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Eccentricity Properties of Boolean Function Graph $B(\overline{G}, \overline{K_q}, INC)$ of a Graph

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Abstract: *Let G be a simple graph with vertex set V(G) and edge set E(G).The Boolean Function Graph $B(\overline{G}, \overline{K_q}, INC)$ of G is a simple graph with vertex set V(G)∪E(G) and two vertices in $B(\overline{G}, \overline{K_q}, INC)$ are adjacent if and only if they correspond to two nonadjacent vertices of G or to a vertex and an edge incident to it in G. For simplicity, this graph is denoted by $BF_1(G)$. In this paper, eccentricity properties of $BF_1(G)$ are studied.*

Keywords: *Boolean Function Graph, eccentricity, self-centered*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. For two vertices u and v in a graph G, the distance d(u, v) from u to v is the length of a shortest u-v path in G. For a vertex in a connected graph G, the eccentricity e(v) or ecc(v) of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of G is its radius and maximum eccentricity is its diameter, which are denoted by rad(G) and diam(G), respectively. A vertex v in G is a central vertex if e(v) = rad(G) and the subgraph induced by the central vertices of G is the center Cen(G) of G. If every vertex of G is a central vertex, then Cen(G) = G and G is called self-centered. As for every graph (undirected, uniformly weighted) there exists an adjacency (0, 1) matrix, we call the general operation a Boolean operation. Boolean operation on a given graph uses the adjacency relation between two vertices or two edges and incidence relationship between vertices and edges and defines new structure from the given graph. This extracts information from the original graph and encodes it into a new structure.

Whitney [22] introduced the concept of the line graph L(G) of a given graph G in 1932. The first characterization of line graphs is due to Krausz. The Middle graph M(G) of a graph G was introduced by Hamada and Yoshimura[5]. Chikkodimath and Sampathkumar [3] also studied it independently and they called it the semi-total graph $T_1(G)$ of a graph G. Characterizations were presented for middle graphs of any graph, trees and complete graphs in [1]. The concept of total graphs was introduced by Behzad [2] in 1966. Sastry and Raju [21] introduced the concept of quasi-total graphs and they solved the

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graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. Janakiraman et al., introduced the concepts of Boolean and Boolean function graphs [7 - 20].

The Boolean function graph $B(\overline{G}, \overline{K_q}, INC)$ of G is a graph with vertex set V(G)$\cup$E(G) and two vertices in $B(\overline{G}, \overline{K_q}, INC)$ are adjacent if and only if they correspond to two nonadjacent vertices of G or to a vertex and an edge incident to it in G. For brevity, this graph is denoted by $BF_1(G)$. In this paper eccentricity properties of $BF_1(G)$ are studied. Here, G is a graph with p vertices and q edges. The definitions and details not furnished in this paper may be found in [4].

2. Main Results

In this section, eccentricities of vertices of $BF_1(G)$ are found. It is observed that diameter of $BF_1(G)$ is atmost 4. The graphs G for which $BF_1(G)$ is self-centered with radius 2 and 3 are characterized. Also, the graphs G for which $BF_1(G)$ is bi-eccentric with radius 2 and diameter 3 are characterized.

Observation:

2. 1. V $(BF_1(G))$ = V(G)$\cup$V(L(G)), where L(G) is the line graph of G.

2.2. $BF_1(G)$ is always connected.

2.3. $\overline{G}$ and $\overline{K_q}$ are induced subgraphs of $BF_1(G)$.

2.4. The number of vertices in $BF_1(G)$ is p + q and the number of edges in $BF_1(G)$ is $\frac{p(p-1)}{2} + q$.

2.5. The degree of u$\in$V(G) in $BF_1(G)$ is p-1 and degree of e$\in$E(G) in $BF_1(G)$ is 2

2.6. If p = 3, then $BF_1(G)$ is C_3, C_4 or C_5.

2.7. If p ≥ 4, then $BF_1(G)$ is bi-regular.

In the following, eccentricities of vertices in V$(BF_1(G))\cap$V(G) are found.

Theorem 2.1:

Eccentricity of the vertex of $BF_1(G)$ corresponding to a vertex of G is atmost 3.

Proof:

Let u$\in$V(G). Then u$\in$V$(BF_1(G))$

(a) Distance between any two vertices of V$(BF_1(G))\cap$V(G).

Case(1): $deg_G(u) = 0$

Then u is adjacent to all the vertices of V(G) in $BF_1(G)$.

Therefore, d(u, v) = 1 in $BF_1(G)$, for all v$\in$V(G).

Case(2): $deg_G(u) \geq 1$.

Then there exists a vertex $v \in V(G)$ adjacent to u in G.

Let $e = (u, v) \in E(G)$. Then $e \in V(BF_1(G))$ and $d_{BF1(G)}(u, v) = 2$, since uev is a geodesic path in $BF_1(G)$. From Case(i) and Case(ii), distance between any two vertices of G in $BF_1(G)$ is either 1 or 2.

