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b-Continuity Properties of Square of Tadpole Graphs

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Abstract: *A b-coloring of a graph G by k colors is a proper vertex coloring such that in each color class, there exists a vertex adjacent to at least one vertex in every other color class and the b-chromatic number* $\chi_b(G)$ *of G is the largest integer k such that there is a b-coloring. A graph G is b-continuous if G has a b-coloring by k colors for every integer k satisfying* $\chi(G) \leq k \leq \chi_b(G)$*. The b-spectrum* $S_b(G)$ *of G is the set of all integers k for which G has a b-coloring by k colors. The graph T(m, n) is the graph obtained by joining any vertex of cycle* C_m *to a pendant vertex of path* P_n *by an edge. In this paper, we find the b-chromatic number of square of Tadpole graphs. Also, b-continuity properties of these graphs are discussed.*

Keywords: *b-coloring, b-chromatic number, b-continuity, b-spectrum, Tadpole graph, Square of Tadpole graph.*

1. Introduction

All graphs considered in this paper are finite, simple, and undirected. For those terminologies not defined in this paper, the reader may refer to [2]. A proper k-coloring of a graph G is an assignment of k-colors to the vertices of G such that no two adjacent vertices are assigned the same color. Equivalently a proper k-coloring of G is a partition of the vertex set V(G) into k independent sets $V_1, V_2, \ldots, V_k$. The sets V_i $(1 \leq i \leq k)$ are called color classes with color i. The chromatic number $\chi(G)$ is the minimum k for which G admits a proper k-coloring. Later, new types of vertex coloring were introduced and one such coloring is b-coloring. The concept of b-coloring was introduced by Irving and Manlove in 1991 [3]. A b-coloring of G by k-colors is a proper k-coloring such that in each color class, there exists a vertex adjacent to at least one vertex in every other color class. Such a vertex is called a color dominating vertex. Hence, if G has a b-coloring with k colors, then it has at least k color dominating vertices. Consequently, G has at least k vertices of degree at least k-1. The b-chromatic number of G, denoted by $\chi_b(G)$, is the largest integer k such that G has a b-coloring with k colors. To determine the upper bound of $\chi_b(G)$, the term t-degree of G, denoted by t(G) was defined as $t(G) = \max\{i : 1 \leq i \leq |V(G)|$, G has at least i vertices of degree at least $i - 1\}$.

Hence, the inequality $\chi_b(G) \leq t(G)$ follows. After introducing b-coloring, in the same paper, Irving and Manlove introduced the concept of b-continuity. For each integer k such that $\chi(G) \leq k \leq \chi_b(G)$, if G has a b-coloring by k-colors, then G is said to be b-continuous.

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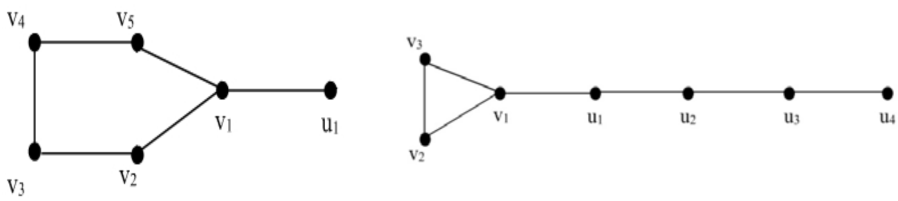
To check the b-continuity property of a graph, a set called b-spectrum was defined. The b-spectrum $S_b(G)$ of G is the set of all integers k for which G has a b-coloring by k colors. If $S_b(G)$ contains all the integers from $\chi(G)$ to $\varphi_b(G)$, then G is b-continuous.

A Tadpole graph T(m, n) [6] is the graph obtained by joining a cycle C_m, $m \geq 3$ to a path P_n, $n \geq 1$ with a bridge. Throughout this paper, in T(m, n), $\{v_1, v_2, \ldots, v_m\}$ denotes the vertex set $V(C_m)$ and $\{u_1, u_2, \ldots, u_n\}$ represents vertex set $V(P_n)$ and P_n is joined to C_m at v_1 by the edge u_1v_1.

