International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 7, No. 3, July – September 2016, pp. 78 - 85

Extended results of triple connected [1,2] domination number of a graph

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Abstract: *A set $S \subseteq V$ of vertices in a graph G = (V, E) is called [1,2] dominating set, if every vertex $v \in V - S, 1 \leq |N(v) \cap S| \leq 2$, that is, every vertex in V – S is adjacent to ateast one vertex and at most two vertices in S. A [1, 2] dominating set is said to be a triple connected [1,2] dominating set if < S > is triple connected. The minimum cardinality taken over all the triple connected [1,2] dominating sets is called the triple connected [1,2] domination number and is denoted by $\gamma_{[1,2]tc}$ (G). In this paper, we extend the study of this parameter.*

Keywords: *Triple connected [1,2] dominating set, Triple connected [1,2] domination number, star related graphs AMS Subject Classification. 05C69*

1. Introduction.

By a graph we mean a finite, simple, connected and undirected graph G = (V, E). A subset *S* of *V* of a nontrivial graph *G* is called a dominating set of *G* if every vertex in *V* – *S* is adjacent to at least one vertex in *S*. The domination number γ(G) is the minimum cardinality taken over all dominating sets in G. The concept of triple connected graphs was introduced by Paulraj Joseph etc., in [4] . A graph G is said to be triple connected if any three vertices are lie on a path in G. A set S ⊆ V of vertices in a graph G = (V, E) is called [1,2] dominating set, if every vertex $v \in V - S, 1 \leq |N(v) \cap S| \leq 2$, that is, every vertex in V – S is adjacent to ateast one vertex and at most two vertices in S. A [1, 2] dominating set is said to be a triple connected [1,2] dominating set if < S > is triple connected. The minimum cardinality taken over all the triple connected [1, 2] dominating sets is called the triple connected [1,2] domination number and is denoted by $\gamma_{[1,2]tc}$(G). The corona $G_1 \odot G_2$ is defined as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2. For all the basic definitions and standard classes of graphs, we refer to [1].

1.1 Main Result.

In this paper we obtained the sharpness of the bound for triple connected [1, 2] dominating set and triple connected [1,2] domination number for star related graphs. For

Received: 20 June, 2016; Revised: 16 July, 2016; Accepted: 16 August, 2016

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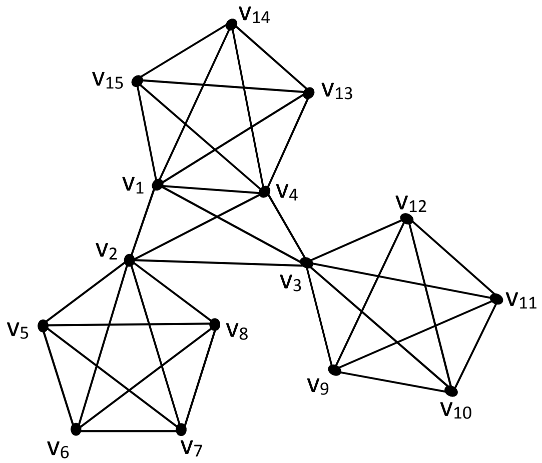
any connected graph G, γb_c (G) ≤ γ_{tc} (G) ≤ $\gamma_{tc.com}$ (G) ≤ γ_{ptc} (G) ≤ γ_{tct}(G) ≤ γ_{ntc}(G) ≤ γ_{cptc}(G) ≤ $\gamma_{[1,2]tc}$(G) ≤ γ_{dstc}(G) ≤ $\gamma_{rtc(G)}$. From this sequence of domination parameter, For any connected graph G of order p, γ_c (G) ≤ $\gamma_{[1,2]tc}$(G) ≤ p . We describe the sharpness of this bound by standard graphs as follows. We can characterize the graphs satisfying γ_c(G) ≤ $\gamma_{[1,2]tc}$ (G) and $\gamma_{[1,2]tc}$ (G) ≤ p.

Definition 1.2: A [1,2] dominating set is said to be triple connected [1,2] dominating set if the $\langle S \rangle$ is triple connected. The minimum cardinality taken over all the triple connected [1,2] dominating set is called the triple connected [1,2] domination number and is denoted by $\gamma_{[1,2]tc}$ (G).