(b) Distance between $u \in V(G)$ and $e \in V(L(G))$ in $BF_1(G)$

Let $e \in V(L(G))$. Then $e \in E(G)$. If e is incident with u in G, then $d_{BF1(G)}(u, e) = 1$. Let e be not incident with u in G and let $e = (v, w) \in E(G)$, where $v, w \in V(G)$ and $u \neq v, w$.

(i) If u is not adjacent to atleast one of v and w, say v, then uve is a geodesic path in $BF_1(G)$ and hence $d(u, e) = 2$ in $BF_1(G)$.

(ii) Let both uv, $vw \in E(G)$ and let $f = (u, v) \in E(G)$. Then u f v e is a geodesic path in $BF_1(G)$ and hence $d(u, e) = 3$ in $BF_1(G)$.

From (i) and (ii), $d(e, u) \leq 3$ in $BF_1(G)$.

That is, the distance between $u \in V(G)$ and $e \in V(L(G))$ in $BF_1(G)$ in $BF_1(G)$ is atmost 3.
From (a) and (b), eccentricity of a vertex of G in $BF_1(G)$ is atmost 3.

Theorem 2.2:

Let G have at least two edges. Eccentricity of a vertex of $BF_1(G)$ corresponding to an edge of G lies between 2 and 4.

Proof:

Let $e \in E(G)$. Then $e \in V(BF_1(G))$. From (b) of Theorem 2.1., the distance between e and a vertex of G in $BF_1(G)$ is atmost 3.

Let $e_1, e_2 \in E(G)$. In the following, distance between two vertices in $BF_1(G)$ corresponding to e_1 and e_2 is found.

Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$, where $u_1, v_1, u_2, v_2 \in V(G)$.

Case(i): e_1 and e_2 are adjacent in G

If u_2 is the vertex common to both e_1 and e_2, then $e_1u_2e_2$ is a geodesic path in $BF_1(G)$ and hence $d_{BF1(G)}(e_1, e_2) = 2$.

Case(ii): e_1 and e_2 are nonadjacent in G.

(a) If one of u_i is nonadjacent to v_i in G, say $(u_1,v_1) \in E(\overline{G})$, then $e_1u_1 v_1e_2$ is a geodesic path in $BF_1(G)$ and hence $d_{BF1(G)}(e_1, e_2) = 3$.

(b) Let each u_i be adjacent to each v_i $(i = 1, 2)$. Then (u_1,v_1), (u_1,v_2), (u_2,v_1), $(u_2,v_2) \in E(G)$.

Let $x = (u_1, v_1)$. Then $x \in V(BF_1(G))$. Then e_1 u_1 x v_1e_2 is a geodesic path in $BF_1(G)$ and hence $d_{BF1(G)}(e_1, e_2) = 4$. From the above, it can be concluded that eccentricity of a vertex in $BF_1(G)$ corresponding to an edge of G lies between 2 and 4.

In the following, the graphs G for which radius of $BF_1(G)$ is 1 are characterized.

Theorem 2.3:

For any graph G, radius of $BF_1(G)$ is 1 if and only if $G \cong K_2$ or nK_1, $n \geq 2$.

Proof:

If $G \cong K_2$, then the edge $e \in E(K_2)$ has eccentricity 1 in $BF_1(G)$. If $G \cong nK_1$, $n \geq 2$, then each vertex of G has eccentricity 1 in $BF_1(G)$. Therefore, radius of $BF_1(G)$ is 1.

Conversely, assume radius of $BF_1(G)$ is 1. Therefore, there exists a vertex in $BF_1(G)$ of degree $p + q - 1$.

Degree of a vertex in $V(BF_1(G)) \cap V(G)$ is $p - 1$. If $p - 1 = p + q - 1$, then $q = 0$. Therefore, $G \cong nK_1$, $n \geq 2$. Similarly, degree of a vertex in $V(BF_1(G)) \cap V(L(G))$ of $BF_1(G)$ is 2. Therefore, if $2 = p+q-1$, then $p+q = 3$, $q \geq 1$. Therefore, $p = 2$ and $q = 1$ and hence $G \cong K_2$.

Remarks 2.1:

1. $BF_1(G)$ is self-centered with radius 1 if and only if $G \cong nK_1$, $n \geq 2$.
2. For any graph G, $BF_1(G)$ is bi-eccentric with radius 1 if and only if $G \cong K_2$.

In the following, the graphs G for which $BF_1(G)$ is self-centered with radius 2 are characterized.

Theorem 2.4:

For any graph G, $BF_1(G)$ is self-centered with radius 2 if and only if $G \cong K_{1,n} \cup mK_1$, $n \geq 2$, $m \geq 0$ or $G \cong K_2 \cup mK_1$, $m \geq 1$.