Graphs T(5, 1) and T(3, 4) are shown Figure 1.



(a) T(5, 1) (b) T(3, 4)

Figure 1: Examples of Tadpole graphs

Definition 1.1 [1]: The square of a graph G denoted by G^2 is the graph whose vertex set is V(G) and two vertices of G^2 are adjacent if and only if the distance between the vertices is at most 2.

In this paper, we find the b-chromatic number of the square of Tadpole graphs T(m, n) for all m and n and also prove that these graphs are b-continuous.

Notations and Terminologies:

1. Throughout this paper, c is a function which assigns colors to the vertices of a graph in discussion. Hence, if u is any vertex of a graph, then c(u) denotes its color.
2. In figures, the color dominating vertices are circled.
3. We refer to a color dominating vertex as ***cdv***. In particular, if u is a color dominating vertex of color i, then it is referred to as ***i-cdv***.

2. Prior Results and Some Observations

In this section, some properties of the Tadpole graph T(m, n) and some basic results on T(m, n) are discussed.

Observation 2.1 [3, 4]

i. If G admits a b-coloring with k-colors, then G must have at least k vertices of degree at least k – 1.
ii. Any proper coloring with χ colors is a b-coloring.
iii. If G contains an induced path or cycle on at least 5 vertices, then $\varphi_b(G)$ is at least 3.

iv. If G contains an induced K_n, then $\varphi_b(G) \geq n$.
v. For a graph G, $\chi(G) \leq \varphi_b(G) \leq t(G)$

Observation 2.2
For $m \geq 3$ and $n \geq 1$,

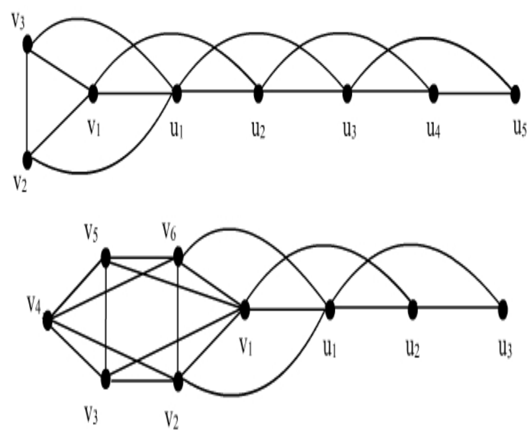
i. T(m, n) has $m + n$ vertices and $m + n$ edges.
ii. T(m, n) has exactly one vertex of degree 3, one vertex of degree 1 and $m + n - 2$ vertices of degree 2.
iii. $\chi(T(m,n)) = \begin{cases} 2, & \text{if m is even} \\ 3, & \text{if m is odd} \end{cases}$

Theorem 2.3 [5]: For $m \geq 3$ and $n \geq 1$,

i. $t(T(m, n)) = 3$
ii. $2 \leq \varphi_b(T(m, n)) \leq 3$.
iii. Tadpole graph T(m, n) is a b-continuous graph.

3. Main results on Square graph of Tadpole graph

This section deals with the properties and b-chromatic number of Square of Tadpole graphs T(m, n). Based on Definition 1.1, the Graphs $T^2(3, 5)$ and $T^2(6, 3)$ are obtained as shown in Figure 2.



(a): $T^2(3, 5)$ (b): $T^2(6, 3)$

Figure 2: Square graphs of Tadpole graphs T(3, 5) and T(6, 3)

Observation 3.1: For $m \geq 3$ and $n \geq 1$,

i. $|V(T^2(m, n))| = |V(T(m, n))| = m + n$

ii. $$|E(T^2(m,n))| = \begin{cases} 2n+4, & \text{if } m = 3 \\ 2n+7, & \text{if } m = 4 \\ 2m+2n+1, & \text{if } m \geq 5 \end{cases}$$

iii. For $m \geq 3$ and $n \geq 1$, $\chi(T^2(m,n)) = \begin{cases} 4, & \text{if } m \neq 5 \\ 5, & \text{if } m = 5 \end{cases}$

Observation 3.2:

i. $$t(T^2(3,n)) = \begin{cases} 4, & \text{if } 1 \leq n \leq 5 \\ 5, & \text{if } n \geq 6 \end{cases}$$

ii. $$t(T^2(4,n)) = \begin{cases} 4, & \text{if } 1 \leq n \leq 3 \\ 5, & \text{if } n \geq 4 \end{cases}$$

iii. $t(T^2(m,n)) = 5$, if $m \geq 5, n \geq 1$.

