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Complementary Tree Nil Domination Number of a Graph

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Abstract: *A set D of a graph G = (V, E) is a dominating set, if every vertex in V-D is adjacent to some vertex in D. The domination number γ (G) of G is the minimum cardinality of a dominating set. A dominating set D of a connected graph G is called a complementary tree nil dominating set if the induced sub graph <V-D> is a tree and V-D is not a dominating set . The minimum cardinality of a complementary tree nil dominating set is called the complementary tree nil domination number of G and is denoted by $\gamma_{ctnd}(G)$. In this paper, bounds for $\gamma_{ctnd}(G)$ and its exact values for some particular classes of graphs are found. Some results on complementary tree nil domination number are also established.*

Key words: *complementary tree domination number, complementary tree nil domination number.*

1. Introduction

Graphs discussed in this paper are finite, undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. For v$\in$V(G), the neighborhood N(v) of v is the set of all vertices adjacent to v in G. N[v] = N(v)$\cup$ {v} is called the closed neighborhood of v. A vertex v$\in$V(G) is called a support if it is adjacent to a pendant vertex. (That is, a vertex of degree one). The concept of domination in graphs was introduced by Ore[7]. A set D$\subseteq$ V(G) is said to be a dominating set of G, if every vertex in V(G) - D is adjacent to some vertex in D. A minimum dominating set in a graph G is a dominating set of minimum cardinality. The cardinality of a minimum dominating set in G is called the domination number of G and is denoted by $\gamma(G)$. Some domination parameters are defined by imposing additional constraint on the complement of a dominating set. Such parameters are called codomination parameters. Based on these, the concept of nonsplit domination in graphs was introduced by Kulli and Janakiram [3]. A dominating set D of a connected graph G is a nonsplit dominating set, if the induced subgraph <V(G) - D> is connected. The nonsplit domination number γ_{ns}(G) of G is the minimum cardinality of a nonsplit dominating set. Another new parameter called complementary nil domination number of a graph was defined and studied by T. Tamil Chelvam and S. Robinson Chellathurai [8]. A set D$\subseteq$ V is said to be a complementary nil dominating set (cnd-set) of a graph G if it is a dominating set and its complement V – D is not a dominating set for G. The minimum cardinality of a cnd-set is called the complementary nil domination number of G and is denoted by γ_{cnd}(G). Muthammai, Bhanumathi and Vidhya[4] introduced the concept of complementary tree dominating set.

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A dominating set $D \subseteq V(G)$ is said to be complementary tree dominating set (ctd-set) if the induced subgraph <V(G) - D> is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$.
We call a set of vertices, a γ – set if it is a dominating set with cardinality $\gamma(G)$. Similarly, a γ_{ns}–set, γ_{cnd} – set and γ_{ctd} - set are defined. The Corona $G_1 \circ G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ be any two graphs. The join of G_1 and G_2 is the graph $G = G_1 + G_2$ with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$. Any undefined terms in this paper may be found in Harary[1]. Here, G is a connected graph with p vertices and q edges.

In this paper, bounds for $\gamma_{ctnd}(G)$ and its exact values for some particular classes of graphs are found. Also, the graphs for which $\gamma_{ctnd}(G) = 2, p, p - 1$ or $p - 2$ are characterized.

Theorem1.1:[4] For any connected graph (p, q) graph G with $\delta(G) \geq 2$, $\gamma_{ctnd}(G) \geq 3p - 2q - 2$.

Theorem1.2:[4] For any connected graph G, $\gamma(G) \leq \gamma_{ctd}(G)$.

Theorem.1.3: [8] For any noncomplete connected graph G, $\gamma(G) < \gamma_{cnd}(G)$.

Theorem.1.4: [7] For any graph, $\gamma(G) \leq p - \Delta(G)$

Theorem 1.5: [7] A dominating set D of a graph G = (V, E) is a minimal dominating set if and only if for each vertex v in D, one of the following two conditions holds.

(a) v is an isolated vertex of D.
(b) There exists a vertex u in V-D such that $N(u) \cap D = \{v\}$.

2. Main Results

In this section, a new parameter called complementary tree nil domination number is defined, minimal complementary tree nil dominating sets are characterized, bounds and exact values of this parameter are found.

Definition 2.1:

A dominating set $D \subseteq V$ of a connected graph G = (V, E) is said to be a complementary tree nil dominating set (ctnd – set) of G, if the induced subgraph <V–D > is a tree and V – D is not a dominating set. The minimum cardinality of a ctnd-set is called the complementary tree nil domination number of G and is denoted by $\gamma_{ctnd}(G)$. A set corresponding to the complementary tree nil dominating number is called a γ_{ctnd}-set of G.

A complementary tree nil dominating set D of G is minimal if no proper subset of D is a complementary tree nil dominating set of G. By a ctnd-set, we mean a complementary tree nil dominating set. Here after, we assume that G is a connected graph.

In the following minimal complementary tree nil dominating sets are characterized.

Theorem 2.1:

A complementary tree nil dominating set D of a connected graph G is minimal if and only if for each vertex v in D, one of the following conditions holds.

(a) v is an isolated vertex of D

(b) There exists a vertex u in V-D such that $N(u) \cap D = \{v\}$.

(c) V- (D - {v}) is a dominating set of G.

(d) V - (D - {v}) either contains cycle or disconnected.

