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The Dominating Graph $D_mG^{bcd}(G)$ of a Graph G

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Abstract: *The dominating graph $D_mG^{bcd}(G)$ of a graph G is obtained from G with vertex set $V' = V(G) \cup S$, where $V = V(G)$ and S is the set of all ⊠-sets of G. Two elements in V′ are said to satisfy property 'a' if $u, v \in V'$ and are adjacent in G. Two elements in V′ are said to satisfy property 'b' if $u = D_1$, $v = D_2 \in S$ such that D_1 and D_2 have a common vertex. Two elements in V′ are said to satisfy property 'c' if $u \in V(G)$, $v = D \in S$ such that $u \in D$. Two elements in V′ are said to satisfy property 'd' if $u, v \in V(G)$ and there exists $D \in S$ such that $u, v \in D$. A graph having vertex set V′ and any two elements in V′ are adjacent if they satisfy any one of the property b, c, d is denoted by $D_mG^{bcd}(G)$. In this paper, we have studied some basic properties of $D_mG^{bcd}(G)$. Also, we have characterized graphs G for which $D_mG^{bcd}(G)$ has some specific properties and we have established some extremal properties of $D_mG^{bcd}(G)$.*

Keywords: *dominating set, minimum dominating set, dominating graph.*
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1. Introduction

Graphs discussed in this paper are undirected and simple. For graph theoretic terminology refer to Harary[7], Buckley and Harary[5]. For a graph, let V(G) and E(G) denote its vertex and edge set respectively. A graph with p vertices and q edges is called a (p, q) graph.

The concept of distance in graph plays a dominant role in the study of structural properties of graphs in various angles using related concept of eccentricity of vertices in graphs.

The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$. The *radius* rad(G) is the minimum eccentricity of the vertices, whereas the *diameter* diam(G) is the maximum eccentricity. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r self-centered graph*. For any connected graph G, rad(G) ≤ diam(G) ≤ 2rad(G). v is a central vertex if e(v) = r(G). The *center* C(G) is the set of all central vertices. For a vertex v, each vertex at a distance e(v) from v is an *eccentric vertex of v*.

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The *girth* $g(G)$ of the graph G, is the length of the shortest cycle (if any) in G.

A graph G is *connected* if every two of its vertices are connected, otherwise G is *disconnected*. The *vertex connectivity* or simply *connectivity* ⊠(G) of a graph G is the minimum number of vertices whose removal from G results in a disconnected or trivial graph. The *edge connectivity* ⊠(G) of a graph G is the minimum number of edges whose removal from G results in a disconnected or trivial graph. A set S of vertices of G is *independent* if no two vertices in S are adjacent. The *independence number* ⊠$_o(G)$ of G is the maximum cardinality of an independent set.

The concept of domination in graphs was introduced by Ore [13]. The concept of domination in graphs is originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters. For details on ⊠(G), refer to [6, 14].

A set $D \subseteq V$ is said to be a *dominating set* in G, if every vertex in $V-D$ is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the *domination number* and is denoted by $\gamma(G)$.

A vertex v is said to be good if there is a ⊠-set of G containing v [16]. If there is no ⊠-set of G containing v, then v is said to be a bad vertex.

In [15], Walikar, Acharya and et al., defined ⊠$_D$(G) as the total number of minimum dominating sets in a graph G.

Kulli, Janakiram and Niranjan [8, 9, 10, 11, 12] introduced the following concept in the field of domination theory.

The *minimal dominating graph* $MD(G)$[9] of a graph G is the intersection graph defined on the family of all minimal dominating sets of vertices of G. The *common minimal dominating graph* $CD(G)$[10] of a graph G is the graph having the same vertex set as G with two vertices adjacent in CD(G) if and only if there exists a minimal dominating set in G containing them [6]. The *vertex minimal dominating graph* $M_vD(G)$[11] of a graph G is a graph with $V(M_vD(G)) = V' = V \cup S$, where S is the collection of all minimal dominating sets of G with two vertices $u, v \in V'$ are adjacent if either they are adjacent in G or $v = D$ is a minimal dominating set of G containing u. The *dominating graph* $D(G)$[12] of a graph $G = (V, E)$ is a graph with $V(D(G)) = V \cup S$, where S is the set of all minimal dominating sets of G and with two vertices $u, v \in V(D(G))$ are adjacent if $u \in V$ and $v = D$ is a minimal dominating set of G containing u.