For example,



In the above figure, S = {v_1,v_2,v_3,v_4} is a minimum triple connected [1,2] dominating set of G and hence $\gamma_{[1,2]tc}$ (G) = 4.

Definition 1.3: For a given graph G = (V, E), by subdividing each edge exactly once and joining all the non-adjacent vertices of G. the graph obtained by this process is called **central graph** and is denoted by C(G).

Definition 1.4: The **middle graph** of G is denoted by M (G) is defined as, M(G) is a graph whose vertex set is V(G) ∪ E(G) and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other an edge incident with it.

Definition 1.5: The **total graph** T (G) of a graph G is the graph whose vertex set is V(G) ∪ E(G) and two vertices are adjacent whenever they are either adjacent or incident in G.

Definition 1.6: Double star $K_{1,p,p}$ is a tree obtained from the star $K_{1,p}$ by adding a new pendant edge of the existing p pendant vertices. It has 2p + 1 vertices and 2p edges. Let $V(K_{1,p,p}) = (v_0) \cup (v_1, v_2, \dots, v_p) \cup (u_1, u_2, \dots, u_p)$ and $E(K_{1,p,p}) = (e_1, e_2, \dots, e_p) \cup (s_1, s_2, s_3, \dots, s_p)$

2. Characterization triple connected [1, 2] domination number of a graph

A dominating set can have all the vertices of G and there must be minimum 3 vertices by the definition of triple connected graphs. Hence $3 \le \gamma_{[1,2]tc}(G) \le p$ and the bounds are sharp. The lower bound attained for, wheel graph, flower graph, sunflower graph, Friendship graph, Book graph etc. The upper bound attained for complete graph.

2.1. Classification of graphs satisfying $\gamma_c(G) < \gamma_{[1,2]tc}(G)$ or $\gamma_c(G) = \gamma_{[1,2]tc}(G)$ or $\gamma_c(G) \le \gamma_{[1,2]tc}(G)$

Observation 2.1.1: For any graph G, $\gamma_c(G) \le \gamma_{[1,2]tc}(G)$.
Observation 2.1.2: If G is either a path or a cycle, then $\gamma_c(G) = \gamma_{[1,2]tc}(G)$
Observation 2.1.3: If G is a corona $H \circ K_1$, then $\gamma_c(G) = \gamma_{[1,2]tc}(G)$
Observation 2.1.4: If T is Tree, then $\gamma_c(G) < \gamma_{[1,2]tc}(G)$.
Observation 2.1.5: For complete bipartite graph, $\gamma_c(G) < \gamma_{[1,2]tc}(G)$.
Observation 2.1.6:

1. In bi star graphs, $\gamma_{[1,2]tc}(G) = 3 > 2 = \gamma_c(G)$ gives $\gamma_c(G) < \gamma_{[1,2]tc}(G)$.
2. In flower graphs and in sunflower graphs, $\gamma_{[1,2]tc}(G) = 3 > 2 = \gamma_c(G)$ gives $\gamma_c(G) < \gamma_{[1,2]tc}(G)$.
3. In Barbell graphs, Lollipop graphs, Tadpole graphs, cocktail graphs, $\gamma_{[1,2]tc}(G) = 3 > 2 = \gamma_c(G)$ gives $\gamma_c(G) < \gamma_{[1,2]tc}(G)$.
4. In Turan graph (complete multi bipartite graph), $\gamma_{[1,2]tc}(G) = 3 > 2 = \gamma_c(G)$ gives $\gamma_c(G) < \gamma_{[1,2]tc}(G)$.
5. In grid graphs, $\gamma_{[1,2]tc}(G) = \gamma_c(G)$.

2.2. Classification of graphs satisfying $\gamma_{[1,2]tc}(G) < p$ (or) $\gamma_{[1,2]tc}(G)=p$ (or) $\gamma_{[1,2]tc}(G) \le p$

Theorem 2.2.1: For any tree T, $\gamma_{[1,2]tc}(G) = p - e$, where e is the end vertices of T, if and only if T is a caterpillar.

