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The Upper Detour Domination Number of a Graph.

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Abstract: *A detour dominating set S in a connected graph G is called a minimal detour dominating set of G if no proper subset of S is a detour dominating set of G. The upper detour domination number $\gamma_d{}^+(G)$ of G is the maximum cardinality of a minimal detour dominating set of G. Some general properties satisfied by this concept are studied. For a connected graph G of order p with upper detour domination number p is characterized. It is shown that for every two positive integers a and b, with $2 \leq a \leq b$, there exists a connected graph G with $\gamma_d(G) = a$ and $\gamma_d{}^+(G) = b$.*

Keywords: *detour set, detour dominating set, upper detour dominating set.*

Mathematical subject classification 05C12

1. Introduction

For a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. We consider connected graphs with at least two vertices. For basic definitions and terminologies we refer to [1,4].

For vertices u and v in a connected graph G, the *detour distance* $D(u, v)$ is the length of the longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ *detour*. It is known that the detour distance is a metric on the vertex set $V(G)$. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance form v to a vertex of G.The detour *radius*, $rad_D G$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D G$ of G is the maximum detour eccentricity among the vertices of G. These concept were studied by Chartrand et al.[2].

A vertex x is said to lie on a $u - v$ detour P if x is a vertex of u-v detour path P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a *minimum detour set* of G. These

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concepts were studied by G. Chartrand et al.[3].Let $G = (V, E)$ be a connected graph with at least two vertices. A set $S \subseteq V(G)$ is called a dominating set of Gif every vertex in $V(G)$-S is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum order of its dominating sets and any dominating set of order $\gamma(G)$ is called γ - set of G

The following theorem is used in the sequel.

Theorem 1.1: [5] Every end vertex of G belongs to every detour dominating set of G.

Theorem 1.2: [5] For a non trivial tree, $\gamma_d(G) = k$, where k is the number of end vertices of G.

Definition 2.1:

Let $G = (V, E)$ be a connected graph with at least two vertices .A detour dominating set S in a connected graph G is called a *minimal detour dominating set* of G if no proper subset of S is a detour dominating set of G.The upper detour dominating number $\gamma_d^+(G)$ of G is the maximum cardinality of a minimal detour dominating set of G.

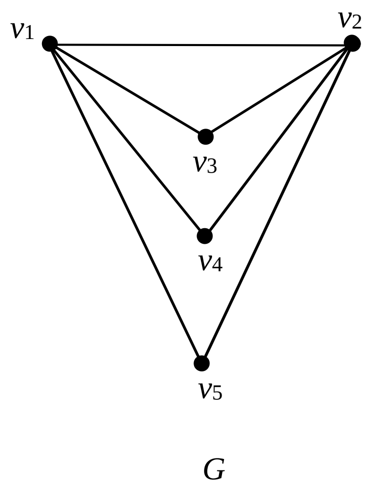

Figure 2.1

Example 2.2:

For the graph G given in Figure 2.1, $S_1 = \{v_1, v_2\}, S_2 = \{v_1, v_3\}, S_3 = \{v_1, v_4\}, S_4 = \{v_1, v_5\}, S_5 = \{v_2, v_3\}, S_6 = \{v_2, v_4\}$, and $S_7 = \{v_2, v_5\}$ are the only seven detour dominating sets of G so that $\gamma_d(G) = 2$.Also $S_8 = \{v_3, v_4, v_5\}$ is a upper detour dominating set of of G.Since no proper subset of S_8 is a detour dominating set of G, S_8 is a minimal set of Gso that $\gamma_d^+(G) \geq 3$. It is easily verified that no four elements subset of G is a detour dominating set of $G$and so $\gamma_d^+(G) = 3$.

Remark 2.3:

Every minimum detour dominating set of G is a minimal detour dominating set of G, but the converse need not be true. For the graph G given in Figure 2.1, $S_8 = \{v_3, v_4, v_5\}$is a minimal detour dominating set of G but not a minimum detour dominating set of G.

Theorem 2.4:

For a connected graph $G, 2 \leq \gamma_d(G) \leq \gamma_d^+(G) \leq p$.

Proof: A detour dominating sets needs at least two vertices so that $\gamma_d(G) \geq 2$. Since every minimal detour dominating set is also a detour dominating set, $\gamma_d(G) \leq \gamma_d^+(G)$. Since the $V(G)$ set is a detour dominating set of G. we have $\gamma_d^+(G) \leq p$.Thus $2 \leq \gamma_d(G) \leq \gamma_d^+(G) \leq p$.