The *Total graph* $T(G)$ of a graph G is the graph with vertex set $V(G) \cup E(G)$ and two vertices of T(G) are adjacent if and only if the corresponding elements of G are adjacent or incident in G.

The *Quasi-total graph P(G)* has a vertex set $V(G) \cup E(G)$ and two vertices are adjacent in P(G) if and only if they correspond to two non-adjacent vertices of G, two adjacent edges of G or to a vertex and an edge incident to it in G.

In [1, 2, 3, 4], we have defined and studied the Dominating Graph, $DG^{bcd}(G)$, $DG^{abc}(G)$ and $ED_mG^{abc}(G)$ of a graph G.

In this paper, we define the dominating graph $D_mG^{bcd}(G)$. We find $D_mG^{bcd}(G)$ for some families of graphs, and some basic properties of $D_mG^{abc}(G)$ are studied. Also, characterization of $D_mG^{abc}(G)$ are established.

The following results are needed to study the dominating graph $D_mG^{bcd}(G)$ of a graph G.

Theorem 1.1[11]: A graph G is Eulerian if and only if every vertex of G is of even degree.

Theorem 1.2[17]: (i) $\gamma_D(P_{3k}) = 1$.

(ii) $\gamma_D(P_{3k+1}) = (k^2+5k+2)/2$.

(iii) $\gamma_D(P_{3k+2}) = k+2$.

Theorem 1.3 [17]: (i) $\gamma_D(C_{3k}) = 3$.

(ii) $\gamma_D(C_{3k+1}) = (3k+1)(k+2)/2$.

(iii) $\gamma_D(C_{3k+2}) = 3k+2$.

2. The Dominating Graph $D_mG^{bcd}(G)$ of a Graph G

Definition 2.1: The dominating graph $D_mG^{bcd}(G)$ of a graph G is obtained from G with vertex set $V' = V(G) \cup S$, where V = V(G) and S is the set of all ⊠-sets of G. Two elements in V' are said to satisfy property 'a' if $u, v \in V'$ and are adjacent in G. Two elements in V' are said to satisfy property 'b' if $u = D_1$, $v = D_2 \in S$ such that D_1 and D_2 have a common vertex. Two elements in V' are said to satisfy property 'c' if $u \in V(G)$, $v = D \in S$ such that $u \in D$. Two elements in V' are said to satisfy property 'd' if $u, v \in V(G)$ and there exists $D \in S$ such that $u, v \in D$. A graph having vertex set V' and any two elements in V' are adjacent if they satisfy any one of the property b, c, d is denoted by $D_mG^{bcd}(G)$.

The dominating graph $DG^{bcd}(G)$ of a graph G is obtained from G with vertex set $V' = V(G) \cup S$, where V = V(G) and S is the set of all minimal dominating sets of G and any two elements in V' are adjacent if they satisfy any one of the property b, c, d.

Remark 2.1:

(i) $|S| =$ ⊠$_D(G)$.

(ii) $D_mG^{bcd}(G) \subseteq DG^{bcd}(G)$.

(iii) Number of vertices in $D_mG^{bcd}(G)$ is $p+\gamma_D(G)$.

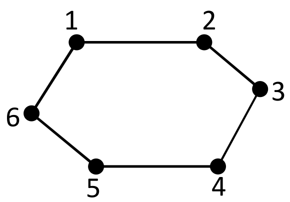
(iv) Number of edges in $D_mG^{bcd}(G)$ is $q' < \dfrac{p^2 - p + 2\gamma\gamma_D + \gamma^2_D - \gamma_D}{2}$.

(v) Degree of a point vertex v_i in $D_mG^{bcd}(G)$ is $\leq n(v_i)\gamma(G)$, where $n(v_i)$ is the number of γ-sets containing v_i in G. This bound is sharp if γ-sets of G are mutually disjoint.