Proof: If T is a caterpillar, all the stalk is to be taken as the $\gamma_{[1,2]tc}(G)$ set, thus the induced sub graph of $\gamma_{[1,2]tc}(G)$ is a path satisfying triple connected graph. Thus, $\gamma_{[1,2]tc}(G) = p - e$. Conversely, Let $\gamma_{[1,2]tc}(G) = p - e$, gives the removal of all end vertices makes $\gamma_{[1,2]tc}(G)$ set, thus $p - e$ is a connected graph and is a path. Thus, G is a caterpillar.

Observation 2.2.2: For complete graph and ladder graph $\gamma_{[1,2]tc}(G) = p$.

Observation 2.2.3: For prism and crossed prism graph $\gamma_{[1,2]tc}$ (G) = p/2 < p

Theorem 2.2.4: For any cubic graph G, if g(G) ≤ $\frac{p}{2}$ then

$$\gamma_{[1,2]tc}(G) \leq \begin{cases} \frac{p}{2} \; if \; g(G) \leq \frac{p}{2} \\ p \; if \; g(G) > \frac{p}{2} \end{cases} \text{ for } p \geq 6 \text{ and g is the girth of G.}$$

Proof: Let S be the $\gamma_{[1,2]tc}$ set of G. The inequality is true for g(G) = 3 except for vertex 4. If g (G) ≥ 4 the vertex of v_1 ∈ D dominates three vertices in N(u) = {v_2,v_3,v_4} and since triple connected any one of N(u) also in D. Hence $\gamma_{[1,2]tc}$ (G) ≤ $\frac{p}{2}$

Let G be a cubic graph with vertex set V(G) = {v_1,v_2,......,v_n,......,v_{2n}}. Let S be the $\gamma_{[1,2]tc}$ set of G. The graph with maximum girth is obtained by joining v_1 with v_2, v_{2n}, and v_n (that is, middle vertex). And the remaining vertices v_2,v_3,....,v_{2n} are also should join as well then g(G) = p/2 . Hence g(G) ≤p/2. Each vertex of a cubic graph dominates three of its neighboring vertices and for satisfying triple connected condition in between vertices are also taken as $\gamma_{[1,2]tc}$(G) ≥ p/2 . Consider the cubic graph of six vertices then $\gamma_{[1,2]tc}$ (G) ≤ p/2. Thus, $\gamma_{[1,2]tc}$(G) = p/2.

$\gamma_{[1,2]tc}$ (G) + g(G) ≤ p/2 +p/2.

$\gamma_{[1,2]tc}$ (G) + g(G) ≤ p .

2.3. Characterization of cubic graphs on which triple connected [1, 2] domination number equals chromatic number

In this section, we obtained some cubic graphs whose chromatic number equals [1,2] triple connected domination number of graphs.

Theorem 2.3.1: If G is a connected graph of 8 vertices then $\gamma_{[1,2]tc}$ (G) = χ(G) = 3 if and only if G ≅ G_1,G_2 and G_3

Proof: Let $G = (V, E)$ be a connected cubic graph of order p with $\gamma_{[1,2]tc}$ = ⊠ = 3. We consider cubic graphs for which, $\gamma_{[1,2]tc}$ = ⊠ = 3. But $\gamma_{[1,2]tc} \geq \gamma \geq \left\lceil \frac{P}{\Delta+1} \right\rceil$.

That is, $\gamma_{[1,2]tc} \geq \left\lceil \frac{P}{\Delta+1} \right\rceil$.

Thus, $\gamma_{[1,2]tc} \geq \left\lceil \frac{P}{4} \right\rceil$. Since, $\gamma_{[1,2]tc}$ = 3, we have $6 < p \leq 15$ and $p \neq 14$.

Also, G is cubic, p is even then p = 8, 10 or 12.

