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On Eccentric Domination in Graphs

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Abstract: *A set $D \subseteq V(G)$ is an eccentric dominating set if D is a dominating set of G and for every $v \in V-D$, there exists at least one eccentric vertex of v in D. The minimum cardinality of eccentric dominating set is called the eccentric domination number and is denoted by $\gamma_{ed}(G)$. In this paper, we have provided some new bounds for $\gamma_{ed}(G)$ and established the relation between $\gamma_{ed}(G)$, $\alpha_0(G)$ and $\beta_0(G)$. We have also characterized graphs for which $\gamma_{ed}(G) = p-1$ and $p-2$.*

Keywords: *eccentric dominating set, minimum eccentric dominating set.*

1. Introduction

Graphs discussed in this paper are undirected and simple. For graph theoretic terminology refer to Harary[5], Buckley and Harary[3]. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. A graph with p vertices and q edges is called a (p, q) graph.

The concept of distance in graph plays a dominant role in the study of structural properties of graphs in various angles using related concept of eccentricity of vertices in graphs. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, The *eccentricity*e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$.The *radius* rad(G) is the minimum eccentricity of the vertices, whereas the *diameter* diam(G) is the maximum eccentricity.

A vertex cover of a graph G is a set of vertices that covers all the edges. The vertex covering number $\alpha_0(G)$ of G is minimum cardinality of a vertex cover.

A set S of vertices of G is independent if no two vertices in S are adjacent. The independent number $\beta_0(G)$ of G is the maximum cardinality of an independent set.

The concept of domination in graphs is originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters.

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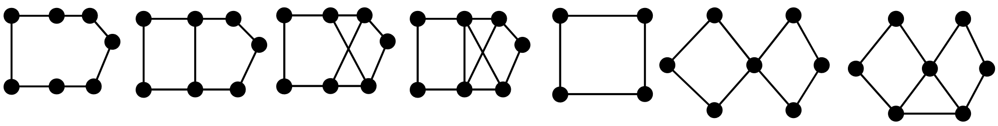
A set $D \subseteq V$ is said to be a *dominating set* in G, if every vertex in $V-D$ is adjacent to some vertex in D. The cardinality of minimum dominating set is called the*domination number* and is denoted by $\gamma(G)$. For details on $\gamma(G)$, refer to [4].

Janakiraman, Bhanumathi and Muthammai [2, 6] introduced and studied the concept of eccentric dominating set. In [1], they have studied the eccentric domination in trees. A set $D \subseteq V(G)$ is an *eccentric dominating set* if D is a dominating set of G and for every $v \in V-D$, there exists at least one eccentric vertex of v in D. The minimum cardinality of an eccentric dominating set is called the *eccentric domination number* and is denoted by $\gamma_{ed}(G)$. An eccentric dominating set with cardinality $\gamma_{ed}(G)$ is known as *minimum eccentric dominating set.* An eccentric dominating set D is a *minimal eccentric dominating set* if no proper subset $D'' \subseteq D$ is an eccentric dominating set.

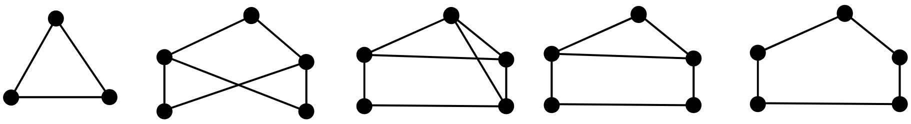
The following results are needed to study the eccentric dominating set of a graph G.

Theorem: 1.1[5] **:**For any graph G with even order n and no isolated vertices, $\gamma(G) = n/2$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.

Theorem: 1.2[5] **:**If G is a connected graph with order n and$\delta(G) \geq 2$ and $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$, then $G \in A \cup B$.



Graphs in family A.



Graphs in family B.

Figure 1.1

Theorem: 1.3 [5] **:**$\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

Theorem: 1.4[5]**:**$\gamma(C_n) = \begin{cases} \frac{n}{3}, & if\ n=3k. \\ \left\lceil \frac{n}{3} \right\rceil, & if\ n=3k+1 or\ 3k+2. \end{cases}$

Theorem: 1.5[7]:$\gamma_{ed}(K_n) = 1$.

Theorem: 1.6[7]:$\gamma_{ed}(K_{m,\ n}) = 2$.

Theorem: 1.7[7]:$\gamma_{ed}(P_n) = \gamma(P_n)$ or $\gamma(P_n) + 1$.

Theorem: 1.8[7]: (i) $\gamma_{ed}(C_n) = n/2$ if n is even.

$$\text{(ii) } \gamma_{ed}(C_n) = \begin{cases} \frac{n}{3} & \text{if } n = 3m \text{ and is odd.} \\ \left\lceil \frac{n}{3} \right\rceil & \text{if } n = 3m+1 \text{ and is odd.} \\ \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n = 3m+2 \text{ and is odd.} \end{cases}$$

Theorem: 1.9[3]:If G is a connected graph with n vertices then $\gamma_{ed}(G) \leq \left\lfloor \frac{2n}{3} \right\rfloor$.

Theorem: 1.10[7]:$\gamma_{ed}(\overline{C_4}) = 2$, $\gamma_{ed}(\overline{C_5}) = 3$ and$\gamma_{ed}(\overline{C_n}) = \left\lceil \frac{n}{3} \right\rceil$, $n \geq 6$.

2. Some New Results onEccentric Domination in Graphs

Let G be a (p, q) graph. First, we shall find the relation between $\gamma_{ed}(G)$, $\alpha_0(G)$ and $\beta_0(G)$.

