^2 - 2n + 4}{2} & \text{if n is even.} \end{cases}$$

3. Clique partition of $BF_3(G)$

In the following, clique partition number of path, cycle, star and wheel graphs are found.

Theorem 3.1:

For the path P_n on n vertices ($n \geq 5$), $cp(BF_3(P_n)) = n^2 - 2n + 2$.

Proof:

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_{n-1}$ be the edges of P_n, where $e_i = (v_i, v_{i+1})$, $(1 \leq i \leq n - 1)$. Then $v_1, v_2, v_3, \ldots, v_n, e_1, e_2, \ldots, e_{n-1} \in V(BF_3(P_n))$ and $|V(BF_3(P_n))| = 2n - 1$. $|E(BF_3(P_n))| = |E(L(P_n))| + n(n - 2) = n^2 - 2n + 2$.

The clique number of $BF_3(P_n)$ is 3.

$$E(BF_3(P_n)) = E(L(P_n)) \bigcup F, \text{ where } F = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n-1}\left(v_j, e_i\right)\right); |F| = n(n-1).$$

The edge set of $BF_3(P_n)$ is decomposed into K_3 and K_2's.

Vertex sets of K_3's is given by $B = \{\{e_i, e_{i+1}, e_{i+1}\}$, for each i, $1 \leq i \leq n-2\}$.

These cover all the edges of $L(P_n)$ and $2(n - 2)$ edges of F. The remaining $(n^2 - 3n + 4)$ edges in F are covered by K_2's.

Therefore, $BF_3(P_n) = (n - 2)K_3 \bigcup (n^2 - 3n + 4)K_2$ and hence

$cp(BF_3(P_n)) = n - 2 + n^2 - 3n + 4 = n^2 - 2n + 2$.

Theorem 3.2:

For the cycle C_n on n vertices ($n \geq 5$), $cp(BF_3(C_n)) = n^2 - n$.

Proof:

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices and $e_1, e_2, \ldots, e_n$ be the edges of C_n, where $e_i = (v_i, v_{i+1})$, for $(1 \leq i \leq n - 1)$ and $e_n = (v_n, v_1)$.

$V(BF_3(C_n)) = V(C_n) \bigcup E(C_n)$. Then $|V(BF_3(C_n))| = 2n$ and

$|E(BF_3(C_n))| = |E(L(C_n))| + n^2 = n^2 + n$. The clique number of $BF_3(C_n)$ is 3.

$$E(BF_3(C_n)) = E(L(C_n)) \bigcup F, \text{ where } F = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n}\left(v_j, e_i\right)\right); |F| = n^2.$$

The edge set of $BF_3(C_n)$ is decomposed into K_3 and K_2's.

Vertex sets of K_3's is given by

$C = \{\{e_i, e_{i+1}, v_{i+1}\}$, for each i, $1 \le i \le n$, $v_{n+1} = v_1$, $e_0 = e_1\}$.

These sets cover all the edges of $L(C_n)$ and 2n edges of F. The remaining n(n – 2) edges are covered by K_2's. Therefore $BF_3(C_n) = nK_3 \bigcup (n(n-2))K_2$ and

hence $cp(BF_3(C_n)) = n + n(n-2) = n^2 - n$.

Theorem 3.3:

For the star $K_{1,n}$ on n vertices ($n \ge 6$), $cp(BF_3(K_{1,n})) = \begin{cases} \dfrac{3n^2 + 14n - 1}{4} & \text{if n is odd.} \\ \dfrac{3n^2 + 12n + 8}{4} & \text{if n is even.} \end{cases}$

Proof:

Let v be the central vertex and $v_1, v_2, v_3, \ldots, v_n$ be the pendant vertices and $e_1, e_2, \ldots, e_n$ be the edges of $K_{1,n}$, where $e_i = (v, v_i)$, $(1 \le i \le n)$.

Then $v, v_1, v_2, v_3, \ldots, v_n, e_1, e_2, \ldots, e_n = V(BF_3(K_{1,n}))$ and $|V(BF_3(K_{1,n}))| = 2n + 1$ and $|E(BF_3(K_{1,n}))| = E(L(K_{1,n})) + n(n+1) = \dfrac{n(3n+1)}{2}$ and the clique number is $\dfrac{n}{2}$.