Remark: From Observation 3.1, we get $\chi(T^2(m, n)) = 5$ only for $m = 5$ and in all the other cases, it is equal to 4. From Observation 3.2, it follows that $t(T^2(m, n)) = 4$ or 5. Hence, for any $m, n \geq 1$, we have $4 \leq \varphi_b(T^2(m, n)) \leq 5$. Suppose, $\chi(T^2(m, n)) = t(T^2(m, n)) = k$, then it is obvious that $\varphi_b(T^2(m, n)) = k$. From Observation 3.1, as any proper coloring with χ colors is also a b-coloring with χ colors, in the following theorem, in all the cases where $\chi(T^2(m, n)) = 4$, as $T^2(m, n)$ will obviously have a b-coloring with 4 colors, we discuss only whether $T^2(m, n)$ has a coloring with 5 colors or not.

In the following theorem, we discuss the b-spectrum of the square graphs $T^2(m, n)$, where $m, n \geq 1$.

Theorem 3.3:

(a) $$S_b(T^2(3,n)) = \begin{cases} \{4\}, & \text{if } 1 \leq n \leq 5 \\ \{4,5\}, & \text{if } n \geq 6 \end{cases}$$

(b) $S_b(T^2(5,n)) = \{5\}$, *if* $n \geq 1$

(c) For $m = 4$ and $m = 6$, $S_b(T^2(m,n)) = \begin{cases} \{4\}, & \text{if } 1 \leq n \leq 3 \\ \{4,5\}, & \text{if } n \geq 4 \end{cases}$

(d) $$S_b(T^2(7,n)) = \begin{cases} \{4\}, & \text{if } n = 1 \\ \{4,5\}, & \text{if } n \geq 2 \end{cases}$$

(e) For $m \geq 8$ and $n \geq 1$, $S_b(T^2(m, n)) = \{4, 5\}$

Proof: ***Proof of (a)*:** Here, we consider two cases.

Case (i): $1 \leq n \leq 5$

It follows from Observation 3.1 and 3.2 that $\chi(T^2(3, n)) = t(T^2(3, n)) = 4$. Therefore, as $\chi(T^2(3, n)) \leq \chi_b(T^2(3, n)) \leq t(T^2(3, n))$, the result follows trivially.

Case (ii): $n \geq 6$

Since $\chi(T^2(3, n)) = 4$ and $t(T^2(3, n)) = 5$, $4 \leq \chi_b(T^2(3, n)) \leq 5$. As, $T^2(3, 6)$ is an induced sub graph of $T^2(3, n)$, for $n \geq 6$ and any b-coloring of $T^2(3, 6)$ with k colors can be extended to a b-coloring of $T^2(3, n)$ with k colors, it is enough to prove that $T^2(3, 6)$ has a b-coloring with 5 colors.

Let c be a proper coloring of $T^2(3, 6)$ with 5 colors. Without loss of generality, let $c(v_1) = 1$, $c(v_2) = 4$, $c(v_3) = 5$, $c(u_1) = 2$, $c(u_2) = 3$, $c(u_3) = 4$, $c(u_4) = 5$, $c(u_4) = 5$, $c(u_5) = 1$ and $c(u_6) = 2$. (Refer Figure 3). From Figure 3, it is clear that v_1 is a 1-cdv, u_1 is a 2-cdv, u_2 is a 3-cdv, u_3 is a 4-cdv, u_4 is a 5-cdv. Hence, $T^2(3, 6)$ has a b-coloring with 5 colors. Now repeat the coloring scheme of u_1, u_2, u_3 in a cyclic order from u_4 onwards. Then, this generates a b-coloring of the remaining vertices. Hence, c is a b-coloring of $T^2(3, n)$ and therefore, $\chi_b(T^2(3, n)) = 5$, which gives $S_b(T^2(3, n)) = \{4,5\}$.