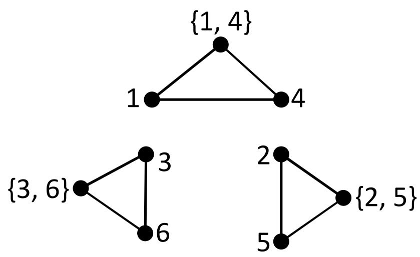
(vi) Degree of a set vertex D_i in $D_mG^{bcd}(G)$ is $\leq \gamma_D(G)-1+|D_i| \leq \gamma_D(G)-1+\gamma(G) \leq \gamma_D(G)-1+p$, $1 \leq i \leq \gamma_D$.

Remark 2.2: If each minimal dominating set is a γ-set of G, then $DG^{bcd}(G) = D_mG^{bcd}(G)$.

Example 2.1:



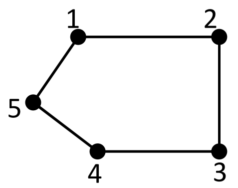
G



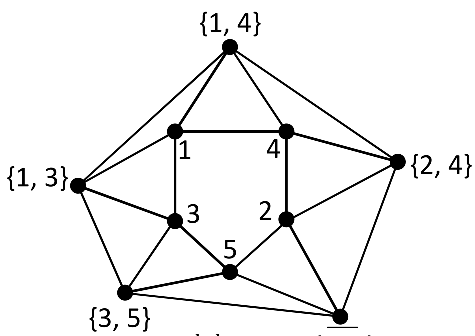
$D_mG^{bcd}(G) \cong T(3K_2)$

{1, 4}, {2, 5} and {3, 6} are γ-sets of G.

Example 2.2:



G



$D_mG^{bcd}(G) \cong T(\overline{C_5})$

{1, 4}, {1, 3}, {2, 4}, {2, 5} and {3, 5} are γ-sets of G.

Theorem 2.1: If $G = K_p$, then $D_mG^{bcd}(G)$ is pK_2.

Proof: Suppose $G = K_p$. Then each vertex forms a γ-set. By the definition of $D_mG^{bcd}(G)$, $D_mG^{bcd}(G)$ is pK_2.

Theorem 2.2: If G = $\overline{K_p}$, then $D_mG^{bcd}(G)$ is a complete graph with 1+p vertices.

Proof: Suppose G = $\overline{K_p}$. Then the whole vertex set is a γ-set. By the definition $D_mG^{bcd}(G)$ is a complete graph with 1+p vertices.

Theorem 2.3: If G = $K_{1, p-1}$, $p \geq 3$, then $D_mG^{bcd}(G)$ is $K_2 \cup (p-1)K_1$.

Proof: Suppose G = $K_{1, p-1}$, $p \geq 3$. Let u be the central vertex. $\{u\}$ = D is a γ-set of G. In $D_mG^{bcd}(G)$, D is adjacent to u only and other vertices are isolated. Hence $D_mG^{bcd}(G)$ is $K_2 \cup (p-1)K_1$.

Theorem 2.4: If G = $K_{m,n}$, $2 < m \leq n$, then $D_mG^{bcd}(G) \cong T(K_{m,n})$, which is a 2 self-centered graph.

Proof: Let G = $K_{m, n}$, $2 < m \leq n$. V(G) = $V_1 \cup V_2$. $|V_1|$ = m and $|V_2|$ = n. D = {u, v}, $u \in V_1$ and $v \in V_2$ is a γ-set of G. Thus, we get mn such γ-sets of G. Since G is an induced sub graph of $D_mG^{abc}(G)$, e(u) = e(v) = 2. Suppose x, y $\in$ S. Then x = D_1 and y = D_2 are two γ-sets of G. If D_1 and D_2 have a common vertex, then, in $D_mG^{bcd}(G)$, $d(D_1, D_2) = 1$. Suppose D_1 and D_2 are disjoint. Then there exists a γ-set D_3 such that D_3 is adjacent to both D_1 and D_2. Then, in $D_mG^{abc}(G)$, $d(D_1, D_2) = d(D_1, D_3)+d(D_3, D_2) = 2$. Suppose x $\in$V(G), y = D_4 is a γ-set of G. If x $\in D_4$, then in $D_mG^{abc}(G)$, d(x, y) = 1.
If x $\notin D_4$, then there exists a vertex x' such that x' dominates x, and $x' \in D_4$. Then it follows that, in $D_mG^{abc}(G)$, $d(x, y) = d(x, x')+d(x', y) = 2$. Thus, the eccentricity of point vertices is 2 and eccentricity of set vertices is also 2. Hence, $D_mG^{bcd}(G)$ is a 2 self-centered graph.

