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A Note on Vertex-Disjoint Diametral Path Set of a Graph

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Abstract: *The diametral path of a graph is the shortest path between two vertices which has length equal to diameter of that graph. Vertex - disjoint diametral path set of a graph is introduced as a collection of vertex - disjoint diametral paths of a graph so that every vertex of the graph appears in exactly one diametral path. Vertex - disjoint diametral path number d_v is the cardinality of such a set. The vertex - disjoint diametral path index D_v is the number of such sets that can be formed for a graph. In this paper, a few results on d_v and D_v in certain classes of graphs are presented and characterization is made for the graphs which admit this property.*

Keywords: *Diameter, Radius, Peripheral vertices, Central vertices, Diametral Path.*

Mathematics Subject Classification (2010): 05C12, 05C70

1. Introduction

The concept of partition of graphs with respect to edges or vertices has been extensively studied by researchers in literature [2, 3]. The challenge taken up in this study is to partition in such a way that all the vertices of the graph are uniquely placed in groups of peripheral vertices with their connecting vertices. In other words, it is about forming teams with diversity by bringing together two remotely related people and the connecting people as a team in an organization. The motivation for this study is the importance of these concepts in addressing problems in team formation, designing and distribution.

In this paper, a study on a collection of diametral paths such that every vertex of the graph appears in exactly one of the diametral paths is undertaken. The study is undertaken in simple, connected, undirected and unweighted graphs. In Section 2, preliminaries relevant to this study are discussed. In Section 3, vertex - disjoint diametral path set is introduced. Also vertex - disjoint diametral path number and index for some classes of graphs are discussed. In section 4, bounds on number of edges and d_v are proposed.

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2. Preliminaries

The definitions and results are in accordance with [1, 4]. The length of a path is the number of edges on the path. The distance between two vertices in a graph is the length of shortest path between them. The eccentricity of a vertex is the maximum of distances from it to all the other vertices of that graph. While diameter is the maximum of the eccentricities of all vertices of that graph, the radius is minimum of these. Peripheral vertices are vertices of maximum eccentricity and central vertices are of minimum eccentricity. The diametral path of a graph is the shortest path between two vertices which has length equal to diameter of that graph.

Given below are a few standard results in certain classes of graphs.

1. Complete Graph K_n: $diam(K_n) = 1$ where $n \geq 2$.
2. Wheel W_n: $diam(W_n) = 2$ where $n \geq 5$.
3. Star $K_{1,n}$: $diam(K_{1,n}) = 2$ where $n \geq 2$.
4. Complete Bipartite Graph $K_{r,s}$: $diam(K_{r,s}) = 2$ where $r \geq 2$ or $s \geq 2$.
5. Path P_n: $diam(P_n) = n - 1$ and $Rad(P_n) = \lfloor n/2 \rfloor$.
6. Cycle C_n: $diam(C_n) = \lfloor n/2 \rfloor$ where $n \geq 3$.

3. Results on d_v and D_v

Definition 3.1: Vertex - disjoint diametral path set of a graph is a collection of vertex - disjoint diametral paths of a graph so that every vertex of the graph appears in exactly one diametral path.

Definition 3.2: Vertex - disjoint diametral path number d_v is the cardinality of such a set.

Definition 3.3: Vertex - disjoint diametral path index D_v is the number of such sets that can be formed for a graph.

Proposition 3.1: Every path P_n $(n \geq 2)$ has a vertex - disjoint diametral path set with $d_v(P_n) = 1 = D_v(P_n)$.

Proof: Since every path is the diametral path itself, the set has only one element which is the path itself. Also there is only one vertex - disjoint diametral path set.

Hence $d_v(P_n) = 1 = D_v(P_n)$.

Proposition 3.2: Cycle C_n $(n \geq 3)$ has no vertex - disjoint diametral path set.

Proof: Since any two diametral paths of C_n have atleast one common vertex, C_n has no vertex - disjoint diametral path set.