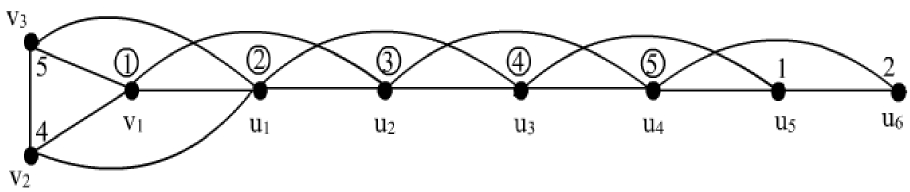


Figure 3: A b-coloring of $T^2(3, 6)$ with 5 colors

Proof of (b): Suppose that $m = 5$ and $n \geq 1$. Then the result follows trivially from the fact that $\chi(T^2(5, n)) = 5 = t(T^2(5, n))$.

Proof of (c): Let $m = 4$. We prove the result by considering two cases.

Case (i): $1 \leq n \leq 3$.

Then $\chi(T^2(4, n)) = 4 = t(T^2(4, n))$ and hence the result follows.

Case (ii): $n \geq 4$.

We have $4 \leq \chi_b(T^2(4, n)) \leq 5$. As discussed earlier, since $T^2(4, 4)$ is an induced subgraph of $T^2(4, n)$, for $n \geq 4$ and any b-coloring of $T^2(4, 4)$ with 5 colors can be extended to a b-coloring of $T^2(4, n)$ with 5 colors, it suffices to prove that $T^2(4, 4)$ has a b-coloring with 5 colors.

Let c be a proper coloring of $T^2(4, 4)$ with 5 colors defined by $c(v_1) = 1$, $c(v_2) = 4$, $c(v_3) = 3$, $c(v_4) = 5$, $c(u_1) = 2$, $c(u_2) = 3$, $c(u_3) = 4$, $c(u_4) = 5$ (Refer Figure 4). Then it can be seen from Figure 4 that v_1 is a 1-cdv, u_1 is a 2-cdv, u_2 is a 3-cdv, v_2 is a 4-cdv, v_4 is a 5-cdv. Hence, $T^2(4, 4)$ has a b-coloring with 5 colors. By a similar argument as in the proof of (a), the remaining vertices of P_n in $T^2(4, n)$ can be colored by repeating the coloring scheme of u_1, u_2, u_3 in a cyclic order from u_4 onwards. Hence, we get a b-coloring of $T^2(4, n)$ with 5 colors. Consequently, $\chi_b(T^2(4, n)) = 5$ and hence $S_b(T^2(4, n)) = \{4, 5\}$.

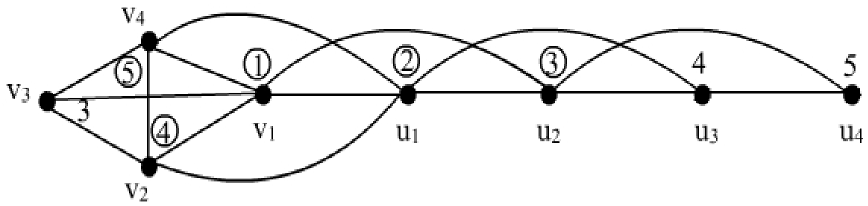


Figure 4: A b-coloring of $T^2(4, 4)$ with 5 colors

Let m = 6. We consider the following two cases.