Theorem 2.5: If G = $W_p = C_p+K_1$, $p \geq 4$, then $D_mG^{bcd}(G)$ is $K_2 \cup pK_1$.

Proof: Suppose G = $W_p = C_p+K_1$. Let u be the central vertex. $\{u\}$ = D is a γ-set of G. In $D_mG^{bcd}(G)$, D is adjacent to u only and other vertices are isolated. Hence, $D_mG^{bcd}(G)$ is $K_2 \cup pK_1$.

Theorem 2.6: If G is a connected graph with $\Delta(G) = p-1$, then $D_mG^{bcd}(G)$ is disconnected.

Proof: Suppose G is a connected graph with $\Delta(G) = p-1$. Then rad(G) = 1, and if u is a central vertex, it forms a γ-set. In $D_mG^{bcd}(G)$, a central vertex is adjacent to the corresponding γ-set only. Hence $D_mG^{bcd}(G)$ is disconnected .

Corollary 2.6: (i) If $\Delta(G) = p-1$, then K_2 is a component of $D_mG^{bcd}(G)$.

(ii) Let G be a graph with $\Delta(G) = p-1$. If t is the number of central vertices, then $D_mG^{bcd}(G)$ is $tK_2 \cup (p-t)K_1$.

Theorem 2.7: Let G be a connected graph. If any two γ-sets are disjoint, then $D_mG^{bcd}(G)$ is disconnected.

Proof: Let G be a connected graph. Suppose any two γ-sets are disjoint. Then, in $D_mG^{bcd}(G)$, set vertices and their corresponding point vertices are adjacent and set vertices are not adjacent . Hence, $D_mG^{bcd}(G)$ is disconnected.

Theorem 2.8: Let G be a connected graph. If every vertex of G is contained in at least one γ-set and any two γ-sets intersect, then $D_mG^{bcd}(G)$ is connected.

Proof: Let G be a connected graph. Suppose, there exists a vertex $u \in V(G)$ such that u does not lie on any γ-set. Then, by the definition, u is an isolated vertex in $D_mG^{bcd}(G)$. Hence every vertex of G is contained in at least one γ-set.

Suppose there is a γ-set D_1 such that D_1 is not intersecting other γ-sets. Then in $D_mG^{bcd}(G)$, D_1 and the point vertices in D_1 form a clique, which is one of the components of $D_mG^{bcd}(G)$. Hence $D_mG^{bcd}(G)$ is disconnected.

Theorem 2.9: For any graph G, if $D_mG^{bcd}(G)$ is connected, then the distance between any two vertices in $D_mG^{bcd}(G)$ is at most three.

Proof: Let $u, v \in V'$. Consider the following cases:

Case (i): Suppose $u, v \in V(G)$.

There exists a γ-set D such that D contains u and v. This implies that, in $D_mG^{bcd}(G)$, $d(u, v) = 1$.

Suppose $u \in D_1$ and $v \in D_2$. If there exists a vertex $x \in V(G)$ such that $x \in D_1$ and D_2, then, in $D_mG^{bcd}(G)$, $d(u, v) = d(u , x) + d (x , v) = 2$.

Suppose $u \in D_1$ and $v \in D_2$. If there exists vertices $y, z \in V(G)$ such that $y \in D_1$, $z \in D_2$ and there exists D_3 which contains y, z, then in $D_mG^{bcd}(G)$,

$d(u , v) = d(u, y) + d(y ,z)+ (z , v) = 3$.

Case (ii): Suppose $u \in V(G)$ and $v \in S$, $v = D'$ is a γ-set of G.

Suppose $u \in D'$. Then, in $D_mG^{bcd}(G)$, $d(u , v) = 1$. Suppose $u \notin D'$, $u \in D''$. If there exists a vertex $u' \in D'$ and D'', then it follows that, in $D_mG^{bcd}(G)$, $d(u, v) = d(u, u')+d(u', v') = 2$ (or) $d(u, D') = d(u, D'') + d(D'', D') = 2$.

