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Further results on b-domination in graphs

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Abstract: *A vertex v is a boundary vertex of u if d(u, w) ≤ d(u, v) for all w ∈ N(v). A vertex u has more than one boundary vertex at different distance levels. A vertex v is called a boundary neighbor of u if v is a nearest boundary of u. A set S ⊆ V(G) is a b-dominating set if every vertex in V–S has at least one neighbor in S and at least one boundary neighbor in S. The minimum cardinality of the b-dominating set is called the b- domination number and is denoted by* $\gamma_{bd}(G)$.

In this paper we present several bounds on the b-domination number of graph G and we found the exact value of $\gamma_{bd}(G)$ *for some particular classes of graphs and trees.*

Keywords: *Boundary vertex, boundary neighbor, b-dominating set, b-domination number.*
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1. Introduction

Graphs discussed in this paper are undirected and simple. For graph theoretic terminology refer to Harary[7], Buckley and Harary[5]. For a graph G, let V(G) and E(G) denote its vertex and edge set respectively. A graph with p vertices and q edges is called a (p, q) graph. The length of any shortest path between any two vertices u and v of a connected graph G is called the distance between u and **v** and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the eccentricity e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max{d(u, v) : u ∈ V}. The radius rad(G) is the minimum eccentricity of the vertices, whereas the diameter diam(G) is the maximum eccentricity. If these two are equal in a graph, that graph is called self-centered graph with radius r and is called an r self-centered graph. For any connected graph G, rad(G) ≤ diam(G) ≤ 2rad(G). The vertex v is a central vertex if e(v) = r(G). The center C(G) is the set of all central vertices. For a vertex v, each vertex at a distance e(v) from v is an eccentric vertex of v.

A subgraph of G is a graph having all of its vertices and edges in G. It is a spanning subgraph if it contains all the vertices of G. If H is a subgraph of G, then G is a super graph of H. For any set S of vertices in G, the induced subgraph < S > is the maximal subgraph with vertex set S. A vertex v in a graph G is called a complete vertex or an extreme vertex if the sub graph induced by its neighborhood is complete.

The corona $G_1 \circ G_2$ of two graphs G_1 and G_2 was defined as the graph G obtained by taking one copy of G_1(which has n vertices) and n copies of G_2 and then joining the i^{th} vertex

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of G_1 to every vertex in the i^{th}copy of G_2.Let G be a graph with p vertices. Take p copies of path P_3. Attach an end vertex of i^{th} copy of P_3 to i^{th} vertex of G. The new graph obtained is known as 2-corona of G.

A vertex v is a boundary vertex of u if d(u, w) ≤ d(u, v) for all w ∈ N(v). A vertex u can have more than one boundary vertex at different distance levels. The set of all vertices S = {v ∈V(G)/v is a boundary vertex of some u ∈ V(G)} is known as the set of all boundary vertices of G. A vertex v is called a boundary neighbor of u if v is a nearest boundary of u. The number of boundary neighbor of u is called the boundary degree of u [6].

A set D ⊆ V is said to be a dominating set in G, if every vertex in V−D is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number and is denoted by γ(G) [9]. A dominating set D is said to be an eccentric dominating set if for every v ∈V−D, there exists at least one eccentric vertex of v in D. The minimum cardinality of an eccentric dominating set of G is called the eccentric domination numberγ_{ed}(G) [1, 2, 8]of G.

In 2010, Janakiraman, Bhanumathi and Muthammai[8] defined Eccentric domination in graphs. Motivated by this, we have defined b-domination number of a given graph and study the parameter γ_{bd}(G) in [3].

Theorem 1.1[9]: If D is any dominating set then $|V-D| \leq \sum_{u \in D} \deg(u)$ for all u∈D.

Theorem 1.2[9]: If G is a connected graph with p vertices, then γ(G) ≤ p/2.

Theorem 1.3[8]: $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$.

2. b-domination in graphs

In [3], we have defined the b-domination number of a graph G as follows:

Definition 2.1[3]:A set S⊆V(G) is a b-dominating set if every vertex in V−S has at least one neighbor in S and at least one boundary neighbor in S. The minimum cardinality of the b-dominating set is called the b-domination number and is denoted by γ_{bd}(G).