Proof:

Suppose D is a minimal ctnd-set. On the contrary, if there exists a vertex $v \in D$ such that v does not satisfy any of the given conditions. Then by (a) and (b), D' = D - {v} is a dominating set of G; by (c), V - D' is connected and is a dominating set. By (d), <V-D'> is a tree. This implies that D' is a complementary tree nil dominating set of G, which is a contradiction. Therefore, D satisfies one of the conditions (a), (b), (c) and (d).

Conversely, suppose D is a ctnd-set, and for each vertex v in D, one of the four stated conditions holds. Now we prove that D is a minimal ctnd-set. Suppose D is not a minimal ctnd-set. Then there exists a vertex v in D, such that D - {v} is a ctnd-set. Thus, v is adjacent to atleast one vertex in D - {v}. Therefore, condition (a) does not hold. Also if D - {v} is a dominating set, then every vertex in V- (D – {v}) is adjacent to atleast one vertex in D - {v}. Therefore, (b) does not hold. Since, D - {v} is a ctnd-set, <V - (D - {v}> is a tree, which contradicts conditions (c) and (d) . Therefore, there exists a vertex v in D not satisfying conditions (a), (b), (c) and (d), a contradiction to the assumption. Therefore, D is a minimal ctnd-set of G.

Bounds and some exact values of complementary tree nil domination number

Observation 2.1.

Since every ctnd-set is a cnd-set as well as a ctd-set, $\gamma_{cnd}(G) \le \gamma_{ctnd}(G)$ and $\gamma_{ctd}(G) \le \gamma_{ctnd}(G)$. But, $\gamma(G) < \gamma_{cnd}(G)$. Therefore, $\gamma(G) < \gamma_{cnd}(G) \le \gamma_{ctnd}(G)$

Example 2.1: For the graph G_1 given in Figure 2.1., $\gamma_{ctd}(G_1) < \gamma_{ctnd}(G_1)$

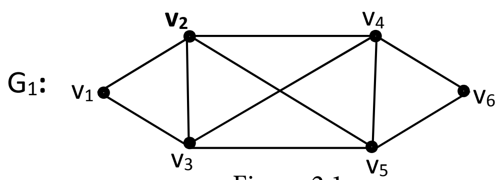


Figure 2.1.

For this graph G_1, $\{v_2, v_4\}$ is a γ_{ctd} -set, $\{v_2, v_4, v_5, v_6\}$ is a γ_{ctnd} -set. Therefore, $\gamma_{ctd}(G_1) = 2$, $\gamma_{ctnd}(G_1) = 4$. Hence, $\gamma_{ctd}(G_1) < \gamma_{ctnd}(G_1)$.

Example 2.2: For the graph G_2 given in Figure2.2., $\gamma_{ctd}(G_2) = \gamma_{ctnd}(G_2)$

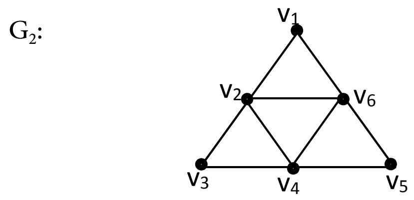


Figure 2.2.

For this graph G_2, $\{v_1, v_2, v_6\}$ is a γ_{ctd} -set as well as a γ_{ctnd} -set. Therefore, $\gamma_{ctd}(G_2) = 3 = \gamma_{ctnd}(G_2)$. Hence, $\gamma_{ctd}(G_2) = \gamma_{ctnd}(G_2)$

Example 2.3: For the graph G_3 given in Figure2.3., $\gamma_{cnd}(G_3) < \gamma_{ctnd}(G_3)$

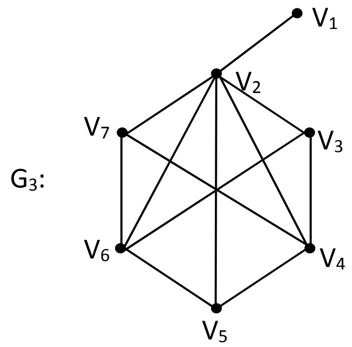


Figure 2.3.

For this graph G_3, $\{v_1, v_2\}$ is a γ_{cnd} -set, $\{v_1, v_2, v_6, v_7\}$ is a γ_{ctnd} - set. Therefore, $\gamma_{cnd}(G_3) = 2$ and $\gamma_{ctnd}(G_3) = 4$ and hence $\gamma_{cnd}(G_3) < \gamma_{ctnd}(G_3)$

Example 2.4: For the graph G_4 given in Figure 2.4., $\gamma_{cnd}(G_4) = \gamma_{ctnd}(G_4)$

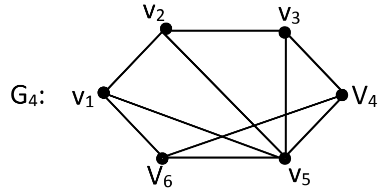


Figure 2.4.

For this graph G_4, $\{v_3, v_4, v_5\}$ is a γ_{cnd}-set as well as a γ_{ctnd}-set. Therefore, $\gamma_{cnd}(G_4) = 3 = \gamma_{ctnd}(G_4)$. Hence, $\gamma_{cnd}(G_4) = \gamma_{ctnd}(G_4)$.

In the following, complementary tree nil domination number for some graphs are given.