Suppose u $\notin D'$, u $\in D''$. D' and D'' are two disjoint γ-sets of G and there is no γ-set containing u and elements of D'. There exists a γ-set D_3 such that D_3 contains $x' \in D'$ and $y' \in D''$. This follows that, in $D_mG^{bcd}(G)$,

$$d(u , v) = d(u , y')+d(y', x') + d(x', v) = 3 \text{ (or)}$$

$$d(u , v) = d(u, D'') + d(D'', D_3) + d (D_3 , D') = 3 \text{ (or)}$$

$$d(u , v) = d(u , y') + d(y' , D_3) + d(D_3 , D') = 3.$$

Case (iii): Suppose u, v $\in$ S, u = D_4 and v = D_5 are two γ-sets of G.

If D_4 and D_5 have a common vertex, then, in $D_mG^{bcd}(G)$, d(u , v) = 1.

If D_4 and D_5 are disjoint, then there exists a γ-set D_6 such that D_6 is adjacent to D_4 and D_5. Thus, if follows that, in $D_mG^{bcd}(G)$, $d(D_4 ,D_5) = d(D_4 , D_6) + d(D_6, D_5) = 2$.

Hence, distance between any two vertices in $D_mG^{bcd}(G)$ is at most three.

Theorem 2.10: (i) If G = P_{3k}, k $\geq$ 2, then $D_mG^{bcd}(G) = K_{k+1} \cup 2kK_1$.

(ii) If G = P_{3k+1}, k $\geq$ 1, then $D_mG^{bcd}(G)$ is a 2 self-centered graph.

(iii) If G = P_{3k+2}, k $\geq$ 1, then $D_mG^{bcd}(G)$ has k+1 components, one component is bi-eccentric with radius 2 and the remaining k components are isolated vertices.

Proof: Let $V(P_n)$ = {1, 2, 3, ..., n}.

(i) Let G = P_{3k}. Then ⊠(G) = $\left\lceil \frac{n}{3} \right\rceil$, ⊠$_D$(G) = 1. D = {2, 5, 8, 11, ..., 3k−4, 3k−1} is the only ⊠-set of G with |D| = k and the remaining 2k vertices are bad vertices. The set D and its vertices form complete graph with k+1 vertices in $D_mG^{bcd}(G)$. Therefore, $D_mG^{bcd}(G)$ is disconnected, one component is a complete graph with k+1 vertices and others are 2k isolated vertices. Hence, $D_mG^{bcd}(G) = K_{k+1} \cup 2kK_1$.

(ii) Let G = P_{3k+1}. Then ⊠(G) = $\left\lceil \frac{n}{3} \right\rceil$, ⊠$_D$(G) = $(k^2+5k+2)/2$ by Theorem 1.2. Here, all vertices are good. That is, for u $\in$ V(G), there exists at least one ⊠-set containing u. By Theorem 2.9 d(u, v) = 3 for u, v $\in$ V(G) and d(u, v) = 3 for u $\in$ V(G) and v $\in$ S are not possible in $D_mG^{bcd}(G)$. Therefore, eccentricity of point vertices is 2 and eccentricity of set vertices is also 2.

(iii) Let G = P_{3k+2}. Then ⊠(G) = $\left\lceil \frac{n}{3} \right\rceil$, ⊠$_D$(G) = k+2. Here, k vertices are bad. In this case, for u, v $\in$ V(G), u $\in D_1$ and v $\in D_2$ only and D_1 and D_2 are disjoint. By Theorem 2.9, in $D_mG^{bcd}(G)$, d(u, v) = 3, $d(u, D_2) = 3 = d(v, D_1)$ and $d(D_1, D_2) = 2$. Hence, $D_mG^{bcd}(G)$ is disconnected, one component is bi-eccentric with radius 2 and the remaining k components are isolated vertices.

Theorem 2.11: If $G = K_{2p} - F$, F is a 1-factor, then $D_m G^{bcd}(G)$ is 2 self-centered.