Remark 2.5:

The bounds in Theorem 2.4 are sharp. For the complete bipartite graph $G = K_{m,n}$ $\gamma_d(G) = 2$ and for the star $= K_{1,p-1}$, $\gamma_d(G) = \gamma_d^+(G)$. Also for $G = K_2$,$\gamma_d^+(G) = 2 = p$. Also the bounds in Theorem 2.4 are strict. For the graph G given in Figure 2.2, $\gamma_d(G) = 3, \gamma_d^+(G) = 4$ and $p = 6$. Thus $2 < \gamma_d(G) < \gamma_d^+(G) < p$.

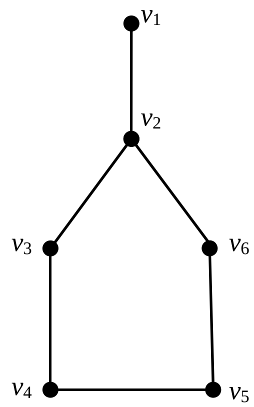

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Figure 2.2

In the following we determine the upper detour domination number of some standard graphs.

Theorem 2.6:

For a star $G = K_{1,p-1}$, $\gamma_d^+(G) = p - 1$.

Proof: Let $G = K_{1,p-1}$. Let S be the set of all end vertices of G.Then by Theorem 1.1 , S is a subset of every detour dominating set of G and so $\gamma_d(G) \geq p - 1$. It is clear that S is a detour dominating set of G so that $\gamma_d^+(G) = p - 1$.

Theorem 2.7:

If G is a double star, then $\gamma_d^+(G) = p - 2$.

Proof. The proof is similar to that of proof of Theorem 2.6.

Theorem 2.8:

For the path $G = P_p(p \geq 2), \gamma_d^+(G) = \begin{cases} \left\lceil \frac{p}{2} \right\rceil & if\ p \geq 5 \\ 2 & if\ p = 2,3\ or\ 4 \end{cases}$

Proof: Let P_p be $v_1, v_2 \dots \dots v_p$. If $p = 2,3\ or\ 4$, then $\{v_1, v_p\}$ is a minimum detour dominating set of G and it is clear that no proper subset of S is a detour dominating set of G so that $\gamma_d^+(G) = 2$. Let $p \geq 5$. Then $S = \left\lceil \frac{p-4}{2} \right\rceil$ is a minimum dominating set of P_{p-4} and $S' = S \cup \{v_1, v_p\}$ is a minimal detour dominating set of G and so $\gamma_d^+(G) \geq \left\lceil \frac{p-4}{2} \right\rceil + 2 = \left\lceil \frac{p}{2} \right\rceil - 2 + 2 = \left\lceil \frac{p}{2} \right\rceil$. It is easily verified that there is no minimal detour dominating set of cardinality $\geq \left\lceil \frac{p}{2} \right\rceil$. Therefore $\gamma_d^+(G) = \left\lceil \frac{p}{2} \right\rceil$.

Theorem 2.9:

For the complete graph $= K_p(p \geq 2)$,$\gamma_d^+(G) = 2$.

Proof: Let u, v be two vertices of G. Then $S = \{u, v\}$ is a detour dominating set of G so that $\gamma_d^+(G) \geq 2$. We have to show that $\gamma_d^+(G) = 2$. Suppose that $\gamma_d^+(G) \geq 3$.Then there exists a minimal detour dominating set S' of G such that $|S'| \geq 3$. Since G is complete, the element of S' are adjacent in G. Then it follows that S' contains a detour dominating set of cardinality two, which is contradiction to S' a minimal detour dominating set of G. Therefore $\gamma_d^+(G) = 2$.

Theorem 2.10:

For the Complete bipartite graph

$$\boldsymbol{G = K_{m,n}, \gamma_d^+(G) = \begin{cases} 2 & if\ m = n = 1 \\ n - 1 & if\ m = 1, n \geq 2 \\ max\{m, n\} & if\ 2 \leq m \leq n. \end{cases}}$$

Proof: If $m = n = 1$, then result follows from Theorem 2.6. If $m = 1, n \geq 2$, then the result follows from Theorem 2.6. If $2 \leq m \leq n$, then let U and W be two bipartite sets of G such that $|U| = m$ and $|W| = n$. Let $S = W$. Then S is a detour dominating set of G. Since no proper subset of S is a detour dominating set of G, S is a minimal detour dominating set of G and so $\gamma_d^+(G) \geq n$. We have to show that $\gamma_d^+(G) = n$. Suppose that $\gamma_d^+(G) \geq n + 1$. Then there exists a detour dominating set S' such that $|S'| \geq n + 1$. Hence it follows that $S' \subseteq U \cup W$. Let $x, y, z \in S'$ such that $x, y \in U$ and $z \in W$. Then$S'' = \{x, y, z\}$ is a detour dominating set of G such that $S'' \subset S'$, which is a contradiction to S' a minimal detour dominating set of G Hence $\gamma_d^+(G) = n$.