Observation:

2.2. For any path P_p on p vertices, $\gamma_{ctnd}(P_p) = p - 2$, $p \geq 5$.

$\gamma_{ctnd}(P_3) = 2$ and $\gamma_{ctnd}(P_4) = 3$

2.3. For any cycle C_p on p vertices, $\gamma_{ctnd}(C_p) = p - 2$, $p \geq 5$ and $\gamma_{ctnd}(C_3) = \gamma_{ctnd}(C_4) = 3$. If u, v be any two adjacent vertices of degree 2 in P_p(or C_p), then $V(P_p) - \{u, v\}$ (or $V(C_p) - \{u, v\}$) is a γ_{ctnd}-set of P_p(or C_p).

2.4. For any star $K_{1, p-1}$, $\gamma_{ctnd}(K_{1, p-1}) = p-1$, $p \geq 3$.

The set consisting of p-2 pendant vertices and the central vertex forms a γ_{ctnd}-set of $K_{1, p-1}$.

2.5. For any complete bipartite graph $K_{m, n}$, $\gamma_{ctnd}(K_{m, n}) = m + n - 1$.

2.6. $\gamma_{ctnd}(\overline{mK_2}) = 2m - 1$, $m \geq 2$.

2.7. For the graph K_p- e, $\gamma_{ctnd}(K_p - e) = p - 1$, where e is an edge in K_p.

2.8. For the graph $K_{m, n}$- e, $\gamma_{ctnd}(K_{m, n} - e) = \gamma_{ctnd}(\overline{K_{m,n} - e}) = m + n - 2$, where e is an edge in $K_{m, n}$.

2.9. $\gamma_{ctnd}(P_p \circ K_1) = \gamma_{ctnd}(C_p \circ K_1) = p+1$. The set consisting of p pendant vertices and a vertex of C_p (or P_p) forms a γ_{ctnd}-set.

2.10:

If G is a connected graph and H is any connected spanning subgraph G, then the inequality $\gamma_{ctnd}(G) \leq \gamma_{ctnd}(H)$ is not true in general. This is illustrated by the following examples.

Example 2.5:

In the graphs given in Figure 2.5., H is a spanning subgraph of G and $\gamma_{ctnd}(G) < \gamma_{ctnd}(H)$

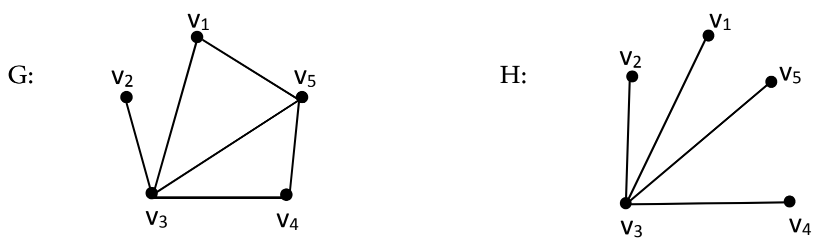


Figure 2.5.

For the graph G, $\{v_2, v_3\}$ is a γ_{ctnd} -set of G and hence γ_{ctnd} (G) = 2. H is a spanning subgraph of G, $\{v_1, v_2, v_3, v_5\}$ is a γ_{ctnd} -set of H and γ_{ctnd} (H) = 4. Therefore, $\gamma_{ctnd}(G) < \gamma_{ctnd}(H)$.

Example 2.6:

In the graphs given in Figure 2.6., H is a spanning subgraph of G and $\gamma_{ctnd}(G) > \gamma_{ctnd}(H)$.

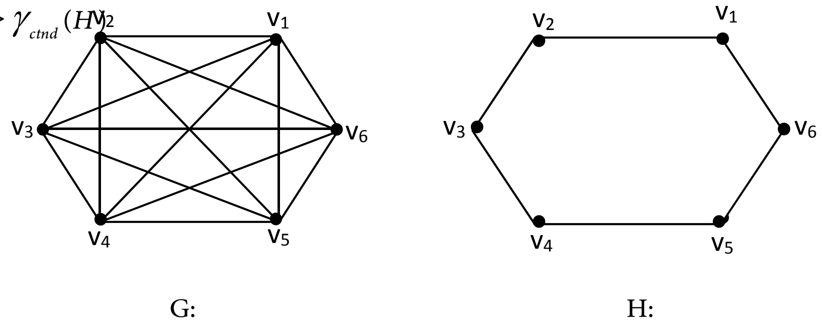


Figure 2.6

For the graph G, $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ is a γ_{ctnd} -set of G and hence γ_{ctnd} (G) = 6. H is a spanning subgraph of G, $\{v_1, v_2, v_3, v_4\}$ is a γ_{ctnd} -set of H and γ_{ctnd} (H) = 4. Therefore, $\gamma_{ctnd}(G) > \gamma_{ctnd}(H)$.

Observation 2.11:

If G is a connected graph and H is a connected induced subgraph G, then inequality $\gamma_{ctnd}(G) \leq \gamma_{ctnd}(H)$ is also not true in general. This is illustrated by the following examples.

Example 2.7:

In the graphs given in Figure 2.7., H is an induced subgraph of G and $\gamma_{ctnd}(G) < \gamma_{ctnd}(H)$.