Lemma: 2.1: Let G be a connected graph with rad(G) = 1 and diam(G) = 2. Then $\gamma_{ed}(G) \leq \frac{p - \beta_0 + 3}{2} = \frac{\alpha_0 + 3}{2}$.

Proof: If G is connected, then $\gamma(G) \leq \beta_0(G)$. Any maximum independent set dominate the graph G. Let S be a maximum independent set.Then S is a dominating set and V−S is also a dominating set.

Let G be a connected graph with rad(G) = 1 and diam(G) = 2. e(u) = 2 whenever u ∈ S. If v ∈ V−S, then it is adjacent to atleast one element of S. Suppose v is adjacent to every element of S and if e(v) = 1, then u ∈ S is an eccentric vertex of v. If e(v) = 2 and is adjacent to every element of S, then eccentric vertex of v is in V−S. Hence, at most $\frac{|V - S|}{2}$ vertices from V−S are needed to dominate G eccentrically. Thus, $\gamma_{ed}(G) \leq \beta_0 + \frac{p - \beta_0}{2} = \frac{p + \beta_0}{2}$.

S is a dominating set. Let v∈ V−S such that e(v) = 1. v dominates G. Since S is independent, any w ∈ S is eccentric to all the vertices of S. Consider the remaining

$p-1-\beta_0$ vertices of V−S. They may have eccentric vertices in V−S itself. Let D be a subset of V−S, which contains eccentric vertices of elements of V−S. Then {u, w} ∪ D is an eccentric dominating set of G. Thus, $\gamma_{ed}(G) \leq 2+\frac{p-1-\beta_0}{2}$. That is $\gamma_{ed}(G) \leq \frac{p-\beta_0+3}{2}$.

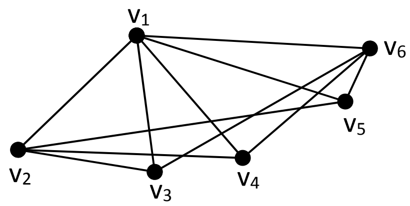
V−S is also a dominating set for G and w ∈ S is eccentric to all vertices of S. Hence, (V−S) ∪ {w} is an eccentric dominating set of G. Thus, $\gamma_{ed}(G) \leq p-\beta_0+1$.

So, $\gamma_{ed}(G) \leq \min\left\{\frac{p+\beta_0}{2}, p-\beta_0+1, \frac{p-\beta_0+3}{2}\right\}$.

Hence, $\gamma_{ed}(G) \leq \frac{p-\beta_0+3}{2} = \frac{\alpha_0+3}{2}$.

This bound is sharp, for the following graph G.

Example: 2.1:



G

Figure 2.1

$D_1 = \{v_1, v_2, v_4\}$ is a minimum eccentric dominating set of G.

$D_2 = \{v_3, v_4, v_5\}$ is an independent set of G. Here, $\gamma_{ed}(G) = 3$, $\beta_0(G) = 3$, $p = 6$.

Lemma: 2.2: Let G be a 2-self-centered graph. Then $\gamma_{ed}(G) \leq \alpha_0(G)$.

Proof: Let S be a maximum independent set. D = V−S dominates all the vertices of G and u ∈ S is eccentric to all other vertices of S. Hence, $\gamma_{ed}(G) \leq |D|+1 = p-\beta_0+1 = \alpha_0+1$.

Case (i): If V−S is also independent, then G is a bipartite graph. Also, G is 2-self-centered. Hence, G is a complete bipartite graph. Therefore, $\gamma_{ed}(G) = 2$.

Case (ii): If V−S is not independent, there exists u, v ∈ V−S such that uv ∈ E(G) and every vertex in S is adjacent to at least two vertices of V−S. Hence, $\gamma(G) \leq |D|-1$. This implies that $\gamma_{ed}(G) \leq |D|-1+1 = |D|$. Therefore, $\gamma_{ed}(G) \leq p-\beta_0 = \alpha_0$. Hence, in all the cases, $\gamma_{ed}(G) \leq \alpha_0$, where G is 2 self-centered.

Example: 2.2:

Let G = C_5. Let v_1 v_2 v_3 v_4 v_5 v_1 represent the cycle C_5. D_1 = {v_1, v_3, v_4} is a minimum eccentric dominating set and also a vertex covering of G. $\gamma_{ed}(G) = \alpha_0(G) = 3$.

Lemma: 2.3: Let G be a graph with diam(G) > 2. Then $\gamma_{ed}(G) \leq \min\left\{\frac{p+\alpha_0}{2}, \frac{p+\beta_0}{2}\right\}$.

Proof: Let G be a graph with diam(G) > 2. Let S be a maximum independent set, S is a dominating set and V−S is also a dominating set. Thus, $\gamma_{ed}(G) \leq \beta_0 + \frac{p-\beta_0}{2}$ and $\gamma_{ed}(G)$ $\leq (p-\beta_0)+(\beta_0/2) = \frac{2p-\beta_0}{2} = \frac{p+(p-\beta_0)}{2} = \frac{p+\alpha_0}{2}$.

Therefore, $\gamma_{ed}(G) \leq \frac{p+\beta_0}{2}$ and $\gamma_{ed}(G) \leq \frac{p+\alpha_o}{2}$.

Hence, $\gamma_{ed}(G) \leq \min\left\{\frac{p+\alpha_o}{2}, \frac{p+\beta_0}{2}\right\}$.