Case 1: Cubic graphs of order 8

Let S = {v_1, v_2, v_3} be a triple connected [1,2] dominating set of G and the complementary set V – S = {v_4, v_5, v_6, v_7, v_8} Therefore $\langle S \rangle = P_3$

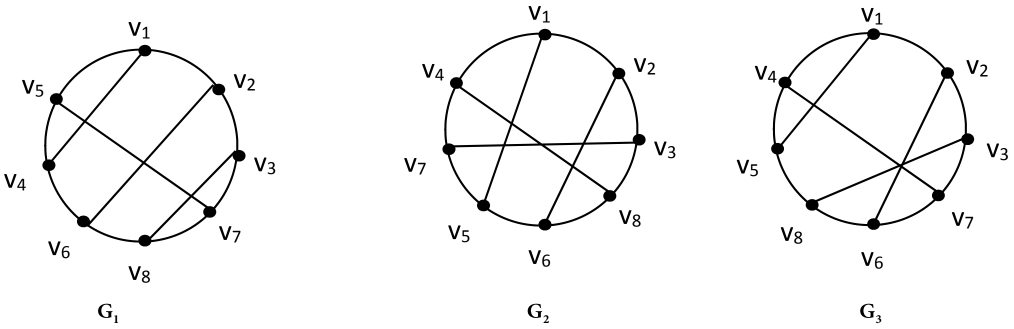


Figure 2.1 Graphs satisfying $\gamma_{[1,2]tc}(G) = \chi(G) = 3$

Fix the vertices of S be adjacent such that v_1 is adjacent to (v_4 and v_5) , v_2 is adjacent to v_6, v_3 is adjacent to v_7 and v_8

Let v_4 is adjacent to

(i) v_5 and v_6 (or v_7 or v_8)

(ii) v_6 and v_7(or v_8)

(iii) v_7 and v_8,

If v_4 is adjacent v_5 and v_6 (or v_7 or v_8) then v_7 is adjacent to v_8 and v_5 (or v_6). Then v_6 must adjacent to v_8. Then, G $\cong$ G_1. If v_4 is adjacent to v_6 and v_7 (or v_8) then v_5 is adjacent to v_8 and v_6 (or v_7) then v_7 must be adjacent to v_8. Then G $\cong$ G_3. If v_4 is adjacent to v_7 and v_8 then v_5 is adjacent to v_6 and v_7 (or v_8), then v_6 must adjacent to v_8 then G $\cong$ G_2.

Case 2: Cubic graphs of order 10 and 12:

Let S = { v_1, v_2, v_3} be triple connected [1,2] dominating set of G and V – S = {v_4, v_5, v_6, v_7, v_8, v_9, v_{10}}, Therefore < S > P_3 or K_3 , Since G is cubic graph and $\gamma_{[1,2]tc}$ = ⊠ = 3, then S is adjacent to only 5 vertices of V – S and the remaining vertices does not belongs to the neighbouring vertices of S. There is no graph exists for the graphs having 10 and 12 vertices.

3. Triple connected [1,2] domination number for star and double star families of graphs

3.1 Total graph of star and double star families of graph

Theorem 3.1.1: Triple connected [1, 2] Domination number of Total graph of Star graph is $\gamma_{[1,2]tc}(T(K_{1,p})) = 3$

Proof: Let $T(K_{1,p})$ be a total graph of star graph has vertex set $V(T(K_{1,p})) = V(K_{1,p}) \cup E(K_{1,p}) = \{v_0\} \cup \{e_i / 1 \le i \le p\} \cup \{v_i / 1 \le i \le p\} = 2p + 1$, in which the vertices $\{v_0\} \cup \{e_i / 1 \le i \le p\}$ induces a clique of order (p + 1) and the vertex v_0 is adjacent to $\{v_i / 1 \le i \le p\}$.

Thus, assign $S = \{v_0, e_i, v_i\}$ forms a [1,2] triple connected domination number of $T(K_{1,p})$ implies $\gamma_{[1,2]tc}(T(K_{1,p})) = 3$

Theorem 3.1.2: $\gamma_{[1,2]tc}(T(K_{1,p,p})) = 2p + 1$.