Proof: Let $G = K_{2p} - F$, F is a 1-factor. Then any two vertices form a ⊠-set of G. Hence, for u, v ∈ V(G), d(u, v) = 1 in $D_m G^{bcd}(G)$. Suppose u ∈ V(G), v ∈ S, v = D is a ⊠-set. If u ∉ D, d(u, D) = 2 in $D_m G^{bcd}(G)$, since point vertices are adjacent in $D_m G^{bcd}(G)$. Suppose u, v ∈ S, u = D_1, v = D_2 are ⊠-sets of G. If D_1 and D_2 are disjoint, then there exists a ⊠-set D_3 such that D_3 is adjacent to both D_1 and D_2. Hence, $d(D_1, D_2) = 2$ in $D_m G^{bcd}(G)$. Therefore, eccentricity of point vertices is 2 and eccentricity of set vertices is also 2. Hence, $D_m G^{bcd}(G)$ is 2 self-centered.

Theorem 2.12: Let G be a γ-excellent graph and G has an isolated vertex. Then $D_m G^{bcd}(G)$ is bi-eccentric with radius 1.

Proof: Let G be γ-excellent and has an isolated vertex. Let w be the isolated vertex. Every γ-set of G contains w. Thus, w is adjacent to all point vertices and also all set vertices in $D_m G^{bcd}(G)$. Hence, eccentricity of w is one and set vertices are adjacent to each other in $D_m G^{bcd}(G)$.

Let u, v ∈ V'. Suppose u, v ≠ w ∈ V(G). If there exists a γ-set D such that D contains u and v, then in $D_m G^{bcd}(G)$, d(u, v) = 1.

If u ∈ D_1 and v ∈ D_2, then w is adjacent to both u and v in $D_m G^{bcd}(G)$, then d(u, v) = 2, $d(u, D_2) = d(u, w) + d(w, D_2) = 2$ in $D_m G^{bcd}(G)$.

Thus, the eccentricity of point vertices except the isolated vertex is 2 and eccentricity of set vertices 2. Hence, $D_m G^{bcd}(G)$ is bi-eccentric with radius 1.

Theorem 2.13: Let G be a disconnected graph without isolated vertices and each component of G is γ-excellent. Then $D_m G^{bcd}(G)$ is 2 self-centered.

Proof: Suppose G is disconnected without isolated vertices. Then G has at least two components G_1 and G_2. Consider the following cases:

Case (i): Let u ∈ $V(G_1)$, v ∈ $V(G_2)$. If there exists a γ-set D such that D contain u and v, then in $D_m G^{bcd}(G)$, d(u , v) =1.

Suppose x, y ∈ $V(G_1)$ such that x ∈ D_1 and y∈ D_2. If there exists a vertex z ∈ D_1 and z ∈ D_2, then it follows that, is $D_m G^{bcd}(G)$, d(x, y) = d(x, z)+d(z, y) = 2.

Case (ii): Suppose a ∈ $V(G_1)$ and b ∈ S. Then b = D_3 is a γ-set of G. If a ∈ D_3, then in $D_m G^{bcd}(G)$, d(a, b) = 1.

Suppose a ∉ D_3. If there exists a vertex c ∈ $V(G_2)$ such that c ∈ D_3.Then, in $D_m G^{bcd}(G)$, d(a, b) = d(a, c)+d(c, b) = 2.

Case (iii): Suppose $u', v' \in S$. Then $u' = D_4$ and $v' = D_5$ are two γ-sets of G. If D_4 and D_5 are not disjoint, then in $D_mG^{bcd}(G)$, $d(D_4, D_5) = 1$.

Suppose D_4 and D_5 are disjoint. Then there exists a γ-set D' such that D' is adjacent to both D_4 and D_5. Thus, it follows that, in $D_mG^{bcd}(G)$,

$d(D_4, D_5) = d(D_4, D') + (D', D_5) = 2$.

Hence eccentricity of point vertices is 2 and eccentricity of set vertices is also 2. Thus $D_mG^{bcd}(G)$ is 2-self-centered.

Theorem 2.14: If $D_1, D_2 \subseteq V(G)$ such that D_1 and D_2 are disjoint γ-sets of G and there exists $v \in D_1$(or D_2) such that there is a γ-set containing v and the elements of D_2 (or D_1), then $D_mG^{bcd}(G)$ is 2-self-centered.