Example: 2.3:

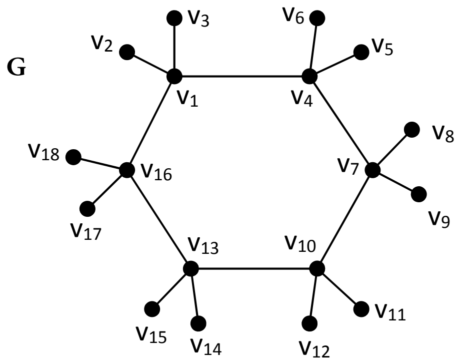


Figure 2.2

D_1 = {v_2, v_3, v_5, v_6, v_8, v_9, v_{11}, v_{12}, v_{14}, v_{15}, v_{17}, v_{18}} is a minimum eccentric dominating set and also an independent set of G. $\gamma_{ed}(G) = \beta_0(G) = 12 = (p+\alpha_0)/2$.

D_2 = {v_1, v_4, v_7, v_{10}, v_{13}, v_{16}} is a vertex covering of G. $\alpha_0(G) = 6$.

From Lemmas 2.1, 2.2 and 2.3, we have the following theorem:

Theorem: 2.1: For any connected graph G, $\gamma_{ed}(G) \leq \min\left\{\frac{p+\alpha_o}{2}, \frac{p+\beta_0}{2}\right\}$.

Bounds of $\gamma_{ed}(G)$ in terms of number of vertices of G is given in the following lemmas.

Lemma: 2.4: Let G be a graph with radius one and diameter two. Then, $\gamma_{ed}(G) \leq p/2$.

Proof: Let G be a graph with radius one and diameter two. Consider the following cases:

Case (i): G has a pendent vertex

Then clearly $\gamma_{ed}(G) = 2$. Thus if $p \geq 4$, then $\gamma_{ed}(G) \leq p/2$.

Case (ii): G has no pendent vertex

Let u be a vertex of minimum degree. Then deg u = $\delta(G) \leq p-2$ and e(u) = 2. In this case $\gamma_{ed}(G) \leq ((\delta(G)-t)/2)+1 \leq ((p-2-t)/2)+1 \leq (p-1)/2$, where t is the number of vertices with eccentricity 1.

Lemma: 2.5: Let G be a self-centered graph of diameter 2 with $p \geq 6$. Then, $\gamma_{ed}(G) \leq p/2$.

Proof: Since G is a self-centered graph with diameter 2, degree of any vertex of G is less than or equal to $p-2$. If degree of every vertex of G is equal to $p-2$, then G = K_p-1 factor and $\gamma_{ed}(G) = p/2$. Hence assume that $\delta(G) \leq p-3$.

Case (i): there exists u ∈ V(G) such that $\delta(G)$ = deg u = p−3.

Since, deg u = $p-3$ there are exactly two vertices w_1, w_2 in $N_2(u)$. Also, degree of those vertices is $\geq p-3$.

Sub case (i): w_1, w_2 in $N_2(u)$ are adjacent.

deg w_1 = $p-3$ or $p-2$. Therefore, w_1 is adjacent to at least $p-4$ vertices of $N_1(u)$. Similarly, w_2 is adjacent to at least $p-4$ vertices of $N_1(u)$. Hence, w_1 and w_2 are non-adjacent to at most one vertex of $N_1(u)$. Thus u with $(p-3)/2$ vertices of $N_1(u)$ form an eccentric dominating set. Thus, $\gamma_{ed}(G) \leq 1+(p-3)/2 = (p-1)/2$.

Sub Case (ii): w_1, w_2 in $N_2(u)$ are non adjacent.

In this case, w_1 and w_2 are adjacent to all the $p-3$ vertices of $N_1(u)$. Hence, again $\gamma_{ed}(G) \leq 1+(p-3)/2 = (p-1)/2$.

Case (ii): there exists u ∈ V(G) such that $\delta(G)$ = deg u < p−3.

Let deg u = $k < p-3$.

Sub Case (i): Suppose k = 2.

Since G is 2 self-centered, any two vertices of G lie on a common cycle of maximum length four or five. Hence, $w \in N_2(u)$ is adjacent to at least one vertex in $N_1(u)$. Thus, exactly three vertices $\{u\} \cup N_1(u)$ dominate G eccentrically. Hence, $\gamma_{ed}(G) \leq 3 \leq p/2$ when $p \geq 6$.

Sub case (ii): Suppose k > 2.

Again, since G is 2 self-centered, any two vertices of G lie on a common cycle of maximum length four or five. Also a vertex and its eccentric vertex lie on a common cycle of length 4 or 5. Let x and y be any two adjacent vertices of u. These three vertices lie on a common cycle of length four or five. If u and x (or y) lie on a cycle of length four, then {u, x} dominates the vertices of this cycle eccentrically, otherwise {u, x, y} dominates the vertices of this cycle eccentrically. Hence, if there are more cycles, additionally at most one (in the case of C_4) or two (in the case C_5) vertices are needed to dominate the vertices of this cycle. So, in general a γ_{ed}-set of G contains at most p/2 vertices. Therefore, $\gamma_{ed}(G) \leq p/2$.

In a similar way, we can prove that $\gamma_{ed}(G) \leq p/2$, if G is any k self-centered graph. Thus, we have the following theorem:

Theorem: 2.2: Let G be a self-centered graph with $p \geq 6$. Then, $\gamma_{ed}(G) \leq p/2$.

Theorem: 2.3: Let G be a cubic graph on 6 vertices. Then $\gamma(G) = \gamma_{ed}(G) = 2$.

Proof: Let $u \in V(G)$. $N_1(u)$ contains three vertices and $N_2(u)$ contains 2 vertices since deg u = 3 for all $u \in V(G)$. Also,$\langle N_1(u) \rangle$ is not complete, since if it is complete, $v \in N_1(u)$ has no adjacent vertices in $N_2(u)$ for all $v \in N_1(u)$, which is a contradiction.

Now, let $N_2(u) = \{w_1, w_2\}$.