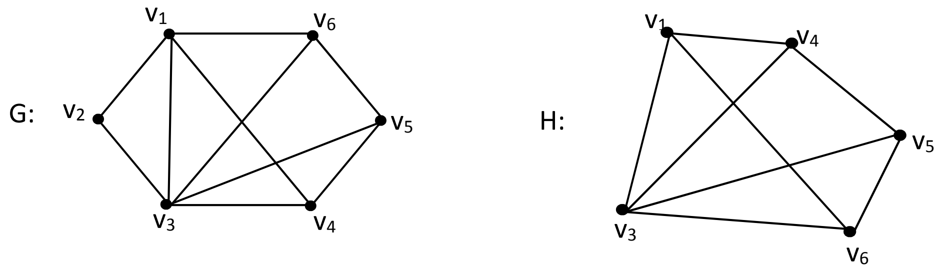


Figure 2.7.

For the graph G, $\{v_1, v_2, v_3\}$ is a γ_{ctnd} -set of G and hence γ_{ctnd} (G) = 3. H is a connected induced subgraph of G, $\{v_1, v_3, v_4, v_6\}$ is a γ_{ctnd} -set of H and γ_{ctnd} (H) = 4. Therefore, $\gamma_{ctnd}(G) < \gamma_{ctnd}(H)$.

Example 2.8:

In the graphs given in Figure 2.8., H is an induced subgraph of G and $\gamma_{ctnd}(G) > \gamma_{ctnd}(H)$

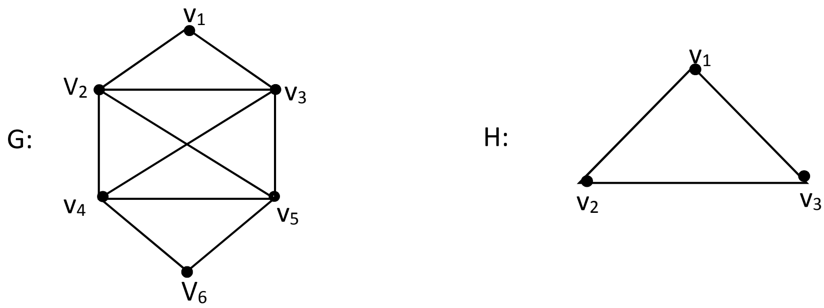


Figure 2.8.

For the graph G, {v_1, v_2, v_3, v_4} is a γ_{ctnd}-set of G and hence γ_{ctnd} (G) = 4. H is a connected induced subgraph of G, {v_1, v_2, v_3} is a γ_{ctnd}-set of H and γ_{ctnd} (H) = 3. Therefore, $\gamma_{ctnd}(G) > \gamma_{ctnd}(H)$.

Theorem 2.2:

For any connected graph G with p vertices, $2 \leq \gamma_{ctnd}(G) \leq p$,where $p \geq 2$.

Proof:

Let D be γ_{ctnd}-set of G such that |D| = 1. Then induced subgraph <V-D> is a tree and V-D is not a dominating set. But this is not possible, since the vertex in D is adjacent to all the vertices of <V-D>. Therefore, $\gamma_{ctnd}(G) \geq 2$. Since all the p vertices of G forms a ctnd-set, $\gamma_{ctnd}(G) \leq p$.

Theorem 2.3:

For any connected graph G, $\delta(G) + 1 \leq \gamma_{ctnd}(G)$.

Proof:

Let D be a γ_{ctnd}-set of G. Since induced subgraph <V - D> is a tree and V-D is not a dominating set, there exists a vertex $v \in D$ which is not adjacent to any of the vertices in V-D. Therefore $N[v] \subseteq D$. Therefore, $|N[v]| \leq |D|$. That is, $\deg_G(v) + 1 \leq |D|$. Therefore, $\delta(G) + 1 \leq \gamma_{ctnd}(G)$.

Equality holds, if $G \cong K_p$, $p \geq 3$.

Theorem 2.4:

Let D be a γ_{ctd}-set of a connected graph G, and S be the set of all pendant vertices in <V – D>. If there exists a vertex $v \in D$ such that $N(v) \cap (V\text{-}D) \subseteq S$, then $\gamma_{ctnd}(G) \leq \gamma_{ctd}(G) + m$, where $m = |N(v) \cap (V - D)|$.

Proof:

The set $D \cup (N(v) \cap (V - D))$ is a ctnd-set of G. Therefore,

$$\gamma_{ctnd}(G) \leq |D| + |N(v) \cap (V - D)| \leq \gamma_{ctd}(G) + m.$$

Theorem 2.5:

Let T be a tree on p vertices ($p \geq 3$) and be not a star. If n and e be respectively the number of cut vertices and the number of end vertices in T, then $\gamma_{ctnd}(T) \geq e+1$ or $\gamma_{ctnd}(T) \geq p - n + 1$.

Proof:

Let T be a tree on p vertices ($p \geq 3$) and be not a star.

Let S be the set of all end vertices of T. Then $|S| = e \geq 2$ and for any support $u \in T$, $S \cup \{u\}$ is a ctnd-set of T. Therefore, $\gamma_{ctnd}(T) \geq |S| + 1 = e + 1$. If n is the number of cutvertices of T, then e = p - n. Therefore, $\gamma_{ctnd}(T) \geq p - n + 1$.

Equality holds, if T is a tree in which each vertex is a support.

Remark 2.1:

Let D be a γ_{ctd}-set of G. If V-D is not a dominating set of G, then D is a ctnd-set. Therefore $\gamma_{ctnd}(G) \leq \gamma_{ctd}(G)$. But, $\gamma_{ctd}(G) \leq \gamma_{ctnd}$ d(G). Hence $\gamma_{ctnd}(G) = \gamma_{ctd}(G)$.

In the following, the connected graphs G for which $\gamma_{ctnd}(G) = 2$ are found.