For a vertex v, let B(v) denote the set of all boundary neighbors of v in G.

Let S⊆V(G). Then S is known as a boundary neighbor set of G if for every vertex v ∈V–S, S has at least one vertex u such that u ∈ B(v). A boundary neighbor set S of G is a minimal boundary neighbor set if no proper subset S′of S is a boundary neighbor set of G.

We define S as a minimum boundary neighbor set if S is a boundary neighbor set with minimum cardinality. Let b(G) denote the cardinality of a minimum boundary neighbor set of G; b(G) can be called as boundary number of G.

Let D be a minimum dominating set of a graph G and S be a minimum boundary neighbor set of G. Clearly, $D \cup S$ is a b-dominating set of a graph G. Hence, $\gamma_{bd}(G) \leq \gamma(G) + b(G)$.

If $G \neq K_n$, $\gamma_{bd}(G) \geq 2$. Hence, if $G \neq K_n$, $2 \leq \gamma_{bd}(G) \leq \gamma(G) + b(G)$.

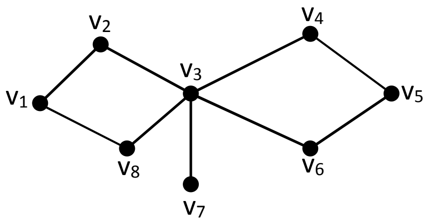

Figure 2.1: G

In Figure 2.1, $D_1 = \{v_1, v_3, v_5\}$ is a minimum dominating set of G. $\gamma(G) = 3$. D_1 is also an eccentric dominating set of G. $\gamma_{ed}(G) = 3$.

$D_2 = \{v_2, v_3, v_5\}$ is a dominating set of G, but it is not an eccentric dominating set of G.

$D_3 = \{v_2, v_8, v_7, v_5\}$ is a b-dominating set but it is not an eccentric dominating set of G.

$D_4 = \{v_1, v_5, v_7\}$ is a minimum b-dominating set of G. $\gamma_{bd}(G) = 3$.

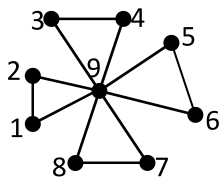

Figure 2.2: G

In Figure 2.2, $D_1 = \{9\}$ is a minimum dominating set of G. Therefore, $\gamma(G) = 1$.

$D_2 = \{1, 2, 9\}$ is a minimum eccentric dominating set. Therefore, $\gamma_{ed}(G) = 3$.

$D_3 = \{1, 3, 5, 7\}$ is a minimum b-dominating set of G. $\gamma_{bd}(G) = 4 = \left\lfloor \frac{9}{2} \right\rfloor$.

Following theorems give the bounds of $\gamma_{ed}(G)$.

Theorem 2.1: Let D be any b-dominating set of G. Then $|V - D| \leq \sum_{u \in D} \deg(u)$ for all $u \in D$. Equality holds for any complete graph.

Proof: Let D be any b-dominating set of G. Then clearly D is a dominating set of G. Thus, the proof follows from Theorem 1.1.

Remark 2.1: For any graph G, $\left\lceil \frac{p}{\Delta+1} \right\rceil \leq \gamma(G) \leq \gamma_{bd}(G)$.

Theorem 2.2: If G is a connected graph with $\delta(G) \geq 2$, then $\gamma_{bd}(G) \leq (p+m)/2$, where m is the number of boundary vertices of G.

Proof: Let G be a graph with m boundary vertices $v_1, v_2, \ldots, v_m$. Attach one pendant vertex to each of these vertices. Let the new graph be G'. Then G' is a connected graph containing p+m vertices. Hence, $\gamma(G') \leq (p+m)/2$. Dominating set of G', containing $v_1, v_2, v_3, \ldots, v_m$ is the b-dominating set of G. Therefore, $\gamma_{bd}(G) \leq (p+m)/2$.

Theorem 2.3: Let G be a graph with radius 1 and diameter 2, then $2 \leq \gamma_{bd}(G) \leq \lfloor p/2 \rfloor$.