Case (i): $1 \le n \le 3$

Since $\chi(T^2(6, n)) = 4$ and $t(T^2(6, n)) = 5$, $4 \le \varphi_b(T^2(6, n)) \le 5$. Let us show that $T^2(6, n)$ has no b-coloring with five colors. On the contrary, let us suppose that $T^2(6, n)$ has a b-coloring c with five colors. In $T^2(6, n)$, v_1, v_2, v_6 and u_1 induce K_4 and hence four colors are used in a proper coloring of K_4. Without loss of generality, let $c(v_1) = 1$, $c(v_2) = 2$, $c(v_6) = 3$, $c(u_1) = 4$. For $1 \le n \le 3$, the vertices u_2 and u_3 are of degree atmost 3 and hence cannot be color dominating vertices in a b-coloring with 5 colors. Hence, one of the vertices v_3, v_4 or v_5 must be a 5-cdv.

Suppose $c(v_3) = 5$. Then, v_3 is adjacent to the vertices with colors 1 and 2 and not adjacent to colors 3 and 4. Hence, colors 3 and 4 must be given to its adjacent vertices which are not yet colored. Therefore, v_4 and v_5 must be given the colors 1 and 3, respectively. But, as both v_4 and v_5 are adjacent to the vertex v_6 of color 3, none of them can be assigned color 3. Therefore, v_3 cannot be a 5-color dominating vertex. By similar argument, it can be shown that v_5 cannot be a 5-cdv.

Finally, suppose $c(v_4) = 5$, then as v_4 is not adjacent to the vertices with colors 1 and 4, colors 1 and 4 can be assigned to its adjacent vertices v_3 and v_5 (which are not yet colored). But, both v_3 and v_5 are adjacent to the vertex v_1 with color 1. Hence, none of them can be assigned color 1. Therefore, v_4 cannot be a 5-color dominating vertex, as well. Hence, $T^2(6, n)$ cannot have a b-coloring with 5 colors. Thus, $\varphi_b(T^2(6, n)) = 4$ and consequently, $S_b(T^2(6, n)) = \{4\}$.

Case (ii): $n \ge 4$

In this case, it is enough to prove that $T^2(6, 4)$ has a b-coloring with 5 colors. By coloring the vertices of $T^2(6, 4)$ as shown in Figure 5, it can be observed that $\varphi_b(T^2(6, n)) = 5$, which gives $S_b(T^2(6, n)) = \{4, 5\}$.

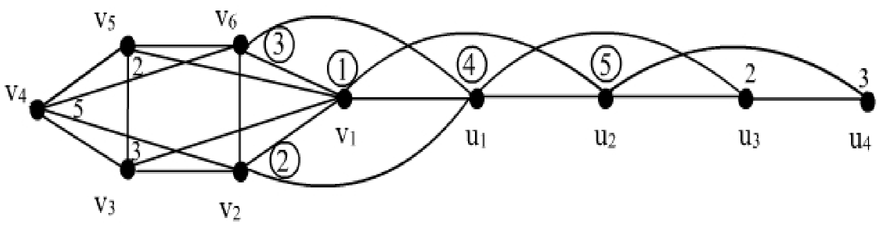


Figure 5: A b-coloring of $T^2(6, 4)$ with 5 colors

Proof of (d): Let m = 7. Then the result can be proved by considering the following two cases.

Case (i) : n = 1

As $\chi(T^2(7, 1)) = 4$ and $t(T^2(7, 1)) = 5$, we have $4 \le \varphi_b(T^2(7, 1)) \le 5$. In $T^2(7, 1)$, the vertices u_1, v_1, v_2, and v_7 induce K_4. Hence, 4 colors are needed to color K_4. Here we show that, the graph $T^2(7, 1)$ has no b-coloring using 5 colors. On the contrary, suppose that $T^2(7, 1)$ has a b-coloring with 5 colors. Without loss of generality, let $c(u_1) = 1$, $c(v_1) = 2$, $c(v_3) = 3$ and $c(v_7) = 4$. In any coloring as $d(u_1) = 3$, u_1 cannot be a color dominating vertex. Hence, one of v_3, v_4, v_5 and v_6 must be 1-cdv. Without loss of generality, let v_3 be a 1-cdv. Since, $d(v_3) = 4$, its neighbors v_1, v_2, v_4 and v_5 must be given the colors 2, 3, 4 and 5. Without loss of generality, let $c(v_1) = 2$, $c(v_2) = 3$, $c(v_4) = 4$ and $c(v_5) = 5$. Since, v_6 and v_7 are adjacent to v_1, none of them can be given color 2. Hence, v_1 must be the 2-cdv. As v_1 is adjacent only to the colors 1 and 3, not adjacent to colors 4 and 5, v_6 and v_7 are adjacent to v_1 and not yet colored, colors 4 and 5 must be given to v_6 and v_7. Since, v_6 and v_7 are adjacent v_5, color 5 cannot be given to them, a contradiction. Therefore, v_3 cannot be a 1-cdv. By a similar argument, it can be shown that v_6 cannot be a 1-cdv. Hence, either v_4 or v_5 must be the 1-cdv.