$E(BF_3(K_{1,n})) = E(L(BF_3(K_{1,n}))) \bigcup F \bigcup H$, where

$F = \{(v, e_i): 1 \le i \le n\}$; $H = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n}\left(v_j, e_i\right)\right)$. $|F| = n$ and $|H| = n^2$.

Case1: n is odd.

The edge set of $BF_3(K_{1,n})$is decomposed into $K_{\frac{n-1}{2}}$, K_3 and K_2's.

Vertex sets of $2K_{\frac{n-1}{2}}$ are listed as elements of the sets A_1 and A_2, where

$A_1 = \{e_1, e_3, \ldots, e_{n-2}\}$; $A_2 = \{e_2, e_4, \ldots, e_{n-1}\}$.

Vertex sets of K_3's are given by

$A_3 = \bigcup_{i=1}^{\frac{(n-3)}{2}} B_i$ where

$B_1 = \{\{v_1, e_i, e_{i+3}\}, i = 1, 2, 3, \ldots, n-3\}$.

$B_2 = \{\{v_2, e_i, e_{i+5}\}, i = 1, 2, 3, \ldots, n-5\}$.

$B_3 = \{\{v_3, e_i, e_{i+7}\}, i = 1, 2, 3, \ldots, n-7\}$.

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$B_{\frac{n-3}{2}} = \{\{v_{\frac{n-3}{2}}, e_i, e_{i+(n-2)}\}, i = 1, 2\}$and $< A_3 > \cong \frac{n^2 - 4n + 3}{4} K_3$.

These cover $\left(\frac{n^2 - 4n + 3}{2}\right)$ edges of $L(K_{1,n})$ and H. The remaining $\left(\frac{n^2 + 9n - 6}{2}\right)$ edges are covered by K_2's.

Therefore, $BF_3(K_{1,n}) = 2K_{\frac{n-1}{2}} \cup \left(\frac{n^2 - 4n + 3}{4}\right) K_3 \cup \left(\frac{n^2 + 9n - 6}{2}\right) K_2$ and hence

$$cp(BF_3(K_{1,n})) = 2 + \left(\frac{n^2 - 4n + 3}{4}\right) + \left(\frac{n^2 + 9n - 6}{2}\right) = \frac{3n^2 + 14n - 1}{4}.$$

Case2: n is even

The edge set of $BF_3(K_{1,n})$ is decomposed into $K_{\frac{n}{2}}$, K_3 and K_2's.

Vertex sets of $2K_{\frac{n}{2}}$ are listed as elements of the sets C_1 and C_2.

$C_1 = \{e_1, e_3, \ldots, e_{n-1}\}$; $C_2 = \{e_2, e_4, \ldots, e_n\}$.

Vertex sets of K_3's are given by

$C_3 = \{\{v_i, e_1, e_{2i+2}\}$, for each i, $1 \le i \le \frac{n-4}{2}\}$ and $< C_3 > \cong \frac{n-4}{2} K_3$.

$C_4 = \bigcup_{i=1}^{\frac{(n-4)}{2}} D_i$ where

$D_1 = \{\{v_1, e_i, e_{i+3}\}, i = 2, 3, \ldots, n - 3\}$.

$D_2 = \{\{v_2, e_i, e_{i+5}\}, i = 2, 3, \ldots, n - 5\}$.

$D_3 = \{\{v_3, e_i, e_{i+7}\}, i = 2, 3, \ldots, n - 7\}$.

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$D_{\frac{n-4}{2}} = \{\{v_{\frac{n-4}{2}}, e_i, e_{i+(n-3)}\}, i = 2, 3\}$ and $< C_4 > \cong \left(\frac{n^2 - 6n + 8}{4}\right) K_3$.

.

These cover $\left(\frac{n^2 - 3n}{2}\right)$ edges of $L(K_{1,n})$ and edges $\left(\frac{n^2 - 4n}{2}\right)$ of H. The remaining $\left(\frac{n^2 + 8n}{2}\right)$ edges are covered by K_2's.