Proof: Consider a graph G with radius 1 and diameter 2. Let D be a minimum boundary neighbor set of G. We have $b(G) \leq \left\lfloor \frac{p}{2} \right\rfloor - 1$, since G has at least one vertex with eccentricity one. Let u be a central vertex of G. Then $\{u\} \cup D$ form a b-dominating set of G. Hence,

$\gamma_{bd}(G) \leq 1 + \left\lfloor \frac{p}{2} \right\rfloor - 1 \leq \left\lfloor \frac{p}{2} \right\rfloor$. ------ (1)

The central vertex u dominates all the vertices of G but u is not a boundary neighbor of all other vertices of G. So we have to include at least one vertex $v(\neq u)$ of G to form a γ_{bd}-set of G. Hence, $\gamma_{bd}(G) \geq 2$. ------ (2)

From (1) and (2), we have $2 \leq \gamma_{bd}(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$.

Remark 2.3:

(i) Lower bound is attained for $G = K_{1,n}$.

(ii) The upper bound is sharp for the graph in Figure 2.2.

Theorem 2.4: If G is of radius 2 with a unique central vertex u and N(u) has no pendant vertex and N(u) has no vertex of degree two, then $\gamma_{bd}(G) \leq n-(\deg(u)/2)$.

Proof: Let G be a graph with radius 2 and has a unique central vertex u. The central vertex u dominates N(u) but vertices in N(u) may have their boundary neighbors in N(u). Let S be a subset of N(u) such that vertices in $N(u)-S$ have their boundary neighbors in S. $(V-N(u)) \cup S$ is a b-dominating set of G. Hence, $\gamma_{bd}(G) \leq n-(\deg(u)/2)$.

Theorem 2.5: If G is a graph with radius greater than two, then $\gamma_{bd}(G) < n-(\Delta(G))/2$.

Proof: Let $u \in V(G)$ with deg u = $\Delta(G)$. Clearly $V-N(u)$ is a dominating set of G. But vertices in N(u) may have their boundary neighbors in N(u). Let S be a subset of N(u) such that vertices in $N(u)-S$ have their boundary neighbors in S. Then $(V-N(u)) \cup S$ is a b-dominating set of G. Hence, $\gamma_{bd}(G) \leq n-(\Delta(G))/2$.

Next, we evaluate the exact value of eccentric domination number of some particular classes of graphs.

Theorem 2.6: If G is 2-corona graph of C_n, then $\gamma_{bd}(G) = \gamma(C_n) + n$, where n is the number of vertices of C_n.

Proof: Let G be a 2-corona graph of C_n. Then G has 3n vertices. In cycle C_n, $\gamma(C_n)$ vertices dominate other vertices of C_n. The graph G has n pendant vertices. These pendant vertices dominate their support vertices in G. Also, the set of all pendant vertices are the boundary neighbor set of G. Hence, $\gamma_{bd}(G) = \gamma(C_n) + n$.

Theorem 2.7: Let n be an even integer and let G be obtained from the complete graph K_n by deleting edges of a linear factor. Then $\gamma_{bd}(G) = n/2$.

Proof: Let u and v be a pair of non adjacent vertices in G. Then u and v are boundary neighbors to each other.

Therefore, $\gamma_{bd}(G) \geq n/2$. ------ (1)

Consider $D \subseteq V(G)$ such that $\langle D \rangle = K_{n/2}$. D contains n/2 vertices such that each vertex in $V-D$ is adjacent to at least one element in D and each element in $V-D$ has its boundary neighbor in D. Hence $\gamma_{bd}(G) \leq n/2$. ------ (2)

From (1) and (2), $\gamma_{bd}(G) = n/2$.

Theorem 2.8: If G is a graph on n vertices and $H = G \circ mK_1$, $m > 1$, then $\gamma_{bd}(H) = 2n$.

Proof: Let G be a graph on n vertices. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$. Attach mK_1 with every vertex of G. Consider the vertices adjacent to u_1 as $u_1^1, u_1^2, \ldots, u_1^m$.Let S be a set of n pendant vertices such that S contains exactly one pendant vertex adjacent to u_i for i = 1, 2, 3, …, n. Then $V \cup S$ is a b-dominating set. Since $m > 1$, $V \cup S$ is a minimum b-dominating set. Therefore, $\gamma_{bd}(H) = 2n$.

Theorem 2.9: If G is a graph with n vertices and $H = G \circ K_2$, then $\gamma_{bd}(H) = n$.