Without loss of generality, let v_4 be the 1-cdv. With the existing coloring, v_4 is adjacent only to color 3. Hence, the vertices v_3, v_5, v_6 must be given the colors 2, 4 and 5. Since, v_3 and v_6 are adjacent to a vertex of color 2, v_5 is given the color 2. Then, v_3 and v_6 must be given the colors 4 and 5. Again, v_6 is adjacent to v_7 and $c(v_7) = 4$ and hence v_6 must be given the color 5. Thus, v_3 is given the color 4. Hence, all the vertices of $T^2(7, 1)$ are colored. Therefore, v_2 must be the 3-cdv. But in the new assignment of colors v_2 is not adjacent to the color 5 and so, both v_4 and v_5 cannot be 1-cdv. Therefore, $T^2(7, 1)$ has no b-coloring with 5 colors.

Case (ii): n ≥ 2

Since $\chi(T^2(7, n)) = 4$ and $t(T^2(7, n)) = 5$, we have $4 \le \varphi_b(T^2(7, n)) \le 5$. It can be shown that $T^2(7, 2)$ has a b-coloring with 5 colors. By assigning colors to the vertices of $T^2(7, 2)$ as in Figure 6, it can be seen that $T^2(7, 2)$ has a b-coloring with 5 colors. Hence, $\varphi_b(T^2(7, n)) = 5$, which gives $S_b(T^2(7, n)) = \{4,5\}$.

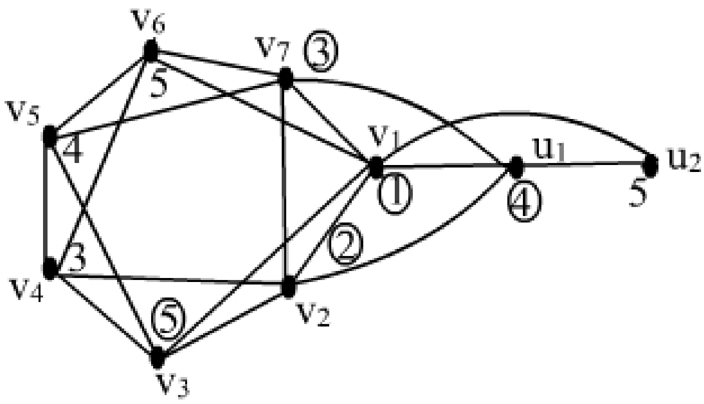


Figure 6: A b-coloring of $T^2(7, 2)$ with 5 colors

Proof of (e): Let $m \geq 8$ and $n \geq 1$.

Since $\boxtimes(T^2(m, n)) = 4$ and $t((T^2(m, n)) = 5$, $4 \leq \boxtimes_b(T^2(m, n)) \leq 5$. We assign colors 1, 2, 3, 4, 5 to the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7 \ldots v_m$ of C_m in a cyclic order. Then five cases arise.

Case (i): $m \equiv 3 \pmod 5$.

Let $c(u_1) = 5$. Then v_2, v_3, v_4, v_5, v_6 are 2-color, 3-color, 4-color, 5-color, 1-color dominating vertices respectively. Assign colors 4, 3, 2, 1 and 5 to the vertices u_i, $i \geq 2$ in a cyclic order. Then, we get a b-coloring with 5 colors (refer Figure 7) and hence, $\boxtimes_b(T^2(8, 1)) = 5$.