Therefore, $BF_3(K_{1,n}) = 2K_{\frac{n}{2}} \cup \left(\frac{n^2-4n}{4}\right)K_3 \cup \left(\frac{n^2+8n}{2}\right)K_2$ and hence

$$cp(BF_3(K_{1,n})) = 2 + \left(\frac{n^2-4n}{4}\right) + \left(\frac{n^2+8n}{2}\right) = \frac{3n^2+12n+8}{4}.$$

$$\text{Therefore, } cp(BF_3(K_{1,n})) = \begin{cases} \dfrac{3n^2+14n-1}{4} & \text{if } n \text{ is odd.} \\ \dfrac{3n^2+12n+8}{4} & \text{if } n \text{ is even.} \end{cases}.$$

Theorem 3.4:

For the wheel W_{n+1} on (n+1) vertices ($n \ge 6$), $cp(BF_3(W_{n+1})) = 2n^2 - n + 1$.

Proof:

Let v be the central vertex of W_{n+1} and $v_1, v_2, v_3, \ldots, v_n$ be the vertices of cycle C_n. Let $e_i = (v_i, v_j)$, $1 \le i \le n$ and $j \equiv (i+1) \pmod n$ and $f_i = (v, v_i)$, $1 \le i \le n$.

Then $V(BF_3(W_{n+1})) = V(W_{n+1}) \cup E(W_{n+1})$. $|V(BF_3(W_{n+1}))| = (n+1) + (2n) = 3n + 1$.

$|E(BF_3(W_{n+1}))| = |E(L(W_{n+1}))| + 2n(n+1) = 3n + \dfrac{n(n-1)}{2} + 2n(n+1) = \dfrac{n(5n+9)}{2}$ and

the clique number of is n.

Then $|E(BF_3(W_{n+1}))| = |E(L(W_{n+1})| \cup |E(K_n)| \cup F \cup H$, where

$$F = \bigcup_{i=1}^{n}\left\{(v,e_i),(v,f_i)\right\};\ |F| = 2n \text{ and } H = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n}\left\{(v_j,e_i),(v_j,f_i)\right\}\right);\ |H| = 2n^2.$$

The edge set of $BF_3(W_{n+1})$ is decomposed into K_n, K_3 and K_2's.

$V(K_n) = \{f_1, f_2, \ldots, f_n\}$;

Vertex sets of K_3 's are given by

$B_1 = \{(e_i, e_{i+1}, v_i), 1 \le i \le n\}$ and

$B_2 = \{\{e_i, f_{i+1}, v_{i+1}\}, 1 \le i \le n, v_{n+1} = v_1, f_{n+1} = f_1\}$and

$B_3 = \{(e_{i+1}, f_{i+1}, v_{i+3}), 1 \le i \le n, e_{n+1} = e_1, v_{n+3} = v_3, f_{n+1} = f_1\}$ and

$< B_1 > \cong < B_2 > \cong < B_3 > \cong nK_3$.

The sets $V(K_n)$, B_1, B_2 and B_3 cover all the edges of K_n, $L(W_{n+1})$ and 6n edges of F. The remaining $2n^2 - 4n$ edges are covered by K_2's. Therefore $BF_3(W_{n+1}) = K_n \cup (3n)K_3 \cup (2n^2 - 4n)K_2$ and hence $cp(BF_3(W_{n+1})) = 1 + 3n + 2n^2 - 4n = 2n^2 - n + 1$.

In the following, clique partition number of $P_n \circ K_1$ and $C_n \circ K_1$ are found.

Theorem 3.5:

For the graph $P_n o K_1$ $(n \geq 6)$, $cp(BF_3(P_n o K_1)) = 4n^2 - 5n + 4$.

Proof:

Let v_i $(1 \leq i \leq n)$ be the vertices of P_n with v_1 and v_n as pendant vertices and let $e_i = (v_i, v_{i+1})$, $(1 \leq i \leq n-1)$ be the edges of P_n. Let u_i be the pendant vertex adjacent to v_i $(1 \leq i \leq n)$ and let $f_i = (v_i, u_i)$, $(1 \leq i \leq n)$. $V(BF_3(P_n o K_1)) = V(P_n o K_1) \cup E(P_n o K_1)$.

Therefore $|V(BF_3(P_n o K_1))| = 2n + 2n - 1 = 4n - 1$.