Proof: Let G be a graph with n vertices and let $H = G \circ K_2$. Let $u_1, u_2, \ldots, u_n$ be the vertices of G. Let $u_1', u_2', \ldots, u_n'$ and $u_1'', u_2'', \ldots, u_n''$ be the newly added vertices of H. Then the sets $\{v_1, v_2, v_3, \ldots, v_n\}$, where $v_i = u_i'$ or $u_i''$are b-dominating sets of G. Hence $\gamma_{bd}(H) = n$.

Theorem 2.10: If G is a friendship graph, then $\gamma_{bd}(G) = r$, where r is the number of wings of G.

Proof: Let G be a friendship graph with r wings. Let S be a set, which contains exactly one vertex from each wing of G. This set is a minimum b-dominating set. Hence, $\gamma_{bd}(G) = r$.

Theorem 2.11: $\gamma_{bd}(\overline{C_4}) = 2$, $\gamma_{bd}(\overline{C_5}) = 3$ and $\gamma_{bd}(\overline{C_n}) = \gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$, $n \geq 6$.

Proof: Clearly, $\gamma_{bd}(\overline{C_4}) = 2$, $\gamma_{bd}(\overline{C_5}) = 3$.

Now assume that $n \geq 6$. Let $v_1, v_2, \ldots, v_n, v_1$ form C_n. Then $\overline{C_n} = K_n - C_n$ and each vertex v_i is adjacent to all other vertices except v_{i-1} and v_{i+1} in $\overline{C_n}$. In $\overline{C_n}$, the eccentric vertices are same as the boundary neighbors. Hence, $\gamma_{bd}(\overline{C_n}) = \gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$ by Theorem 1. 3.

Theorem 2.12: Let $C_n^{(t)}$, $t \geq 2$, be the one point union of t cycles of length n ($n \geq 5$),

If n is even, $\gamma_{bd}(C_n^{(t)}) = \left(\frac{n-2}{2}\right)t + 1.$

If n is odd, $\gamma_{bd}(C_n^{(t)}) = \begin{cases} \left(\frac{n}{3}\right)t & if \quad n = 3m \\ \left\lceil \frac{n}{3} \right\rceil t & if \quad n = 3m+1 \\ \left(\left\lceil \frac{n}{3} \right\rceil + 1\right)t & if \quad n = 3m+2 \end{cases}$

Proof: Let $C_n^{(t)}$ be the one point union of t cycles of length n and u be a central vertex which is a common vertex of all C_n. G has $t(n-1)+1$ vertices. Let the vertex set of k^{th} cycle in $C_n^{(t)}$ be $V_k = \{u, u_{k1}, u_{k2}, \ldots, u_{k(n-1)}\}$, $k = 1, 2, \ldots, t$.

Case (i): n is even, n = 2m.

For each vertex $v_i \in V(C_n^{(j)})$ v_i has its boundary neighbor in the same cycle $C_n^{(j)}$. The middle vertices of $C_n^{(k)}$, $k = 1, 2, 3, \ldots, j-1, j+1, \ldots, t$ are the boundary neighbors of the middle vertex of $C_n^{(j)}$.

Subcase (i)a: m is odd.

Let $D_k = \{u_{k2}, u_{k4}, \ldots, u_{k(2m-2)}\}$, $k = 1, 2, \ldots, t-1$. $D_t = \{u_{t1}, u_{t3}, \ldots, u_{tm}, \ldots, u_{t(2m-1)}\}$ and $D = (\bigcup_{k=1}^{t-1} D_k) \cup D_t$, where $u_m \in D_t$ is the boundary neighbor of u_{mk}, $k = 1, 2, \ldots, t-1$. The vertices of V−D have their neighbors and boundary neighbors in D and vertices from

$V_t - D_t$ have neighbors and boundary neighbors in D_t. Hence, D is a minimum b-dominating set of G and $\gamma_{bd}(G) = |D| = \left(\frac{n-2}{2}\right)t+1$.

Sub case (i)b: m is even.