In view of Theorem 2.4, we have the following realization result.

Theorem 2.11:

For positive integer a and b with $2 \leq a \leq b$, there exists a connected graph with $\gamma_d(G) = a$ and $\gamma_d^+(G) = b$.

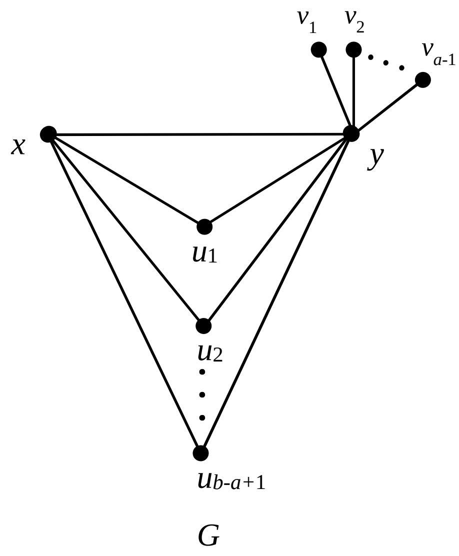

Figure 2.3

Proof:

Case (i): If $a = b$, Let $G = K_{1,a}$. Then by Theorem 1.2, $\gamma_d(G) = a$ and by Theorem 2.6, $\gamma_d^+(G) = a$.

Case (ii): $2 \leq a < b$. Let $P: x, y$ be the path of order 2. Let G be a graph obtained from p by adding b new vertices $v_1, v_2, \dots, v_{a-1}, u_1, u_2, \dots, u_{b-a+1}$ and joining each $v_i (1 \leq i \leq a-1)$ with y and joining each $u_i (1 \leq i \leq b-a+1)$ with x and y. The graph G is shown in Figure 2.3. We show that $\gamma_d(G) = a$. Let $S = \{v_1, v_2, \dots, v_{a-1}\}$ be the set of all end vertices of G. Then by Theorem 1.1, S is a subset of every detour dominating set of G and so $\gamma_d(G) \geq a$. Now $S' = S \cup \{x\}$ is a detour dominating set of G so that $\gamma_d^+(G) = a$. Next we show that $\gamma_d^+(G) = b$. By Theorem 1.1, S is a subset of every detour dominating set of G. It is clear that S is not a detour dominating set of G. Let $S_1 = S \cup \{u_1, u_2, \dots, u_{b-a+1}\}$. Then S_1 is a detour dominating set of G. We show that S_1 is a minimal detour dominating set of G. Suppose that S_1 is not a minimal detour dominating set of G. Then there exists detour dominating set M such that $M \subseteq S_1$. Therefore there exists $x \in S_1$ Such that $x \notin M$. By Theorem 1.1, $x \neq v_i$ $(1 \leq i \leq a-1)$ Then $x = u_i$ for some i $(1 \leq i \leq b-a+1)$ without loss of generality let $x = u_1$,Then u_1 is not adjacent to any vertex of M. Hence M is not a dominating set of G, which is contradiction. Therefore S_1 is a minimal detour dominating set of G and so $\gamma_d^+(G) \geq b$.We have to prove $\gamma_d^+(G) = b$. Suppose that $\gamma_d^+(G) \geq b+1$. Then there exists a minimal detour dominating set M_1 such that $|M_1| \geq b+1$. By Theorem 1.1, M_1 contains each

$v_i(1 \leq i \leq a-1)$.Since S' is a detour dominating set of G, $x \notin M_1$. Therefore $M_1 = S_1 \cup \{y\}$. Since S_1 is a detour dominating set of G, it follows that M_1 is not a minimal detour dominating set of G, which is a contradiction Therefore $\gamma_d^+(G) = b$.

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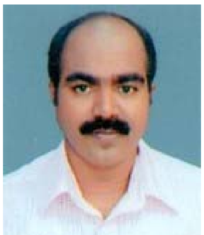
Authors' Profile:



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