Let $F = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n}\left\{(v_j, f_i), (u_j, f_i)\right\}\right)$; $H = \bigcup_{j=1}^{n}\left(\bigcup_{i=1}^{n-1}\left\{(v_j, e_i), (u_j, e_i)\right\}\right)$ and

$|F| = 2n^2$, $|H| = 2n(n-1)$. Then $E(BF_3(P_n o K_1)) = E(L(P_n o K_1)) \cup F \cup H$.

$$|E(BF_3(P_n o K_1))| = |E(L(P_n o K_1))| + 2n(2n-1) = (2n-1)(2n-1) + \frac{(9n-10)}{2} = 4n^2 + n - 4.$$

The clique number of $BF_3(P_n o K_1)$ is 3.

Edge set of $BF_3(P_n o K_1)$ is decomposed into K_3 and K_2′s.

Vertex sets of K_3′s are given by

$C_1 = \{\{e_i, e_{i+1}, v_i\}, \}, 1 \leq i \leq n-2\}$

$C_2 = \{\{e_i, f_{i+1}, v_{i+1}\}, 1 \leq i \leq n-1\}$ and

$C_3 = \{\{e_i, u_i, f_{i+1}\}$, for each i, $1 \leq i \leq n-1\}$ and $< C_1 > \cong (n-2)K_3$,

$< C_2 > \cong < C_3 > \cong (n-1)K_3$.

The sets C_1, C_2 and C_3 cover all the edges of $L(P_n o K_1)$. $2(n-2)$ edges of H. $(n-1)$ edges of F are covered by C_1 and C_2 respectively. $2(n-1)$ edges of H and $(n-1)$ edges of F are covered both by C_2 and C_3. The remaining $(4n^2 - 8n + 8)$ edges are covered by K_2′s.

Therefore $BF_3(P_n o K_1) = (3n-4)K_3 \cup (4n^2 - 8n + 8)K_2$ and hence

$cp(BF_3(P_n o K_1)) = 3n - 4 + 4n^2 - 8n + 8 = 4n^2 - 5n + 4$.

Theorem 3.6:

For the graph $C_n \circ K_1$ $(n \geq 6)$, $cp(BF_3(C_n o K_1)) = 4n^2 - 3n$.

Proof:

Let v_i $(1 \leq i \leq n)$ be the vertices of C_n and let u_i $(1 \leq i \leq n)$ be the pendant vertex adjacent to v_i. Let $e_i = (v_i, v_{i+1})$, $(1 \leq i \leq n-1)$, $e_n = (v_n, v_1)$ and $f_i = (v_i, u_i)$; $(1 \leq i \leq n)$.

$V(BF_3(C_n o K_1)) = V(C_n \circ K_1) \cup E(C_n \circ K_1)$.

Therefore $|V(BF_3(C_n o K_1))| = 2n + 2n = 4n$.

Let $F = \bigcup_{j=1}^{n}\left\{\bigcup_{i=1}^{n}(v_j, e_i), (v_j, f_i), (u_j, e_i), (u_j, f_i)\right\}$. Then $E(BF_3(C_n o K_1)) = E(L(C_n \circ K_1)) \cup F$.

$$|E(BF_3(C_n o K_1))| = \frac{8n^2 + 6n}{2} = n(4n+3).$$

The clique number of $BF_3(C_n \circ K_1)$ is 3.

Edge set of $BF_3(C_n \circ K_1)$ is decomposed into K_3 and K_2's.

Vertex sets of K_3's are given by

$B_1 = \{\{e_i, e_{i+1}, v_i\}$, for each i, $1 \le i \le n$, $e_{n+1} = e_1\}$ and

$B_2 = \{\{e_i, v_{i+1}, f_{i+1}\}$, for each i, $1 \le i \le n$, $f_{n+1} = f_1\}$.

$B_3 = \{\{e_{i+1}, u_i, f_{j+1}\}$, for each i, $1 \le i \le n$, $e_{n+1} = e_1$, $f_{n+1} = f_1\}$ and

$< B_1 > \cong < B_2 > \cong < B_3 > \cong nK_3$.

The sets B_1, B_2 and B_3 cover all the edges of $L(C_n \circ K_1)$ and 6n edges of F. The remaining n(4n – 6) edges of F are covered by K_2's.

Therefore, $BF_3(C_n \circ K_1) = (3n)K_3 \cup (4n^2 - 6n) K_2$ and hence

$cp(BF_3(C_n \circ K_1)) = 3n + 4n^2 - 6n = 4n^2 - 3n$.

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