Case (i):$\langle N_2(u) \rangle$ is independent.

w_1 is not adjacent to w_2. Therefore w_1 is adjacent to all vertices of $N_1(u)$ and is eccentric to u and w_2. Similarly, w_2 is adjacent to all of $N_1(u)$ and is eccentric to u and w_1. Hence, elements in $N_1(u)$ has eccentric vertices in $N_1(u)$ only and for $v \in N_1(u)$, v is adjacent to w_1 and w_2 and v is adjacent to u. Hence, deg v in $N_1(u) = 3-2-1 = 0$. But $N_1(u)$ contains 3 elements. Therefore, v is not adjacent to exactly two elements of $N_1(u)$. Hence, exactly one vertex $w \in N_1(u)$ is an eccentric vertex of elements of $N_1(u)$. Hence, {u, w} is an eccentric dominating set of G. Therefore, $\gamma_{ed}(G) = 2$.

Case (ii):$\langle N_2(u) \rangle = K_2$.

w_1 is adjacent to w_2. w_1 is adjacent to exactly 2 vertices of $N_1(u)$and $w_2$is adjacent to exactly 2 vertices of $N_1(u)$. Hence, a vertex w of $N_1(u)$is adjacent to w_1 and w_2 and has eccentric vertices in $N_1(u)$ only, and for $v \in N_1(u)$, v is adjacent to one of w_1 or w_2 and is adjacent to u. Hence, degree of v in $N_1(u)$ is $3-1-1 = 1$. $N_1(u)$ contains 3 elements.
Therefore, v is not adjacent to exactly one element w of $N_1(u)$. Hence, {u, w} is an eccentric dominating set of G. Therefore, $\gamma_{ed}(G) = 2$.

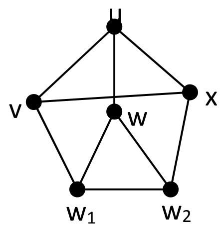


Figure 2.3

Remark: 2.1: Let G be a (p—3) regular graph with p > 5. Then, by Theorem 1.10, $\gamma_{ed}(G) \leq$ p/2, since in this case G = $\overline{C_n}$.

Theorem: 2.4: Let G be a connected p—4 regular graph. Then $\gamma_{ed}(G) \leq p/2$ for $p \geq 6$.

Proof: G is a p—4 regular graph. Therefore p is even.

When p = 6,G = C_6 and $\gamma_{ed}(G) = 3 = p/2$.

Let u $\in$ V(G). Since G is (p—4) regular, $N_1(u)$ contains (p—4) vertices and $N_2(u)$ contains exactly three vertices. (If $N_2(u)$ contains two vertices then $N_3(u)$ contains one vertex, whose degree is one or two, a contradiction). Hence,G is two self-centered, when p $\geq$ 8.

Now, we claim that **<N_1(u)> is not complete.**

Suppose, <$N_1(u)$> is complete, <$N_1(u)$> = K_{p-4} and if v $\in$ $N_1(u)$ then v is adjacent to u. This implies that, deg v $\geq$ p—5+1 = p—4. But deg v = p—4 implies that v has no adjacent vertices in $N_2(u)$. This is true for all v $\in$ $N_1(u)$, which is a contradiction, since G is connected. Hence, <$N_1(u)$> is not complete.

Let $N_2(u) = \{w_1, w_2, w_3\}$. Since deg w_1 = p—4, it is adjacent to all vertices of $N_1(u)$; or is adjacent to w_2 or w_3 and any p—5 vertices of $N_1(u)$; or is adjacent to both w_2 and w_3 and any p—6 vertices of $N_1(u)$.

Case (i):<N_2(u)> is independent.

w_1, w_2 and w_3 are pair wise disjoint. Therefore, w_1 is adjacent to all vertices of $N_1(u)$ and is eccentric to u, w_2 and w_3. Similarly, $w_2(w_3)$ is adjacent to all of $N_1(u)$ and is eccentric to u, w_1 and $w_3(w_2)$. Hence, elements in $N_1(u)$ has eccentric points in $N_1(u)$ only and for v $\in$ $N_1(u)$, v is adjacent to w_1, w_2 and w_3 and v is adjacent to u. Hence, degree of v in $N_1(u)$ = p—4—3—1 = p—8. But $N_1(u)$ contains p—4 elements. Hence, v is not adjacent to exactly three elements of $N_1(u)$.Therefore, at most (p—4)/2 vertices from $N_1(u)$ is needed to dominate G eccentrically. {u} $\cup$ S, where S $\subseteq$ $N_1(u)$ containing (p—4)/2 such vertices is an eccentric dominating set. Hence, $\gamma_{ed}(G) \leq 1+(p-4)/2 \leq p/2$.

Case (ii):<N_2(u)> = $K_2 \cup K_1$. w_1 and w_2 are adjacent.

In this case, w_1 and w_2 are adjacent to exactly p—5 vertices of $N_1(u)$ and is eccentric to u and w_3 is adjacent to all vertices of $N_1(u)$ and is eccentric to u, w_1 and w_2. Hence, elements in $N_1(u)$ has eccentric vertices in $N_1(u)$. Let v $\in$ $N_1(u)$.Consider the following sub cases:

Sub Case (i):v is adjacent to w_1 and w_2 and v is adjacent to u.

In this case, degree of v in $N_1(u)$ is p—4—1—3 = p—8. But $N_1(u)$ contains p—4 elements. Therefore, v is not adjacent to exactly three elements in $N_1(u)$.

Sub Case (ii):v is adjacent to w_1 or w_2 and v is adjacent to u.