Theorem 3.1: If $n = 3k$ ($k \geq 3$), then wheel W_n has a vertex - disjoint diametral path set with $d_v(W_n) = k$. Also $D_v(W_n) = \frac{((n-1)\lceil\frac{n-8}{3}\rceil)}{2}$

Proof: Since $diam(W_n) = 2$ for $n \geq 5$, W_n has $(n - 1)$ peripheral vertices and a central vertex. Also a diametral path has 3 vertices.

Since there are $n = 3k = 3(k-1) + 3$ vertices, we consider k-1 diametral paths involving peripheral vertices and one diametral path through the central vertex. Hence we can conclude that $d_v(W_n) = (k - 1) + 1 = k$.

Each vertex - disjoint diametral path set has one diametral path through the central vertex and the rest through only peripheral vertices. Also the sets are uniquely determined by some of the diametral paths which pass through the central vertex. We form the vertex - disjoint diametral path set in the following manner. For each vertex, the other end vertex of the diametral path through the central vertex can be chosen in $\left\lceil \frac{n-8}{3} \right\rceil$ ways by

a. Excluding the 3 vertices on both sides, itself and the central vertex
b. Identifying the number of ways, a free vertex can be chosen, when paths of length 2 are formed of the remaining vertices.

Hence for each vertex, the number of diametral paths through the central vertex = $\left\lceil \frac{n-8}{3} \right\rceil$.

Since there are $n - 1$ vertices, after excluding the repeated sets of diametral paths the number of vertex - disjoint diametral path sets = $D_v(W_n) = \frac{((n-1)\lceil\frac{n-8}{3}\rceil)}{2}$.

Example 3.1: Consider wheel W_9 in Figure 1.

$diam(W_9) = 2$ and $9 = n = 3k = (3)(3)$. We get $d_v(W_9) = k = 3$ by taking a vertex - disjoint diametral path set {(A,I,E), (B,C,D), (H,G,F)}.

Also $D_v(W_9) = \frac{((9-1)\lceil\frac{9-8}{3}\rceil)}{2} = 4$.

The vertex - disjoint diametral path sets are {(A,I,E), (B,C,D), (H,G,F)}, {(B,I,F), (C,D,E), (A,H,G)}, {(C,I,G), (D,E,F), (B,A,H)} and {(D,I,H), (E,F,G), (C,B,A)}.