Proof : By the definition of total graph, each edge $\{v_0 u_i / 1 \le i \le p\}$ and $\{u_i v_i / 1 \le i \le p\}$ be subdivided by the vertices $\{e_i / 1 \le i \le p\}$ and $\{w_i / 1 \le i \le p\}$ respectively in $T(K_{1,p,p})$ and $V(T(K_{1,p,p})) = \{v_0\} \cup \{e_i / 1 \le i \le p\} \cup \{v_i / 1 \le i \le p\} \cup \{u_i / 1 \le i \le p\} \cup \{w_i / 1 \le i \le p\}$ on which $\{e_i / 1 \le i \le p\}$ is adjacent to $\{u_i / 1 \le i \le p\}$ and $\{w_i / 1 \le i \le p\}$ is adjacent to $\{v_i / 1 \le i \le p\}$.

Thus, $\gamma_{[1,2]tc}(T(K_{1,p,p})) = \{v_0\} \cup \{e_i / 1 \le i \le p\} \cup \{u_i / 1 \le i \le p\} \le 2p + 1$. If $\{v_i / 1 \le i \le p\}$ and $\{w_i / 1 \le i \le p\}$ are taken as triple connected [1,2] dominating set then $\gamma_{[1,2]tc}(T(K_{1,p,p})) = \{v_0\} \cup \{u_i / 1 \le i \le p\} \cup \{v_i / 1 \le i \le p\} \cup \{w_i / 1 \le i \le p\} = 3p + 1 \ge 2p + 1$. Hence, $\gamma_{[1,2]tc}(T(K_{1,p,p})) = 2p+1$.

3.2 Middle graphs of star and double star of graphs

Theorem 3.2.1: Triple Connected [1,2] Domination number for Middle graph of Star graph is $\gamma_{[1,2]tc}(M(K_{1,p})) = p + 1$

Proof: Let $M(K_{1,p})$ be a middle graph of a star graph, each $v_0 v_i$ for $1 \le i \le p$ of $K_{1,p}$ is subdivided by the vertices $\{e_i / 1 \le i \le p\}$. Let S be any [1,2] triple connected dominating set of $M(K_{1,p})$, since $\{v_i / 1 \le i \le p\}$ are pendent vertices then either $\{e_i / 1 \le i \le p\} \in S$ or $\{v_i / 1 \le i \le p\} \in S$. Let $\{e_i / 1 \le i \le p\} \in S$ and $\{v_0\} \cup \{e_i / 1 \le i \le p\}$ induces a clique of order $(p + 1)$. Then $\{v_0\} \cup \{e_i / 1 \le i \le p\}$ forms a [1,2] triple connected dominating set of G. Hence $\gamma_{[1,2]tc}(M(K_{1,p})) \le p+1$. If $\{v_i / 1 \le i \le p\} \in S$ then, $\gamma_{[1,2]tc}(M(K_{1,p})) = \{v_0\} \cup \{e_i / 1 \le i \le p\} \cup \{v_i / 1 \le i \le p\} = 2p + 1 \ge p + 1$. Hence $\gamma_{[1,2]tc}(M(K_{1,p})) = p + 1$.

Theorem 3.2.2: Triple connected [1, 2] domination number for the Middle graph of double Star graph $\gamma_{[1,2]tc}(M(K_{1,p,p})) = 2p+1$

Proof: By the definition of middle graph $M(K_{1,p,p})$, each edge $\{v_0 u_i / 1 \le i \le p\}$ and $\{u_i v_i / 1 \le i \le p\}$ be subdivided exactly once by the vertices $\{e_i / 1 \le i \le p\}$ and $\{w_i / 1 \le i \le p\}$ in $M(K_{1,p,p})$ and $V(M(K_{1,p,p})) = \{v_0\} \cup \{e_i / 1 \le i \le p\} \cup \{v_i / 1 \le i \le p\} \cup \{u_i / 1 \le i \le p\} \cup \{w_i / 1 \le i \le p\}$, $\{v_0\} \cup \{e_i / 1 \le i \le p\} \cup \{w_i / 1 \le i \le p\}$ dominates all the remaining vertices. Thus, $\gamma_{[1,2]tc}(M(K_{1,p,p})) = \{v_0\} \cup \{e_i / 1 \le i \le p\} \cup \{w_i / 1 \le i \le p\} \le 2p + 1$. If $\{e_i / 1 \le i \le p\}$ and $\{w_i / 1 \le i \le p\}$ are taken as triple connected [1,2] dominating set then $\gamma_{[1,2]tc}(M(K_{1,p,p})) = \{v_0\} \cup \{u_i / 1 \le i \le p\} \cup \{e_i / 1 \le i \le p\} \cup \{w_i / 1 \le i \le p\} = 3p + 1 \ge 2p + 1$. Hence $\gamma_{[1,2]tc}(M(K_{1,p,p})) = 2p+1$.