In this case, degree of v in $N_1(u)$ is p—4—1—1—1 = p—7. But $N_1(u)$ contains p—4 elements. Therefore, v is not adjacent to exactly two elements in $N_1(u)$. Therefore, at most (p—4)/2 vertices from $N_1(u)$ is needed to dominate G eccentrically. $\{u\} \cup S$, where $S \subseteq N_1(u)$ containing (p—4)/2 such vertices is an eccentric dominating set. Hence, $\gamma_{ed}(G) \leq 1+(p-4)/2 \leq p/2$.

Case (iii):$<N_2(u)> = K_3$. w_1, w_2 and $w_3$are adjacent to each other.

$<N_2(u)>$ is complete. Therefore, w_1, w_2 and w_3 are adjacent to exactly p—6 vertices of $N_1(u)$ and is eccentric to u. Let $v \in N_1(u)$, consider the following sub cases:

Sub Case (i): v is adjacent to w_1, w_2 and v is adjacent to u.

In this case, degree of v in $N_1(u)$ is p—4—2—1 = p—7. Therefore, v is not adjacent to exactly two vertices in $N_1(u)$.

Sub Case (ii): v is adjacent to w_1 and v is adjacent to u.

In this case, degree of v in $N_1(u)$ is p—4—1—1 = p—6. Therefore, v is not adjacent to exactly one vertex in $N_1(u)$.

Sub Case (iii): v is adjacent to all the three vertices of $N_2(u)$ and v is adjacent to u.

In this case, degree of v in $N_1(u)$ is p—4—3—1 = p—8. Therefore, v is not adjacent to exactly three vertices in $N_1(u)$.

Therefore, at most (p—4)/2 vertices from $N_1(u)$ is needed to dominate G eccentrically. $\{u\}\cup S\cup\{w_1\}$, where $S \subseteq N_1(u)$ containing (p—4)/2 such vertices is an eccentric dominating set. Hence, $\gamma_{ed}(G) \leq 1+1+(p-4)/2 = p/2$.

Case (iv):$<N_2(u)> = K_{1,2}$. $w_2$is adjacent to w_1 and w_3.

In this case, w_1 and w_3 are adjacent to exactly p—5 vertices of $N_1(u)$ and is eccentric to u, w_2 is adjacent to exactly p—6 vertices of $N_1(u)$ and is eccentric to u. Let $v \in N_1(u)$. Consider the following sub cases:

Sub Case (i): v is adjacent to w_1 or w_2 and v is adjacent to u.

In this case, degree of v in $N_1(u)$ is p—4—1—1—1 = p—7. But $N_1(u)$ contains p—4 vertices. Hence, v is not adjacent to exactly two elements of $N_1(u)$.

Sub Case (ii): v is adjacent to w_1 and w_2 and v is adjacent to u.

In this case, degree of v in $N_1(u)$ is p—4—2—1 = p—7. Therefore, v is not adjacent to exactly two vertices in $N_1(u)$.

Sub Case (iii): v is adjacent to all the three vertices of $N_2(u)$.

In this case, degree of v in $N_1(u)$ is p—4—3—1 = p—8. Therefore, v is not adjacent to exactly three vertices in $N_1(u)$.

Hence, at most (p—4)/2 vertices from $N_1(u)$ is needed to dominate G eccentrically. $\{u\} \cup S \cup \{w_2\}$, where $S \subseteq N_1(u)$ containing (p—4)/2 such vertices is an eccentric dominating set. Hence, $\gamma_{ed}(G) \leq 1+1+(p-4)/2 = p/2$

So, in all the cases, $\gamma_{ed}(G) \leq p/2$. Hence the theorem follows.

Remark: Theorem 2.4 can also be proved using Lemma: 2.5

Corollary: 2.4: If G is a connected 4-regular graph on 8 vertices, then $\gamma_{ed}(G) \leq 4$.

In the following theorems, we have characterized graphs for which $\gamma_{ed}(G) = p-1$, $p-2$, where p is the number of vertices of G.

Theorem: 2.5: Let G be a connected graph. Then $\gamma_{ed}(G) = p-1$ if and only if $G \cong K_2$ or $K_{1,2.}$

Proof: If $G = K_2$, then $\gamma_{ed}(G) = 1$. Hence, $\gamma_{ed}(G) = p-1$. If $G = K_{1,2}$, then $\gamma_{ed}(G) = 2$. Hence, $\gamma_{ed}(G) = p-1$.

Conversely, assume that, $\gamma_{ed}(G) = p-1$. Then there exists an eccentric dominating set D containing p—1 vertices. By Theorem 1.9, $\gamma_{ed}(G) \leq (2p)/3$. Therefore, we get, $p-1 \leq (2p)/3$. That is $3p-3 \leq 2p$. Therefore, $p \leq 3$ and it follows that, $G \cong K_2$ or $K_{1,2}$.

Theorem: 2.6: Let G be a connected graph. Then $\gamma_{ed}(G) = p-2$ if and only if G is any one of the following graphs: K_3, C_4, P_5, C_5, K_4-e, $K_{1,3}$, $K_1+K_1+K_2$, $K_1+K_1+K_1+2K_1$, Bull graph, Fan graph(F_4), $\overline{K_2}+K_1+K_1+\overline{K_2}$.

Proof: For all graphs in the theorem, $\gamma_{ed}(G) = p-2$.

Conversely, let G be a connected graph with $\gamma_{ed}(G) = p-2$. By Theorem 1.9, $\gamma_{ed}(G) \leq (2p)/3$.

Thus, we get $p-2 \leq (2p)/3$. That is $3p-6 \leq 2p$. Therefore, $p \leq 6$, and it follows that, G is one of the graph in the stated theorem.

Next, necessary condition for $\gamma_{ed}(G) = p-k$, where k = 1, 2, 3, ... is given.