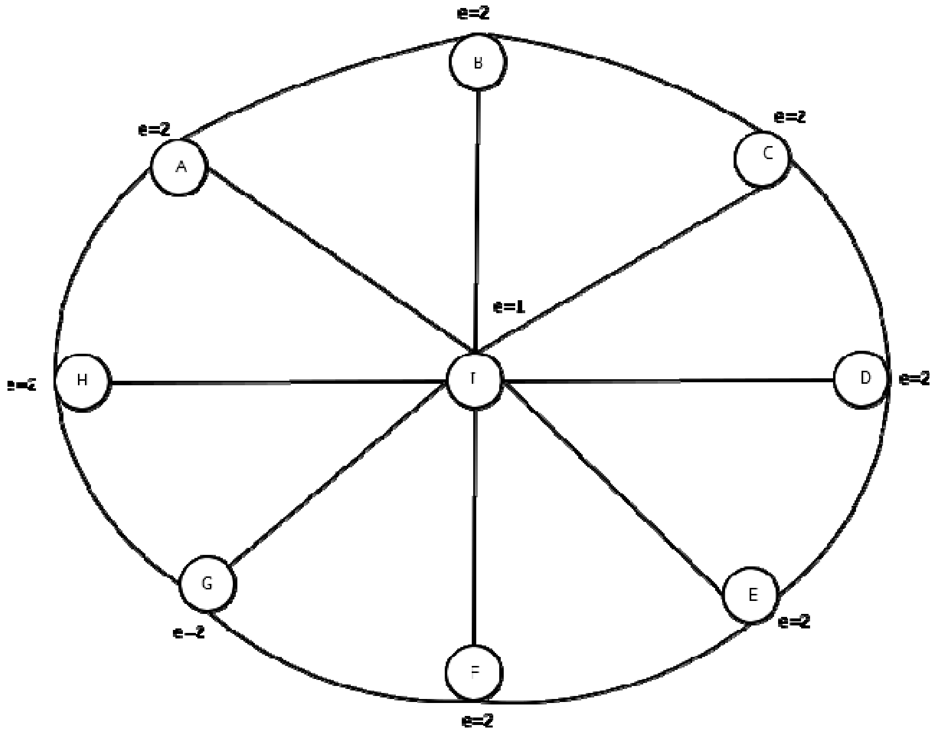

Figure 1 Wheel W_9

Proposition 3.3: Star $K_{1,n}$ ($n \geq 3$) has no vertex - disjoint diametral path set.

Proof: Since all diametral Paths have a common central vertex, star $K_{1,n}$ ($n \geq 3$) has no vertex - disjoint diametral path set.

Theorem 3.2: If n is even, then complete graph K_n($n \geq 2$) has a vertex - disjoint diametral path set with $d_v(K_n) = n / 2$. Also $D_v(K_n) = (n - 1)(n - 3)(n - 5)\ldots1$.

Proof: Since diam(K_n) = 1, every edge is a diametral path. We take two vertices at a time and include the diametral path in the set. Also since n is even, the vertices can be taken in n / 2 pairs and $d_v(K_n) = n / 2$.

In forming a vertex - disjoint diametral path set, we include edges one after the other so that all vertices are represented.

Beginning with any vertex, there are $(n - 1)$ ways in which another vertex is chosen to form an edge. When we include that edge in the set, there are $(n - 2)$ vertices left. Taking any one of the remaining vertices, there are $(n - 3)$ ways in which another vertex is chosen to form an edge. Proceeding this way till all vertices are included, we get all the vertices with corresponding edges in the partition.

Hence the number of vertex - disjoint diametral path sets $D_v(K_n) = (n - 1)(n - 3)(n - 5)\ldots1$.

Example 3.2: Consider complete graph K_4 in Figure 2. K_4 has a vertex - disjoint diametral path set {(A,B), (C,D)} with $d_v(K_4) = 4/2 = 2$.

And $D_v(K_4) = (4 - 1)(4 - 3) = 3$. The vertex - disjoint diametral path sets are {(A,B), (C,D)}, {(A,C), (B,D)} and {(A,D), (B,C)}.

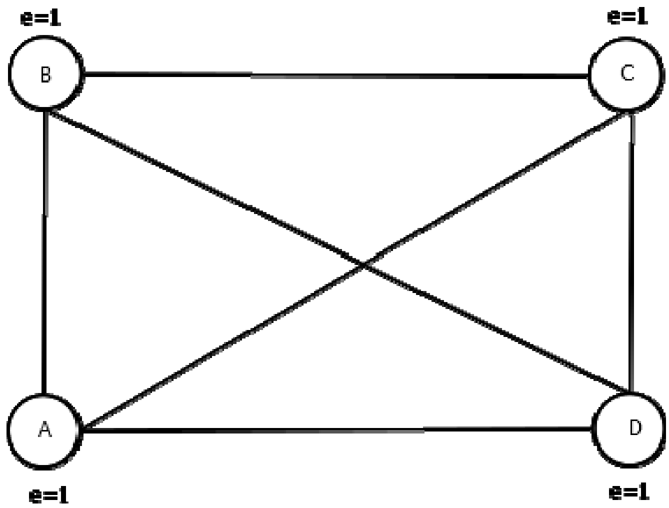

Figure 2: Complete graph K_4

Proposition 3.4: If $r \leq s \leq 2r$ and $r + s = 3k$, then complete bipartite graph $K_{r,s}$ ($r \geq 2$ and $s \geq 2$) has a vertex - disjoint diametral path set with $d_v(K_{r,s}) = k$.