Proof: Let G be a connected graph with rad(G) > 1. Suppose D_1 and D_2 are two disjoint -sets of G, and there exists $v \in D_1$(or D_2) such that there is a -set D_3 containing v and elements of D_2(or D_1). Then, in $D_mG^{bcd}(G)$, $d(v, D_1) = d(v, D_3)+d(D_3, D_1) = 2$. Let v' be the common element of D_1(orD_2) and D_3. Since v and v' are adjacent in $D_mG^{bcd}(G)$, it follows that, $d(v, D_1(orD_2)) = d(v, v')+d(v', D_1) = 2$. By case (i) and case (iii) of Theorem 2.9, eccentricity of all point vertices and set vertices are 2. Hence, $DG^{bcd}(G)$ is 2 self-centered.

Theorem 2.15: If $D_1, D_2 \subseteq V(G)$ such that D_1 and D_2 are disjoint γ-sets and there exists $v \in D_1$(or D_2) such that there is no γ-set containing v and elements of D_2(or D_1), then diameter of $D_mG^{bcd}(G)$ is three.

Proof: Let G be a connected graph with rad(G) > 1. Suppose D_1 and D_2 are -sets and there exists $v \in D_1$(or D_2) such that there is no -set containing v and elements of D_2 (or D_1).

Case (i): Suppose $u, v \in V(G)$. Then as in case (i) of Theorem 2.9, $d(u, v) \leq 3$.

Case (ii): Suppose $v \in D_1$, $v \notin D_2$. Then as in case (ii) of Theorem 2.9, $d(v, D_2) = 3$.

Case (iii): $D_1, D_2 \in S$. Then as in case (iii) of Theorem 2.9, $d(D_1, D_2) = 2$.

So, in all the cases, $diam(D_mG^{bcd}(G))$ is 3.

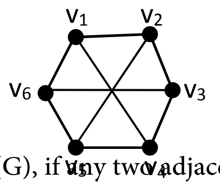
Theorem 2.16: G is an induced sub graph of $D_mG^{bcd}(G)$ if any only if $\gamma(G) = 2$ and every γ-set is connected.

Proof: Assume that $\gamma(G) = 2$ and every γ-set is connected. Suppose D is a γ-set and is connected. Then $\gamma(G) = 2$. Let $D = \{u, v\}$ be a γ-set. Since u and v are adjacent in G, by the definition of $D_mG^{bcd}(G)$, u and v are adjacent in $D_mG^{bcd}(G)$. Also, any two non-adjacent vertices of G is not adjacent in $D_mG^{bcd}(G)$. Hence G is an induced sub graph of $D_mG^{bcd}(G)$.

Conversely assume that G is an induced sub graph of $D_mG^{bcd}(G)$. Thus any two adjacent vertices of G are in a γ-set and also any two non-adjacent vertices are not in a γ-set. Hence, $\gamma(G) \geq 2$.

Suppose a γ-set D contains more than 2 vertices. Then in G, they must form a complete graph. But D is a γ-set implies every vertex of D has a private neighbour in $V-D$. Let $u \in D$ and $x \in V-D$ such that u and x are adjacent. Let $u, v, w \in D$ and x, y, z are private neighbour of u, v, w respectively. By our assumption there exist γ-set D_1 containing x, u. But y and z are private neighbour of v and w implies that v, w or y, z or v, z or y, w must be in that set, which is a contradiction. Therefore, $\gamma(G) = 2$ and every γ-set is connected.

Example 2.3: (i) $G = K_{m,n}$, m, n >2. (ii) **G**



Theorem 2.17: G is a sub graph of $D_mG^{bcd}(G)$, if any two adjacent vertices are in at least one γ-set of G.

Proof: Let G be a connected graph. Suppose any two adjacent vertices are in at least one γ-set. Let u, v be adjacent in G and D contains u, v. Then, in $D_mG^{bcd}(G)$, D is adjacent to u and v, u and v are adjacent. Thus, it follows that G is sub graph of $D_mG^{bcd}(G)$.

Theorem 2.18: If $D_mG^{bcd}(G)$ is connected, then girth of $D_mG^{bcd}(G)$ is three.