Theorem: 2.7: Let G be a connected graph. If $\gamma_{ed}(G) = p-k$, where k = 1, 2, 3, ..., then $p \leq 3k$.

Proof: By Theorem 1.9, $\gamma_{ed}(G) \leq (2p)/3$. Thus, we get $p-k \leq (2p)/3$. That is $3p-3k \leq 2p$. This implies that $p \leq 3k$.

Theorem: 2.8: Let G be a graph with $p \leq 6$ vertices. Then $\gamma_{ed}(G) = p-3$ if and only if G is any one of the graphs given in Figure 2.4 a and Figure 2.4 b.

Proof: Let G be a connected graph with $\gamma_{ed}(G) = p-3$. By Theorem 1.9, $\gamma_{ed}(G) \leq (2p)/3$. Thus, we get, $p-3 \leq (2p)/3$. That is $p \leq 9$. When $p \leq 6$, the graphs given in Figure 2.4 a and Figure 2.4 b are the only graphs with $\gamma_{ed}(G) = p-3 = 3$

In the following theorem, the upper bound of $\gamma_{ed}(G)$ is obtained in terms of order and size of G.

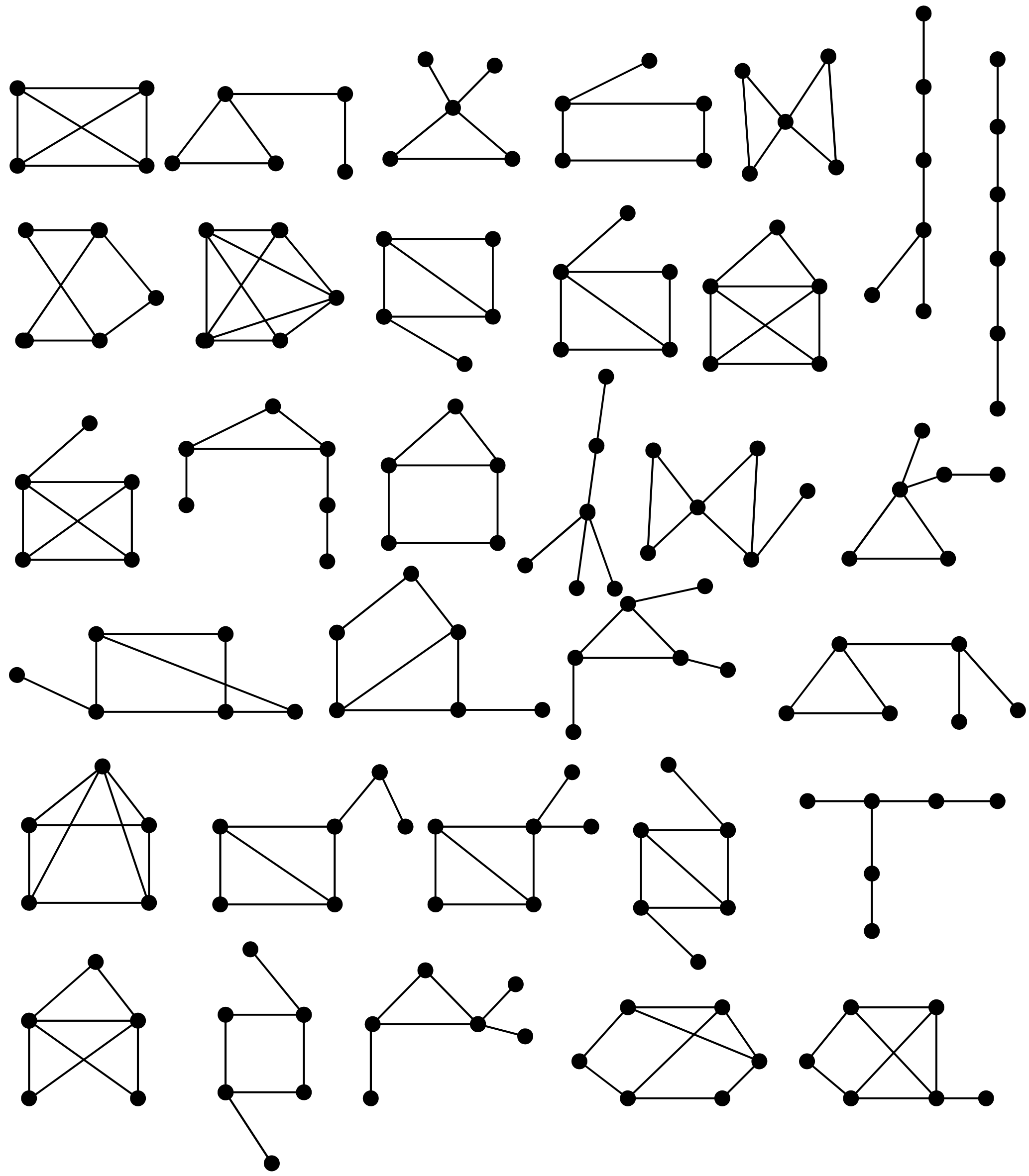


Figure 2.4 a

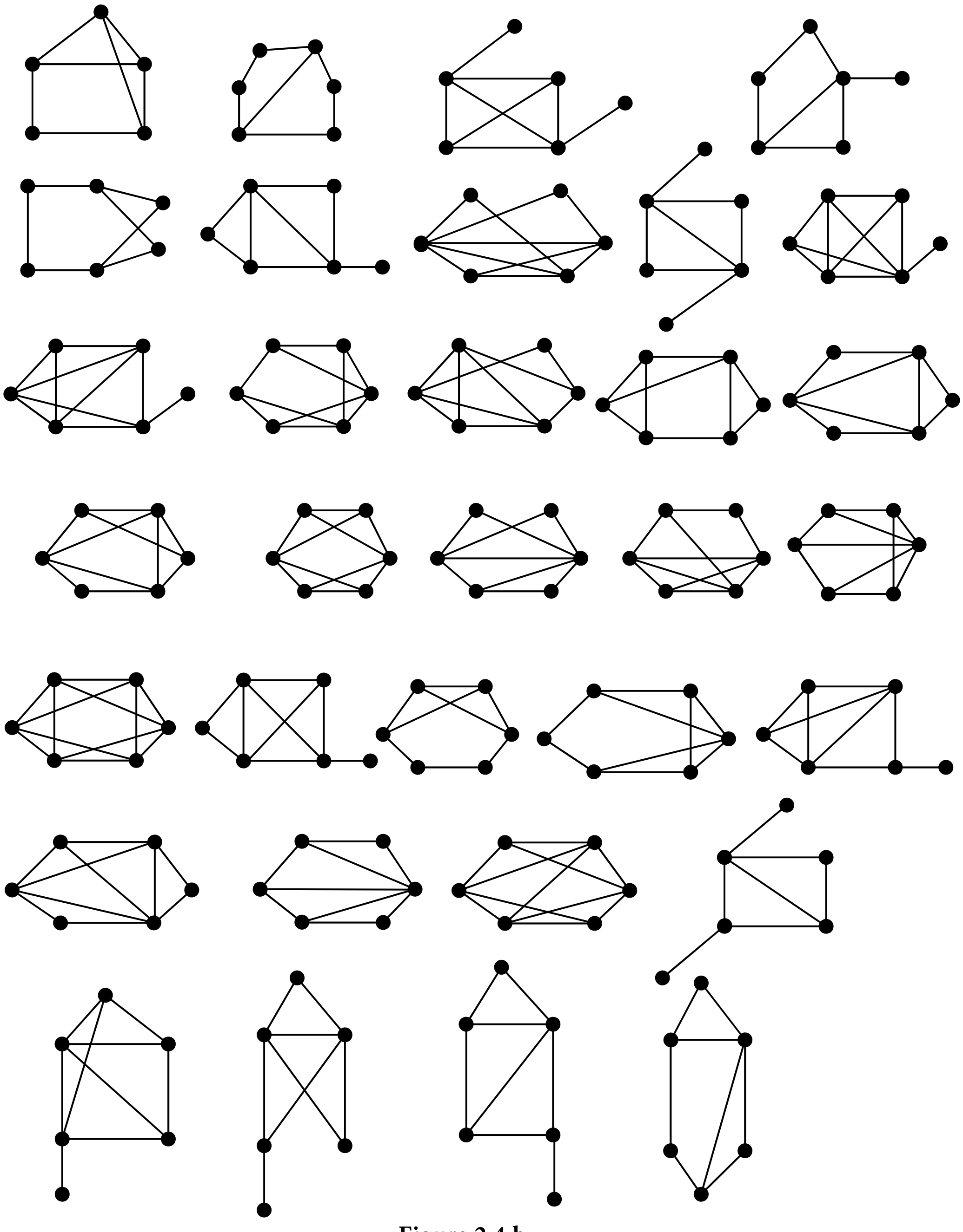


Figure 2.4 b

Theorem: 2.9: For any connected graph G with $p \geq 2$, $\gamma_{ed}(G) \leq 2q-p+1$. Further, the equality holds if and only if $G \cong K_2$ or $K_{1,2}$.

Proof: For any graph G, $\gamma_{ed}(G) \leq p-1 = 2(p-1)-(p-1) \leq 2q-p+1$.

Further, if $\gamma_{ed}(G)$ = 2q—p+1, then 2q—p+1 $\leq$ p—1 and q $\leq$ p—1. Hence, q = p—1. Therefore, G is a tree. By Theorem 2.4, $\gamma_{ed}(G) \cong K_2, K_{1,2}$. Hence, the result is proved.