Theorem 2.6:

Let G be a connected graph with p vertices. Then $\gamma_{ctnd}(G) = 2$ if and only if G is a graph obtained by attaching a pendant edge at a vertex of degree p-1 of $T + K_1$, where T is a tree on (p-1) vertices.

Proof:

Let $\gamma_{ctnd}(G) = 2$. Then there exists a ctnd-set D = {u, v} containing two vertices. Since V-D is not a dominating set, there exists a vertex $u \in D$ such that $N(u) \subseteq D$. Hence u is a pendant vertex in D and also $\deg_G(u) = 1$. Therefore all the vertices of V-D are adjacent to v only. Since <V-D> is a tree, G is a graph obtained by attaching a pendant edge at a vertex of degree p-1 of $T+K_1$ where T is a tree on (p-1) vertices.

Conversely, G is a graph obtained by attaching a pendant edge at a vertex of degree p-1 of $T + K_{1.,}$ where T is a tree on p-1 vertices. Let (u, v) be a pendant edge in G. Then D = {u, v} is a ctnd-set of G. Therefore, $\gamma_{ctnd}(G) \leq 2$. But, $\gamma_{ctnd}(G) \geq 2$. Therefore, $\gamma_{ctnd}(G) = 2$.

In the following, the connected graphs G for which γ_{ctnd} (G) = p are found.

Theroem 2.7:

For any connected graph G with p vertices, γ_{ctnd} (G) = p if and only if G $\cong K_p$, where $p \geq 2$.

Proof:

Assume γ_{ctnd} (G) = p. Let u,v $\in$ V(G) be two nonadjacent vertices in G. Then V(G) - {u} is a ctnd-set of G. Therefore, γ_{ctnd} (G) $\leq$ p – 1, which is a contradiction. Therefore, each vertex in G is adjacent to remaining (p-1) vertices. Hence, G $\cong K_p$.

If G $\cong K_p$, then γ_{ctnd} (G) = p.

Theorem 2.8:

If G is the corona of a graph H and K_1 and if H is tree or a cycle on p vertices, then

γ_{ctnd} (G) = $\frac{p}{2}+1$.

Proof:

Let G be the corona of a graph H and K_1 and let G contain p vertices. Then p is even and H has $\frac{p}{2}$ vertices.

If S is the set of all pendant vertices in H, then |S| = $\frac{p}{2}$

Case (1): H is a tree.

Let v be a pendant vertex in H. Then S $\cup$ {v} is a ctnd-set of G and

γ_{ctnd} (G) $\leq |S|+1 = \frac{p}{2}+1$

Case (2): H is a cycle.

Let u $\in$ V(H). Then S $\cup$ {u} is a ctnd-set of G and γ_{ctnd} (G) $\leq |S|+1 = \frac{p}{2}+1$

But, any ctnd-set D of G contains all the pendant vertices of G. For D to be a nil dominating set, atleast one vertex of V-D must be in D. Hence, $\gamma_{ctnd}(G) \geq \frac{p}{2}+1$.

Definition 2.2:

The Fan F_p on p vertices is defined as, $F_p = P_{p-1} + K_1$, and the Wheel W_p on p vertices is defined as $W_p = C_{p-1} + K_1$, where $p \geq 4$.

In the following, complementary tree domination numbers of Fan and Wheel are found.

Theorem 2.9:

For the Fan F_p, γ_{ctnd} (F_p) = 3, where $p \geq 4$.

Proof:

Let u, v_1, v_2, … , v_{p-1} be the vertices of F_p, where u$\in$V(K_1) and d(v_1) = d(v_{p-1}) = 2 . Then D = {v_1, v_2, u} or {v_{n-2}, v_{n-1}, u} is a dominating set of F_p and $< V - D > \cong P_{p-3}$, a path on (p -3) vertices. Also, D is a ctd-set of F_p and N(v_1)$\cap$(V-D) = N(v_{n-1})$\cap$(V- D)= φ. Therefore, V-D is not a dominating set of G. Hence, D is a ctnd-set of G and γ_{ctnd} (F_p)$\leq$|D| = 3, where $p \geq 4$. By Theorem 2.7., γ_{ctnd} (F_p) $\geq$ 3. Hence, γ_{ctnd} (F_p) $=$ 3, where $p \geq 4$.

Theorem 2.10:

For the Wheel W_p, γ_{ctnd} (W_p) = 4, where $p \geq 5$.

Proof:

Let u, v_1, v_2, …, v_{p-1} be the vertices of W_p, where u$\in W_p$ be the central vertex. Let D_i = {v_{i-1}, v_i, v_{i+1}, u}, i = 1, 2, …, p - 1. Then D_i is a dominating set and V(G) - $D_i \cong P_{p-4}$ and N(v_i)$\cap$(V-D_i) = φ. Therefore, D_i is a ctnd – set of W_p and hence γ_{ctnd} (W_p) $\leq$ 4, where $p \geq 5$. But, γ_{ctnd} (W_p) = 2 and γ_{ctd} (W_p) $\leq$ γ_{ctnd} (W_p), which implies γ_{ctnd} (W_p) $\geq$2. By Theorem 2.7., γ_{ctnd} (W_p) $\neq$2. Therefore γ_{ctnd} (W_p) $\geq$ 3**.** Let D be a ctnd-set of W_p such that |D| = 3. Since <V-D> is a tree, <V-D> is either a path or, n < p. In both cases, V(W_p) - D is a dominating set. Therefore, |D|$\geq$4. Hence, γ_{ctnd} (W_p) = 4, where $p \geq 5$.