3.3 Central graphs of star and double star of graphs

Theorem 3.3.1: Triple Connected [1, 2] Domination number of Central graph of Star graph is $\gamma_{[1,2]tc}(C(K_{1,p})) = 3$.

Proof: Let $\{v_i / 1 \le i \le p\}$ be the pendent vertices of $K_{1,p}$ and let v_0 be the apex of $K_{1,p}$ adjacent to $\{v_i / 1 \le i \le p\}$. Then each edge $\{v_0 v_i / 1 \le i \le p\}$ is subdivided exactly once and joining all the non adjacent vertices of $K_{1,p}$.

In $C(K_{1,p})$, $\{v_i / 1 \le i \le p\}$ induces a clique of order p and $\{e_i / 1 \le i \le p\}$ are independent. Thus, $\{v_0, v_i, e_i\}$ dominates $C(K_{1,p})$. Hence $\gamma_{[1,2]tc}(C(K_{1,p})) = 3$.

Theorem 3.3.2: Triple Connected [1, 2] Domination number of Central graph of double Star graph is $\gamma_{[1,2]tc}(C(K_{1,p,p})) = p + 1$.

Proof: By the definition of central graph, each edge $\{v_0 u_i / 1 \le i \le p\}$ and $\{u_i v_i / 1 \le i \le p\}$ be subdivided by the vertices $\{e_i / 1 \le i \le p\}$ and $\{w_i / 1 \le i \le p\}$ in $C(K_{1,p,p})$ and $V(C(K_{1,p,p})) = \{v_0\} \cup \{e_i / 1 \le i \le p\} \cup \{u_i / 1 \le i \le p\} \cup \{w_i / 1 \le i \le p\} \cup \{v_i / 1 \le i \le p\}$

The vertices $\{u_i / 1 \le i \le p\}$ induces a clique of order p and $\{v_0\} \cup \{v_i / 1 \le i \le p\}$ induces a clique of order p + 1 and $\{w_i / 1 \le i \le p\}$ and $\{e_i / 1 \le i \le p\}$ are the independent vertex set. Thus, v_0 and $\{v_i / 1 \le i \le p\}$ [or $\{w_i / 1 \le i \le p\}$ or $\{u_i / 1 \le i \le p\}$ should taken as a $\gamma_{[1,2]tc}(C(K_{1,p,p}))$ set.

Without loss of generality, let us assume that v_0 and $\{v_i / 1 \le i \le p\}$ as a dominating set. Hence $\gamma_{[1,2]tc}(C(K_{1,p,p})) = \{v_0\} \cup \{v_i / 1 \le i \le p\} = p + 1$. If $\{w_i / 1 \le i \le p\} \in S$, then $\gamma_{[1,2]tc}(C(K_{1,p,p})) = \{w_i / 1 \le i \le p\} \cup \{v_0\} \cup \{v_i / 1 \le i \le p\} = 2p+1 \ge p + 1$ If $\{u_i / 1 \le i \le p\} \in S$, then $\gamma_{[1,2]tc}(C(K_{1,p,p})) = \{u_i / 1 \le i \le p\} \cup \{v_0\} \cup \{e_i / 1 \le i \le p\} = 2p + 1 \ge p + 1$. Hence $\gamma_{[1,2]tc}(C(K_{1,p,p})) = p + 1$.

4. Conclusion

In this paper we have investigated the sharp bound of the triple connected [1,2] domination number and we obtained triple connected [1,2] domination number for star and double star families of graphs. We will extend our results by obtaining more graphs to the sharp of the bound $\gamma_c(G) \le \gamma_{[1,2]tc}(G) \le p$. we plan to find the triple connected [1,2] domination number for special types of graphs.

Acknowledgement

This research work was supported by University Grants Commission, New Delhi under DSA, GRI-DU

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