Theorem: 2.10: For any connected (p, q) graph G, $\gamma_{ed}(G)+\Delta(G) \leq 2p-2$ ($p \geq 2$). Also, the equality holds if and only if $G \cong K_2$ or $K_{1,2}$.

Proof: For any graph with p vertices, $\Delta(G) \leq p-1$ and $\gamma_{ed}(G) \leq p-1$.

Hence, $\gamma_{ed}(G)+\Delta(G) \leq 2p-2$. When $G \cong K_2$ or $K_{1,2}$, $\gamma_{ed}(G)+\Delta(G) = 2p-2$.

Conversely, assume that, $\gamma_{ed}(G)+\Delta(G) = 2p-2$. The only possibility is $\gamma_{ed}(G) = p-1$ and $\Delta(G) = p-1$. Hence, by Theorem 2.4, we get $G \cong K_2$ or $K_{1,2}$.

Theorem: 2.11: For any connected (p, q) graph G, $\gamma_{ed}(G)+\Delta(G) = 2p-3$ if and only if G is any one of the following:K_3, K_4-e, $K_{1,3}$, $K_1+K_1+K_2$ and Fan graph F_4.

Proof: For the graphs given in the theorem, $\gamma_{ed}(G)+\Delta(G) = 2p-3$.

Conversely, assume that $\gamma_{ed}(G)+\Delta(G) = 2p-3$. The possible cases are

(i) $\gamma_{ed}(G) = p-1$ and $\Delta(G) = p-2$ and

(ii) $\gamma_{ed}(G) = p-2$ and $\Delta(G) = p-1$.

Case (i): By Theorem 2.4, $\gamma_{ed}(G) = p-1$ if and only if $G \cong K_2$ or $K_{1,2}$.

In this case, $\Delta(G) = p-1$. Therefore, this case is not possible.