Theorem 2.11:

If G is a connected graph with γ_{ctnd} (G) = 3 and if $\delta(G) \geq 2$, then diam(G) $\leq$4.

Proof:

Let D be ctnd-set of G such that |D| = 3. Then the induced subgraph <V – D> is a tree and there exists a vertex $u \in D$ such that N(u)$\cap$(V-D) = φ. Since G is connected, u is adjacent to a vertex, say v$\in$D. Let D = {u, v, w}, where w$\in$V(G).

Case (1): N(u)$\cap$(V- D) = N(v)$\cap$(V- D) = φ

Since $\delta(G) \geq 2$, both u and v are adjacent to w and each vertex in V-D is adjacent to w. Let $v_i \in$V-D. Then, d(u,v_i) = d(v, v_i) = 2 and d(w, v_i) = 1. Therefore diam(G) = 2.

Case (2): N(u)$\cap$(V-D) = φ.

Since, $\delta(G) \geq 2$, u is adjacent to both v and w.

(a) Let v be adjacent to w. Each vertex in V-D is adjacent to atleast one of v and w and hence for any $v_i \in$V-D, d(u, v_i) = 2, d(v, v_i) = 1 or 2 and d(w, v_i) = 1or 2, . Also, for any $v_j \in$V-D, d(v_i, v_j) $\leq$3, $i \neq j$. Therefore, diam(G) $\leq$3.

(b) Let v be not adjacent to w. Then for any $v_i \in$V-D, d(u, v_i) = 2, d(v, v_i) $\leq$3, d(w, v_i) $\leq$3 and for any $v_j \in$V-D, d(v_i, v_j) $\leq$4, $i \neq j$.

Hence, diam(G) $\leq$4.

Theorem 2.12:

Let D be a γ_{ctnd} – set of a connected graph G with $\delta(G)$ = 1. If |D| = 3 and if the induced subgraph <D> is connected, then diam(G) $\leq$3.

Proof:

Let D be γ_{ctnd} – set of G such that |D| = 3. Then the induced subgraph <V – D> is a tree. Since <D> is connected, <D> $\cong$ P_3, a path on three vertices.

Let D = {u, v, w}, where u, v, w$\in$V(G). Assume u and w are pendant vertices in <D> and N(u)$\cap$(V-D) = φ.

a) If N(v)$\cap$(V-D) = φ, then all the vertices of V-D are adjacent to w.
Therefore, d(u, v) = 1, d(u, w) = 2 and for any v_i, v_j $\in$ V - D, d(u, v_i) = 3, d(v, v_i) = 2, d(w, v_i) = 1 and d(v_i, v_j) = 2. Therefore, diam(G) $\leq$ 3.

b) If N(v)$\cap$(V-D) $\neq\varphi$, then all the vertices of V- D are adjacent to v or w. Therefore, d(u, v) = 1, d(u, w) = 2 and for any v_i, v_j $\in$ V - D, d(u, v_i) = 2 or 3, d(v, v_i) = d(w, v_i) = 1 or 2 and d(v_i, v_j) = 2 or 3. Hence, diam(G) $\leq$ 3.

In the following, the connected graphs G for which γ_{ctnd} (G) = p -1 are found.

Theorem 2.13:

Let G be a connected graph with $p \geq 3$ and $\delta(G)$ =1, then γ_{ctnd} (G) = p -1 if and only if the subgraph of G induced by vertices of degree atleast 2 is K_2 or K_1.

Proof:

Let G' be the subgraph of G induced by vertices of degree atleast 2.

Case (1): $G' \cong K_2$.

Let V(G') = {u, v}. Then V(G) - {u} and V(G) - {v} are the only ctnd-sets of G.

Case (2): $G' \cong K_1$

If w$\in$V(K_1), then V(G) – {w} is a ctnd-set of G. Hence γ_{ctnd} (G) = p - 1.

Conversely, assume γ_{ctnd} (G) = p – 1 and G' contains atleast 2 edges. Since G is connected, G' is also connected. Let e_1= (u_1, v_1), e_2 = (u_2, v_2) $\in$E(G), where u_1, v_1, u_2, $v_2$$\in$ V(G). Then V(G)- {u_1,v_1} or V(G) - (u_2, v_2) is a ctnd-set of G. Therefore, γ_{ctnd} (G) $\leq$p – 2. But γ_{ctnd} (G) = p - 1. Hence G' contains atmost one edge. Therefore, $G' \cong K_2$ (or) K_1.

Remark 2.2:

If G is one of the following graphs γ_{ctnd} (G) = p – 1.

(i) G is the star $K_{1,n}$, n $\geq$2.

(ii) G is the double star $S_{m,n}$(m, n $\geq$1)

Theorem 2.14:

Let G be a connected noncomplete graph such that $\delta(G) \geq 2$. Then γ_{ctnd} (G) = p -1 if and only if each edge of G is a dominating edge.

Proof:

Let G be a connected graph such that $\delta(G) \geq 2$. Assume γ_{ctnd} (G) = p - 1. Let e = (u, v)$\in$E(G) be an edge in G which is not dominating edge of G. Then there exists a vertex, say w$\in$V(G) such that w is adjacent to neither u nor v. Let D = V(G) – {u, v}. Then, V(G) – D = {u, v} and <V(G) - D> $\cong K_2$. Therefore, D is a ctd-set of G. Also, w$\in$D is such that N(w)$\subseteq$ D and hence D is ctnd-set of G. Therefore, γ_{ctnd} (G)$\leq$|D| = p-2. Hence, each edge of G is a dominating edge.