Proof: Since $diam(K_{r,s}) = 2$ and $n = r + s = 3k$ vertices, we consider k diametral paths in the vertex - disjoint diametral path set. Hence $D_v(K_{r,s}) = k$.

Example 3.3: Consider complete bipartite graph $K_{4,5}$ in Figure 3.
Since $4 + 5 = 9 = n = 3k = (3)(3)$, $k = 3$. Hence $K_{4,5}$ has a vertex - disjoint diametral path set {(E,A,F), (B,G,C), (H,D,I)} with $d_v(K_{4,5}) = 3$.

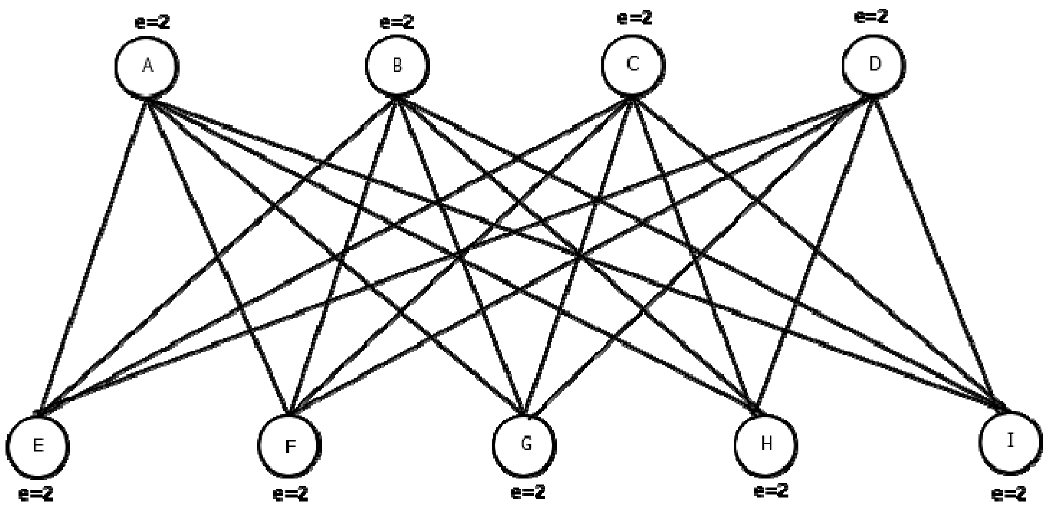

Figure 3: Complete bipartite graph $K_{4,5}$

4. Bounds

Theorem 4.1: If a graph G(n,m) has a vertex - disjoint diametral path set with diam(G) = d and $d_v(G) = k$, then the following results are true.

a. n = k (d + 1).

b. m ≥ kd.

c. If $d_v(G) < \lceil n/2 \rceil$, then G is K_n free.

Proof:

a. Since diam(G) = d and $d_v(G) = k$, each of the k diametral paths has d + 1 vertices. Hence n = k(d + 1).

b. Since diam(G) = d and $d_v(G) = k$, each of the k diametral paths has d edges. Since a repeated edge in two diametral paths would repeat the vertex and repetition of vertices is not allowed, we can conclude that m ≥ kd.

c. Suppose G has K_n. Then ${}^nC_2$ edges of K_n are in G. Since $d_v(G) < \lceil n/2 \rceil$, there is a diametral path with more than two vertices of K_n. Since there is an edge between every pair of vertices in K_n, a diametral path of G cannot have more than two vertices of K_n. This is impossible and hence G cannot have K_n. Hence G is K_n free.

Conclusion:

In this paper, a study has been undertaken on how to partition vertices in such a way that every vertex appears in some diametral path and no vertex appears in two diametral paths in a collection of diametral paths named vertex - disjoint diametral path set. Further to this study, the focus would be to find the number of vertex – disjoint diametral path sets for complete bipartite graphs $K_{r,s}$ where r ≤ s ≤ 2r and r + s = 3k and to identify the applications of these concepts.

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