Let $D_k = \{u_{k1}, u_{k3}, \ldots, u_{k(m-1)}, u_{k(m+2)}, \ldots, u_{k(2m-2)}\}$, k = 1, 2, …, t−1, take $D_t = \{u_{t1}, u_{t3}, \ldots, u_{tm}, u_{t(m+1)}, \ldots, u_{t(2m-2)}\}$ and $D = (\bigcup_{k=1}^{t-1} D_k) \cup D_t$, where $u_m \in D_t$ is the boundary neighbor of u_{mk}, k = 1, 2,…, t−1. The vertices of V−D have neighbors and boundary neighbors in D and vertices from $V_t - D_t$ have neighbors and boundary neighbors in D_t.

Hence, D is a minimum b-dominating set of G and $\gamma_{bd}(G) = |D| = \left(\frac{n-2}{2}\right)t+1$.

Case (ii): n is odd.

Each vertex $v_i \in V(C_n^{(j)})$ has their boundary neighbor in the same cycle $C_n^{(j)}$.

Subcase (ii)a: n = 3m, m is odd.

Let $D_k = \{u_{k1}, u_{k4}, u_{k7} \ldots, u_{km}, u_{k(m+3)}, \ldots, u_{k(2m-1)}\}$, k = 1, 2, 3,…, t and $D = \bigcup_{k=1}^{t-1} D_k \subseteq V(G)$. Then vertices from V−D have their neighbors in D and boundary neighbors in D. Hence, D is a minimum b-dominating set of G and $\gamma_{bd}(G) = |D| = \left(\frac{n}{3}\right)t$.

Sub case (ii)b: n = 3m+1, m is even.

Let $D_k = \{u_{k1}, u_{k4}, u_{k7}, \ldots, u_{k(m+1)}, u_{k(m+3)}, u_{k(m+6)}, \ldots, u_{k(2m)}\}$, k = 1, 2, 3,…, t and $D = \bigcup_{k=1}^{t-1} D_k \subseteq V(G)$. Then vertices from V−D have neighbors in D and boundary neighbor in D. Hence, D is a minimum b-dominating set of G and $\gamma_{bd}(G) = |D| = \left\lceil\frac{n}{3}\right\rceil t$.

Sub case (ii)c: n = 3m+2, m is odd.

Let $D_k = \{u_{k1}, u_{k2}, u_{k5}, u_{k8}, \ldots, u_{km}, u_{k(m+3)}, u_{k(m+4)}, \ldots, u_{k(2m-1)}\}$, k = 1, 2, 3,…, t and $D = \bigcup_{k=1}^{t-1} D_k \subseteq V(G)$. Then vertices from V−D have neighbors in D and boundary neighbors in D. Hence, D is a minimum b-dominating set of G and $\gamma_{bd}(G) = |D| = \left(\left\lceil\frac{n}{3}\right\rceil+1\right)t$.

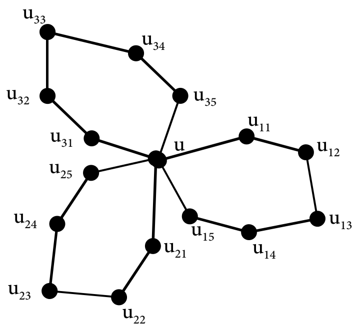

Figure 2.3: G = $C_6^{(3)}$

In Figure 2.3, D = $\{u_{12}, u_{14}\} \cup \{u_{22}, u_{24}\} \cup \{u_{31}, u_{33}, u_{35}\}$ form a minimum b-dominating set of G and $\gamma_{bd}(G) = 7$.