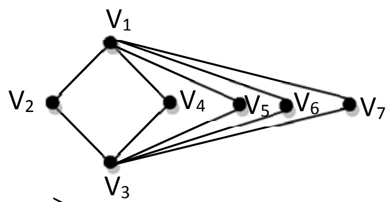
Conversely, assume each edge of G is a dominating edge. Let e$\in$(u, v) be a dominating edge of G. Then each vertex in G is adjacent to atleast one of u and v. Since G is noncomplete, γ_{ctnd} (G)$\leq$p - 1. If there exists a ctnd-set D of G having p-2 vertices, then <V - D>$\cong K_2$ and this edge is also a dominating edge of G and therefore D is a dominating set of G, which is a contradiction. Therefore, γ_{ctnd} (G) > p – 2, which implies γ_{ctnd} (G) $\geq$p - 1. Hence, γ_{ctnd} (G) = p - 1.

Example 2.9:

For the following graphs G, γ_{ctnd} (G) = p-1

a) Leaf graph L_4(G) is a graph having 4 copies of squares with three common vertices and two common edges.

L_4(G):



b) G = $\overline{mK_2}$, $m \geq 2$.

c) G = K_p - e, $p \geq 4$.

d) G = $K_{m,n}$ (m, n $\geq$ 2).

In the following, the trees T for which γ_{ctnd} (T) = p – 2 are found.

Theorem 2.15:

Let T be a tree on p vertices such that γ_{ctnd} (T) $\leq$p -2. Then γ_{ctnd} (T) = p - 2 if and only if T is one of the following graphs.

(i) T is obtained from a path P_n (n$\geq$4 and n < p) by attaching pendant edges at atleast one of the end vertices of P_n.

(ii) T is obtained from P_3 by attaching pendant edges at either both the end vertices or all the vertices of P_3

Proof:

Let T be a tree on p vertices such that γ_{ctnd} (T) $\leq$ p - 2. Let S be the set of all pendant vertices of T together with a support of T.

Let v$\in$V(T)- S be such that $\deg_T(v) \geq 3$ and v is not a pendant vertex of <V(T) - S>. Let u, w be vertices in V(T)-S adjacent to v. Then V(T) - {u, v, w} is a ctnd-set of G. Therefore, γ_{ctnd} (G) $\leq$ p - 3. Hence, if v$\in$V(T) - S, then degree of v in V(T) - S is 1 or 2. That is, <V(T) - S> is a path.

Case (1): <V(T) - S>$\cong$ P_m, where m$\geq$3.

If atleast one vertex of V(T) - S is a support of T, then also γ_{ctnd} (G) $\leq$ p-3. Therefore, no vertex of P_n (n$\geq$3) is a support of T. Hence, T is obtained from P_n(n$\geq$4) by attaching pendant edges at atleast one of the end vertices of P_n.

Case(2): <V(T) - S> $\cong$ K_2.

Then T is a tree obtained from P_3 by attaching pendant edges at atleast one of the vertices of P_3. If pendant edges are attached at exactly one end vertex of P_3, then γ_{ctnd} (T) = p - 1.

Therefore, pendant edges are attached either at both the end vertices or at all the vertices of P_3. Therefore T is one of the graphs given in the Theorem.

Conversely, assume T is a tree obtained by attaching pendant edges at atleast one of the end vertices of P_n, n$\geq$4, n<p.

Let t be the number of pendant vertices in T. Therefore, n + t = p. But, γ_{ctnd} (P_n) = n - 2. Therefore, γ_{ctnd} (P_n)$\geq$n - 2 + t = p - 2. Given, γ_{ctnd} (P_n)$\leq$p - 2. Hence, γ_{ctnd} (P_n) = p - 2. Similarly, if T is a tree obtained from P_3 by attaching pendant edges at both the end vertices or all the vertices of P_3, then the set of all the pendant vertices together with a support is a γ_{ctnd} -set of T. Therefore γ_{ctnd} (T) = p - 2.

Notation 2.1:

Let $\mathcal{G}$ be the class of connected graphs G with δ (G) = 1 having one of the following properties.

(a) There exists two adjacent vertices u, v in G such that $\deg_G(u)$ = 1 and <V(G) - {u, v}> contains P_3 as an induced subgraph such that end vertices of P_3 have degree atleast 2 and the central vertex of P_3 has degree atleast 3.

(b) Let P be the set of all pendant vertices in G and let there exist a vertex v$\in$V(G)-P having minimum degree in V(G) - P and is not a support of G such that V(G) – (N_{V-P}[v] - P) contains P_3 as an induced subgraph such that the end vertices of P_3 have degree atleast 2 and the central vertex of P_3 has degree atleast 3.

Theorem 2.16:

Let G be a connected graph with δ (G) = 1. Assume γ_{ctnd} (G)$\neq$ p -1. Then

γ_{ctnd} (G) = p -2 if and only if G does not belong to the class $\mathcal{G}$ of graphs.