Proof:

Assume $BF_1(G)$ is self-centered with radius 2. Let $v \in V(G)$. Then $v \in V(BF_1(G))$. Eccentricity of v is 2 in $BF_1(G)$, if either

(i) all the vertices of G are adjacent to v (or)

(ii) there exists an edge $e = (u, w)$ in $E(G)$ such that $<\{u, v, w\}> \cong P_3$ or $K_2 \cup K_1$, where $u, w \in V(G)$

Let $e \in E(G)$. Then eccentricity of e in $BF_1(G)$ is 2, if

(iii) all the edges of G are adjacent to e in G.

Therefore, eccentricity of each vertex in $BF_1(G)$ is 2 if [(i) or (ii)] and (iii) hold.

That is , $G \cong K_{1,n} \cup mK_1$, $n \geq 2$, $m \geq 0$ or $G \cong K_2 \cup mK_1$, $m \geq 1$.

Conversely, if $G \cong K_{1,n} \cup mK_1$, $n \geq 2$, $m \geq 0$ or $G \cong K_2 \cup mK_1$, $m \geq 1$, then $BF_1(G)$ is self-centered with radius 2.

In the following, the graphs for which $BF_1(G)$ is self-centered with radius 3 are characterized.

Theorem 2.5:

Let G be a graph with atleast three vertices such that G is not isomorphic to $K_{1,n}\cup mK_1$, $n\geq 1$ and $m\geq 0$ and K_4 is not an induced subgraph of G. Then $BF_1(G)$ is self-centered with radius 3 if and only if each vertex of G lies on a triangle

Proof:

Let $BF_1(G)$ be Self-centered with radius 3. Then each vertex of $BF_1(G)$ has eccentricity 3. Let $v\in V(G)$. Then eccentricity of v in $BF_1(G)$ is 3. But, distance between v and a vertex in $V(BF_1(G))\cap V(G)$ is less than or equal to 2. Also, d(v, e) = 3 in $BF_1(G)$, where $e\in V(BF_1(G))\cap V(L(G))$, if <{v, e}> forms a triangle in G. Therefore, v lies on a triangle in G.

Conversely, assume each vertex of G lies on a triangle in G. Therefore G contains no isolated vertices and radius of $BF_1(G)$ is atleast 2. Let $v_1,v_2\in V(BF_1(G))\cap V(G)$.

Then $d_{BF1(G)}(v_1, v_2)\leq 2$.

Let $v\in V(G)$ and $e = (u, w)\in E(G)$ such that <{u, v, w}> forms a triangle in G. Let $e_1 = (u, v)$. Then v, u, w, e, $e_1\in V(BF_1(G))$ and v e_1 u e is a geodesic path in $BF_1(G)$. Therefore, e(v) = 3 in $BF_1(G)$.

Let e_1, $e_2\in E(G)$. Since K_4 is not an induced subgraph of G, $d(e_1, e_2)\leq 3$ in $BF_1(G)$. But since d(v, e) = 3 in $BF_1(G)$, eccentricity of e is 3 in $BF_1(G)$, for any $e\in V(BF_1(G))\cap V(L(G))$. Therefore $BF_1(G)$ is self-centered with radius 3.

Remark 2.2:

Let G be a graph with atleast four vertices and let G be not isomorphic to $K_{1,n}\cup mK_1$, $n\geq 1$ and $m\geq 0$ and K_4 be not an induced subgraph of G. Then $BF_1(G)$ is bi-eccentric with radius 2 and diameter 3 if and only if there exists atleast one vertex in G not lying on any triangle G.

Theorem 2.6:

Let G be a graph with at least four vertices and let G be not isomorphic to $K_{1,n}\cup mK_1$, $n\geq 1$ and $m\geq 0$. Then $BF_1(G)$ is bi-eccentric with radius 3 and diameter 4 if and only if K_4 is an induced subgraph of G and each vertex of G lies on a triangle in G.

Proof:

Assume K_4 is an induced subgraph of G and each vertex of G lies on a triangle in G. Then vertices of G in $BF_1(G)$ have eccentricity 3. Let e_1, e_2 be two independent edges in K_4. Then $e_1, e_2\in V(BF_1(G))$ and $d(e_1, e_2) = 4$ in $BF_1(G)$. That is, vertices in $BF_1(G)$ corresponding to the edges of G have eccentricity 4. The remaining vertices in $V(BF_1(G))\cap V(L(G))$ have eccentricity less than or equal to 4. Therefore $BF_1(G)$ is bi-eccentric with radius 3 and diameter 4.

Conversely, assume $BF_1(G)$ is bi-eccentric with radius 3 and diameter 4. If there exists a vertex, say v in G not lying in a triangle, then eccentricity of v in $BF_1(G)$ is 2. Similarly, if K_4 is not an induced subgraph of G, then there exists no vertex in $BF_1(G)$ having eccentricity 4. Therefore K_4 is an induced subgraph of G

Remark 2.3:

Let G be a graph with at least four vertices and let G be not isomorphic to $K_{1,n} \cup mK_1$, $n \geq 1$ and $m \geq 0$. Then $BF_1(G)$ has radius 2 and diameter 4 if and only if K_4 is an induced subgraph of G and there exists a vertex not lying on a triangle in G.

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