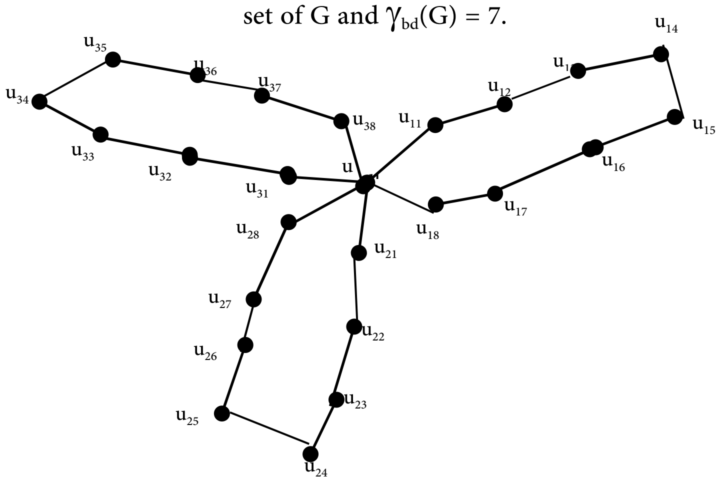

Figure 2.4: G = $C_9^{(3)}$

In Figure 2.4, D = $\{u_{11}, u_{14}, u_{17}\} \cup \{u_{21}, u_{24}, u_{27}\} \cup \{u_{31}, u_{34}, u_{37}\}$ form a minimum b-dominating set of G and $\gamma_{bd}(G) = 9$.

Theorem 2.13: Let $K_n^{(t)}$, $t \geq 2$, be the one point union of t complete graphs of n ($n \geq 2$) vertices, then $\gamma_{bd}(G) = t$.

Proof: Let $K_n^{(t)}$ be the one point union of t complete graphs of n and u be a central vertex which is a common vertex of all K_n. G has $t(n-1) + 1$ vertices. Let the vertex set of k^{th} complete graph in $K_n^{(t)}$ be $V_k = \{u, u_{k1}, u_{k2, \ldots,} u_{k(n-1)}\}$, k = 1, 2, …, t.

Let $D_k = \{u_{k1}\}$, k = 1, 2, …, t and $D = \bigcup_{k=1}^{t-1} D_k \subseteq V(G)$. The vertices from V−D have neighbors and boundary neighbors in D. Hence, D is a minimum b-dominating set and $\gamma_{bd}(G) = |D| = t$.

Following theorems characterize graphs for which $\gamma_{ed}(G) = \gamma_{bd}(G)$, when r(G) = 1 or G is 2 self-centered.

Theorem 2.14: Let G be a 2 self-centered graph. $\gamma_{ed}(G) = \gamma_{bd}(G)$ if and only if G has no complete vertex.

Proof: G has no complete vertex if and only if boundary neighbors of $v \in V(G)$ are its eccentric vertices. Hence the theorem follows.

Theorem 2.15: Let G be a graph of radius one and diameter two. Then $\gamma_{ed}(G) = \gamma_{bd}(G)$ if and only if G has no complete vertex.

Proof: For a vertex of eccentricity one, all other vertices are boundary vertices. G has no complete vertex if and only if boundary neighbors of $v \in V(G)$ with e(v) = 2 are its eccentric vertices. Hence the theorem follows.

3. b-domination in trees

In this section, we study the b-domination number of trees. We found some bounds for $\gamma_{bd}(T)$, where T is a tree.

Theorem 3.1: For any tree T, $\gamma_{bd}(T) \geq \gamma_{ed}(T)$.

Proof: Let T be a tree with n pendant vertices. In a tree T, the set of all pendant vertices is the boundary neighbor set of T. Hence, $\gamma_{bd}(T) \geq \gamma_{ed}(T)$.

In Figure 3.1, $\gamma_{ed}(T) = \gamma_{bd}(T)$.

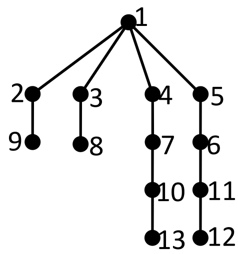

Figure 3.1: T

D_1 = {1, 8, 9, 10, 11} is a minimum dominating set of T. $\gamma(T) = 5$.

D_2 = {2, 3, 6, 7, 12, 13} is a minimum eccentric dominating set of T. $\gamma_{ed}(T) = 6$.

D_2 = {4, 5, 8, 9, 12, 13} is a minimum b-dominating set of T and also a minimum eccentric dominating set of T. $\gamma_{bd}(T) = 6$.

Hence, $\gamma_{ed}(T) = \gamma_{bd}(T) = 6$.

In Figure 3.2, $\gamma_{bd}(T) > \gamma_{ed}(T)$.