Proof:

Assume G is a connected graph with δ (G) = 1 and γ_{ctnd} (G) = p - 2 . Let G$\in \mathcal{G}$. Then G satisfies (a) or (b) of Notation 2.1. Let V(P_3) = {u_1, u_2, u_3} such that deg(u_1), deg(u_3)$\geq$ 2 and deg(u_2)$\geq$3 in G. Then V(G) - {u_1, u_2, u_3} is a ctnd-set of G.

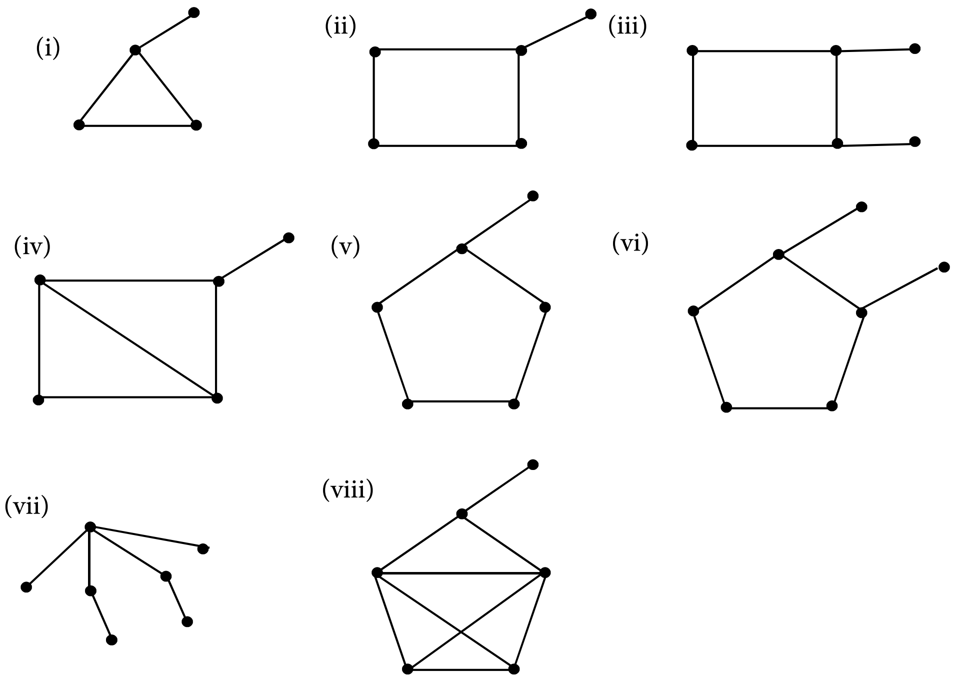
Therefore, γ_{ctnd} (G) $\leq$p - 3. Therefore, G$\notin \mathcal{G}$.

Conversely, assume G$\notin \mathcal{G}$.

By Theorem 2.15., γ_{ctnd} (G) $\leq$p-2, Since, G$\notin \mathcal{G}$, there exists two adjacent vertices u, v$\in$ V(G) such that $\deg_G(u) = 1$ and <V(G) - {u, v}> either contains P_3 as an induced subgraph such that atleast one of the end vertices has degree 1 or the central vertex of P_3 has degree 2 in G; or <V(G) - {u, v}> does not contain P_3 as an induced subgraph. In both cases, γ_{ctnd} (G) $\geq$p - 2. Therefore, γ_{ctnd} (G) = p - 2.

Example 2.10:

The following are some of the graphs G having γ_{ctnd} (G) = p - 2.



(ix) Graphs obtained by attaching a path of length 2 at a vertex of the complete graph K_p, p$\geq$3

Theorem 2.17:

Let G be a connected, noncomplete graph with p vertices (p$\geq$4) and $\delta(G) \geq 2$. Then γ_{ctnd} (G) = p – 2 if and only if G is one of the following graphs.

(a) A cycle on atleast five vertices.

(b) A wheel with six vertices

(c) G is the one point union of complete graphs.

(d) G is obtained by joining two complete graphs by an edges.

(e) G is a graph such that there exists a vertex v$\in$V(G) such that G – v is a complete graph on (p – 1) vertices.

(f) G is a graph such that there exists a vertex v$\in$V(G) such that G – v is K_{p-1} – e, (e$\in$E(K_{p-1})) and N(v) contains atleast one vertex of degree (p – 3) in K_{p-1} – e.

Proof:

Let G be a connected, noncomplete graph with p vertices (p$\geq$4) and $\delta(G) \geq 2$. Assume γ_{ctnd} (G) = p – 2.

Case (1): G has a P_3 as an induced subgraph such that central vertex of P_3 has degree atleast three in G.

If there exists a vertex in V(G) – V(P_3) not adjacent to any of the vertices of P_3, then V(G) – V(P_3) is a ctnd – set of G and hence γ_{ctnd} (G) $\leq$p – 3. Therefore, all the vertices of V(G) – V(P_3) must be adjacent to atleast one vertex of P_3. Hence, G must of the graphs mentioned in (b), (c), (d), (e) and (f) in the Theorem

Case (2): Central vertices of all induced P_3 have degree two in G.

Then G is a cycle on atleast three vertices. If G$\cong$ C_3 or C_4, then γ_{ctnd} (G) = p – 1. Therefore, G is a cycle on atleast five vertices.

Conversely, let G be one of the graphs given in the Theorem. Since G has no dominating edge, by Theorem 2.15., γ_{ctnd} (G) $\leq$p – 2. Also, all the graphs have no ctnd-sets of cardinality atmost (p – 3). Therefore, γ_{ctnd} (G) = p – 2.

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