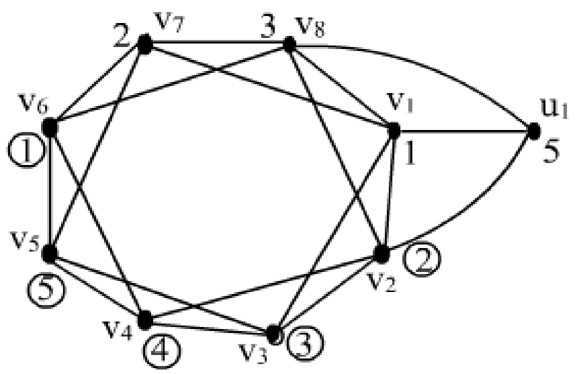


Figure 7: A b-coloring of $T^2(8, 1)$

Case (ii): $m \equiv 4 \pmod 5$.

Here, v_3, v_4, v_5, v_6 and v_7 are 3-color, 4-color, 5-color, 1-color, 2-color dominating vertices respectively. Assign colors 5, 4, 3, 2 and 1 to the vertices u_i, $i \geq 1$ in a cyclic order. Then, we get a b-coloring with 5 colors.

Case (iii) $m \equiv 0 \pmod 5$

As in case (ii), v_3, v_4, v_5, v_6 and v_7 are 3-color, 4-color, 5-color, 1-color, 2-color dominating vertices respectively. Assign colors 4, 3, 2, 5 and 1 to the vertices u_i, $i \geq 1$ in a cyclic order. Then we get a b-coloring with 5 colors.

Case (iv): $m \equiv 1 \pmod 5$

Let $c(u_1) = 5$. Color the vertices up to v_{m-1} and assign color 3 to v_m. Then v_2, v_3, v_4, v_5 and v_6 are 2-color, 3-color, 4-color, 5-color, 1-color dominating vertices respectively. Assign colors 4, 3, 2, 1 and 5 to the vertices u_i, $i \geq 2$ in a cyclic order. Then we get a b-coloring with 5 colors.

Case (v) $m \equiv 2 \pmod 5$.

Let $c(u_1) = 5$. Color the vertices up to v_{m-2} and then assign color 3 to v_{m-1} and color 4 to v_m. Also, assign colors 2, 3, 4, 1 and 5 to the vertices u_i, $i \geq 2$ in a cyclic order. As is case (v), we get a b-coloring with 5 colors.

From all the sub cases, it can be observed that we get a b-coloring with 5 colors. Therefore, $\boxtimes_b(T^2(m, n)) = 5$. Hence $S_b(T^2(m, n)) = \{4, 5\}$.

Corollary 3.4

For $m \geq 3$ and $n \geq 1$, $T^2(m, n)$ is a b-continuous graph.

4. Conclusion

In this paper, we found the b-chromatic number of square of Tadpole graphs and discussed their b-continuity properties. This paper can be further extended to the graphs related to Tadpole graphs and square of Tadpole graphs.

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Saraswathi was born in Manapparai, Tiruchirappalli (Dt.), Tamil Nadu, India, in 1975. She received her B.Sc. degree in Mathematics from the Bharathidasan University , Tiruchirappalli in 1995, and the M.Sc. and M.Phil degrees in Mathematics from the Bharathidasan University, Tiruchirappalli, in 1997 and 2001, respectively. In 1997, she joined the Department of Mathematics, Nehru Memorial College, Tiruchirappalli, as a Lecturer, and in 1998 joined as a lecturer in the Department of Mathematics, Seethalakshmi Ramaswami College, Tiruchirappalli. She became an Assistant Professor in 2008. She is doing her Ph.D. under the guidance of Dr. M. Poobalaranjani, Associate Professor & Head, Department of Mathematics, Seethalakshmi Ramaswami College, Tiruchirappalli. Her main area of research interest is Graph Theory.