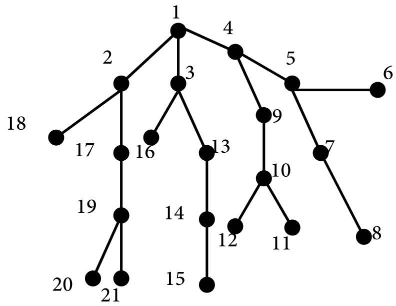

Figure 3.2: T

$D_1 = \{2, 3, 5, 7, 10, 14, 19\}$ is a minimum dominating set. $\gamma(T) = 7$.

$D_2 = \{2, 3, 5, 8, 10, 15, 19\}$ is a minimum eccentric dominating set. $\gamma_{ed}(T) = 7$.

$D_3 = \{4, 6, 7, 10, 11, 14, 16, 18, 19, 21\}$ is a minimum b-dominating set. $\gamma_{bd}(T) = 10$.

In this example, $\gamma(T) = \gamma_{ed}(T)$ but $\gamma(T) \neq \gamma_{bd}(T)$ and $\gamma_{ed}(T) \neq \gamma_{bd}(T)$.

Theorem 3.2: For a tree T, $\gamma_{bd}(T) \leq n-\Delta(T)+1$.

Proof: If T has a vertex u of maximum degree which is not a support, then $V-N(u)$ is a b-dominating set of T. If T has a vertex u of maximum degree which is a support of a pendant vertex v, then $(V-N(u)) \cup \{v\}$ is a b-dominating set of T. Hence the theorem follows.

Theorem 3.3: Let T be a tree with $rad(T) > 2$ and $u \in V(T)$ such that $\deg u = \Delta(T)$. Then $\gamma_{bd}(T) < n - \Delta(T)$.

Proof: Let $\deg u = \Delta(T)$ and $N(u)$ has no pendant vertex. $V-N(u)$ is a b-dominating set of cardinality $n-\Delta(T)$. Since the radius of T is at least three, diameter of T is at least five. Consider a diametral path P. This path contains at least six vertices and includes at most two edges from the sub graph induced by $N[u]$. That is P contains at most three vertices from $N[u]$.

Case (i): All vertices of $P-N[u]$ (except end vertices) are support of some pendant vertices.

In this case, we have to include all the vertices of $P-N(u)$ in a γ_{bd}-set, but we can leave those pendant vertices from $V-N(u)$ to form a γ_{bd}-set. Therefore, $\gamma_{bd}(T) < n-\Delta(T)$.

Case (ii): At least one vertex w of $P-N[u]$ (except end vertices) is not a support.

In this case, we can leave that vertex w from $V-N(u)$ to form a γ_{bd}-set. Therefore, $\gamma_{bd}(T) < n-\Delta(T)$.

Theorem 3.4: For a bi central tree T with radius 2, $\gamma_{bd}(T) \leq 4$.

Proof: Let u and v be the central vertices of T. If N[u] has more than one pendant vertex and $|N(v)| = 2$, then $\gamma_{bd}(T) = 3$. Suppose N(u) and N(v) have more than one pendant vertex then $\gamma_{bd}(T) = 4$. If $T = P_4$, then $\gamma_{bd}(T) = 2$. Hence, $\gamma_{bd}(T) \leq 4$.

Corollary 3.4: If T = $\overline{K_m} + K_1 + K_1 + \overline{K_n}$, m, n > 1 then $\gamma_{bd}(T) = 4$.

Proof: Take G = $\overline{K_m} + K_1 + K_1 + \overline{K_n}$. In G, the central vertices dominate other vertices of G and any two peripheral vertices of G are the boundary neighbor set of G. Therefore, the central vertices and two peripheral vertices form a b-dominating set of G. Hence, $\gamma_{bd}(G) = 4$.

Theorem 3.5: If T is a unicentral tree of radius 2 and N(u) does not have a pendant vertex, then $\gamma_{bd}(T) \leq n-\deg u$, where u is the central vertex.

Proof: If a tree T is of radius 2 with a unique central vertex u and N(u) does not contain a pendant vertex then u dominates N[u]. The vertices in V−N[u] dominates themselves and each vertex in N(u) has boundary vertices in V−N[u] only. Therefore, V−N(u) is a b-dominating set of cardinality n−deg(u), so that $\gamma_{bd}(T) \leq n-\deg u$.

Theorem 3.6: If every non pendant vertex of a tree T is a support vertex, then $\gamma_{bd}(T) \leq l$, where l is the number of pendant vertices.