Case (ii): By Theorem 2.5, $\gamma_{ed}(G) = p-2$ if and only if G is any one of the following graphs:

K_3, C_4, P_5, C_5, K_4-e, $K_{1,3}$, $K_1+K_1+K_2$, $K_1+K_1+K_1+2K_1$, Bull graph, Fan graph F_4, $\overline{K_2}+K_1+K_1+\overline{K_2}$.

If G is any one of the graphs C_4, P_5, C_5, $K_1+K_1+K_1+2K_1$, Bull graph, $\overline{K_2}+K_1+K_1+\overline{K_2}$, then $\Delta(G) \neq p-1$. These cases are not possible. Therefore, G is one of the graphs in the stated theorem.

In the following, Nordhaus-Gaddum type results for eccentric domination number are established.

Theorem: 2.12: Let G be a (p, q)($p \geq 4$) graph such that both G and its complement $\overline{G}$ are connected. Then (i) $4 \leq \gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq 2(p-2)$. (ii) $4 \leq \gamma_{ed}(G) \cdot \gamma_{ed}(\overline{G}) \leq 2(p-2)^2$.

Proof: (i) By Theorem 2.4, $\gamma_{ed}(G) = p-1$ if and only if $G \cong K_2$ or$K_{1,2}$. But in this case, $\overline{G}$ is disconnected. Therefore, $\gamma_{ed}(G) \leq p-2$. Hence, $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq 2(p-2)$.

For the lower bound, $\gamma_{ed}(G) = 1$ if and only if G is a complete graph. In this case, $\overline{G}$ is disconnected. Therefore, $\gamma_{ed}(G) \geq 2$. Hence, $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \geq 4$.

(ii) Proof follows similarly.

Lower bound is attained, if G is a path on 4 vertices and upper bound is attained, if G is a cycle on 5 vertices.

Theorem: 2.13: Let G be a (p, q), $p \geq 4$ graph such that both G and its complement $\overline{G}$ are connected. Then (i) $4 \leq \gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq 4p/3$. (ii) $4 \leq \gamma_{ed}(G) . \gamma_{ed}(\overline{G}) \leq 4p^2/9$.

Proof: (i) By Theorem 1.9, $\gamma_{ed}(G) \leq (2p)/3$. Hence, $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \leq 4p/3$. For the lower bound, $\gamma_{ed}(G) = 1$ if and only if G is a complete graph. In this case, $\overline{G}$ is disconnected. Therefore, $\gamma_{ed}(G) \geq 2$. Hence, $\gamma_{ed}(G) + \gamma_{ed}(\overline{G}) \geq 4$.

(ii) Proof is similar as in case(i).

We have characterized graphs for which $\gamma(G) = \gamma_{ed}(G) = p/2$ in the following theorem:

Theorem: 2.14: For a connected graph G with even number of vertices p and $\delta(G) \geq 2$, $\gamma(G) = \gamma_{ed}(G) = p/2$ if and only if G is C_4 or $H \circ K_1$ for some connected graph H.

Proof: When $G = C_4$, $\gamma_{ed}(G) = 2 = p/2$. Let $G = H \circ K_1$, where H is a connected graph on p/2 vertices. V(H) is a γ-set for G and the set of all pendant vertices in G is a minimum eccentric dominating set. Hence, $\gamma_{ed}(G) = p/2$.

Conversely, assume that $\gamma(G) = \gamma_{ed}(G) = p/2$. Since G is a graph with $\delta(G) \geq 2$, p is even, by Theorem 1.1,we get G is C_4 or $H \circ K_1$ for some connected graph H.

Theorem: 2.15:(i) G is a connected graph with $\delta(G) \geq 2$ and $\gamma(G) = \gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor$ if and only if G is any one of the graphs given in Figure 2.5.

(ii) G is a connected graph with $\gamma(G) = \left\lfloor \frac{p}{2} \right\rfloor$ and $\gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor + 1$ if and only if G is C_5.

Proof: (i) Let G be a connected graph with $\delta(G) \geq 2$ and $\gamma(G) = \gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor$. By Theorem 1.2, $\gamma(G) = \left\lfloor \frac{p}{2} \right\rfloor$ implies G is any one of the graphs in Figure 1.1. Therefore, the graphs given in Figure 2.5 are the only graphs with $\gamma(G) = \gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor$.

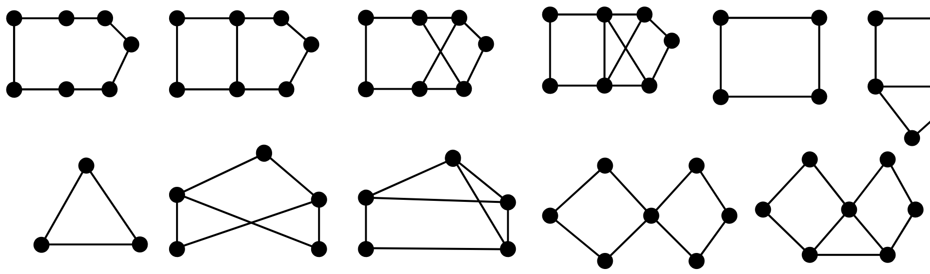


Figure 2.5

(ii)Suppose G is C_5. It is clear that $\gamma(G) = \left\lfloor \frac{p}{2} \right\rfloor$ and $\gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor+1$.

Conversely, assume that $\gamma(G) = \left\lfloor \frac{p}{2} \right\rfloor$ and $\gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor+1$. By Theorem 1.2, C_5 is the only graph with $\gamma(G) = \left\lfloor \frac{p}{2} \right\rfloor$ and $\gamma_{ed}(G) = \left\lfloor \frac{p}{2} \right\rfloor+1$.

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