Proof: Suppose $D_mG^{bcd}(G)$ is connected. Set vertex and its corresponding point vertices are complete in $D_mG^{bcd}(G)$. Then, it follows that girth of $D_mG^{bcd}(G)$ is three.

Theorem 2.19: $D_mG^{bcd}(G)$ is complete if and only if $G = \overline{K_p}$.

Proof: Suppose $G = \overline{K_p}$, then by Theorem 2.2, $D_mG^{bcd}(G)$ is a complete graph.

Conversely, Suppose $D_mG^{bcd}(G)$ is complete. If $D_mG^{bcd}(G)$ is complete, degree of a set vertex is = (⊠$_D$(G)−1)+p. Therefore, every γ-set contains all the vertices of G. Hence, ⊠(G) = p. But ⊠(G) = p if and only if $G = \overline{K_p}$. This proves the theorem.

Theorem 2.20: $D_mG^{bcd}(G)$ is a tree if and only if $G = K_1$.

Proof: Suppose $G = K_1$. Then by the definition, $D_mG^{bcd}(G)$ is K_2. Hence $D_mG^{bcd}(G)$ is a tree.

Conversely assume that $D_mG^{bcd}(G)$ is a tree. To prove that $G = K_1$. On the contrary, assume that $G \neq K_1$.

Case (i): Suppose $\Delta(G) = p-1$. Then rad(G) = 1. From Theorem 2.1 and 2.3, $D_mG^{bcd}(G)$ is disconnected.

Case (ii): Suppose $\Delta(G) < p-1$. Then there exists at least two vertices to form a γ-set. Then $D_mG^{bcd}(G)$ contains a cycle, which is a contradiction. So, G must be K_1.

Remark 2.3: If $D_mG^{bcd}(G)$ is connected, then $D_mG^{bcd}(G)$ has no pendent vertex ($G \neq K_1$).

Theorem 2.21: If G is a (p−2) regular graph, then $DG^{bcd}(G)$ is Eulerian.

Proof: Let G be a (p−2) regular graph. Then any two vertices form a ⊠-set of G. Number of vertices in $D_mG^{bcd}(G)$ is p+p(p−1)/2. Number of γ-sets containing u $\in V(G)$ is p−1. In $D_mG^{bcd}(G)$, degree of point vertices is 2(p−1) and degree of set vertices is also 2(p−1). Thus, every vertex of $D_mG^{bcd}(G)$ has an even degree. Thus, it follows that $D_mG^{bcd}(G)$ is Eulerian.

Theorem 2.22: (i) For any graph G, ⊠($D_mG^{bcd}(G)$) $\leq \min\left\{\min \deg_{D_mG^{bcd}(G)} v_i, \gamma(G)\right\}$, $1 \leq i \leq p$.

(ii) ⊠($D_mG^{bcd}(G)$) $\leq \min\left\{\min \deg_{D_mG^{bcd}(G)} v_i, \gamma(G)\right\}$, $1 \leq i \leq p$.

Proof: (i) We consider the following two cases:

Case (i): Let v $\in$ V(G) and is of minimum degree among all the vertices of $D_mG^{bcd}(G)$. Then by deleting the vertices adjacent to v, the resulting graph is disconnected. Thus, ⊠($D_mG^{bcd}(G)$) $\leq \min\left\{\min \deg_{D_mG^{bcd}(G)} v_i\right\}$, $1 \leq i \leq p$.

Case (ii): Let S be the set of all minimum dominating set of G. Cardinality of each set is $\gamma(G)$. Suppose $\gamma(G) <$ ⊠(G). Then by deleting the vertices adjacent to any one dominating set, the resulting graph is disconnected.

Hence, ⊠($D_mG^{bcd}(G)$) $\leq \min\left\{\min \deg_{D_mG^{bcd}(G)} v_i, \gamma(G)\right\}$,

(ii) Proof is similar to (i).

Conclusion

In this paper, we have defined and studied the dominating graph $D_mG^{bcd}(G)$. Eccentricity properties of $D_mG^{bcd}(G)$ are studied. We have characterised graphs G for which $D_mG^{bcd}(G)$ is complete, a tree, radius one and connected.

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