Proof: Let T be a tree with l pendant vertices. Consider a leaf e = uv in T, where u is a support of v. The boundary neighbor of u is v in T. The pendant vertex is the boundary neighbor of its support vertex. Suppose a support vertex u is adjacent with the pendant vertices $v_1, v_2, v_3, ..., v_s$. Then vertex v_i is the boundary neighbor of $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_s$. Therefore, the set of all pendant vertices form a b-dominating set of T. Hence, $\gamma_{bd}(T) \leq l$.

Theorem 3.7: If T is a bi central tree, then $\gamma_{bd}(T) \leq \gamma(T) + s$, where s is the number of support vertices of T.

Proof: Assume that T is a bi central tree with s support vertices. Let D be a dominating set of T. Each pendant vertex is a boundary neighbor of its support vertex. Suppose a support vertex u_1 has pendant vertices $u_{11}, u_{12}, u_{13}, \ldots, u_{1n}$ then u_{1i} is the boundary neighbor of u_1 and $u_{11}, u_{12}, u_{13}, \ldots, u_{1(i-1)}, u_{1(i+1)}, \ldots, u_{1n}$. Take $S = \{u_{ij}\}$, where u_{ij} is adjacent to u_i, j = 1, 2, …, s. Hence, $D \cup S$ is a b-dominating set. Therefore, $\gamma_{bd}(T) \leq \gamma(T) + s$.

Theorem 3.8: Let G be a caterpillar with base P_n, then $\gamma_{bd}(G) \leq n+s$, where s is the number of support vertices.

Proof: Let G be a caterpillar with base P_n. Then G has at most n support vertices and may have more than n pendant vertices. Let D $\subseteq$V(G) consists of all vertices of P_n and at least one pendant vertex adjacent to each support vertex. Clearly D is a b-dominating set. Hence,$\gamma_{bd}(G) \leq n+s$.

Corollary 3.8: If G is a graph $P_n \circ mK_1$, m > 1 is an integer, then $\gamma_{bd}(G) = 2n$.

Proof: As in Theorem 3.8, we can prove that$\gamma_{bd}(G) = 2n$.

Observations:

3.1 If G is a graph $P_n \circ K_1$, then $\gamma_{bd}(G) = n$.

3.2 If there exists a γ_{ed}-set which contains all the pendant vertices of T, then $\gamma_{bd}(T) = \gamma_{ed}(T)$.

3.3 If there exist a γ-set which contains all the pendant vertices of a tree T, then $\gamma(T) = \gamma_{ed}(T) = \gamma_{bd}(T)$.

3.4 If T has a dominating set D containing boundary neighbors set of T, then $\gamma(T) = \gamma_{bd}(T)$. For example, $\gamma(P_4) = 2 = \gamma_{bd}(P_4)$.

Theorem 3.9: If every non pendant vertex of a tree T is a support vertex, then $\gamma_{bd}(T) \leq n/2$.

Proof: Let T be a tree with n vertices. If u is a support vertex of v in T, then v is a boundary neighbor of u. Suppose that every non pendant vertex of a tree T is a support vertex.

Case (i): Every support vertex has exactly one adjacent pendant vertex in T.

Every pendant vertex is a boundary neighbor of its corresponding support vertex in T. Thus the set of all pendant vertices form a b-dominating set of T. Hence, $\gamma_{bd}(T) = n/2$.

Case (ii): Every support vertex has more than one adjacent pendant vertex in T.

Assume a tree T has s support vertices. Consider a support vertex v_i, i = 1, 2, …, s which is adjacent to the pendant vertices $v_{i1}, v_{i2}, \ldots, v_{im}$. Any one pendant vertex v_{ij} is a boundary neighbor of other pendant vertices $(v_{i1}, v_{i2}, \ldots, v_{i(j-1)}, v_{i(j+1)}, \ldots, v_{im})$ and the support vertex v_i. D = $\{v_i\} \cup \{v_{i1}\}$, i = 1, 2, …, s is a b-dominating set of T. Hence, $\gamma_{bd}(T) \leq n/2$.

Conclusion

In this paper, exact values of b-domination number of some particular classes of graphs and some bounds for b-domination number of graphs and b-domination number of trees are given.

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