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Co-eccentric Eccentric Domination in Graphs

M. Bhanumathi and M. Kavitha

Government Arts College for Women, Pudukkottai-622001, TN, India

Email: *bhanu_ksp@yahoo.com, kavisdev@gmail.com*

Abstract: *A Set $S \subseteq V(G)$ is a co-eccentric eccentric dominating set if S is a dominating set and for every vertex $v \in V-S$, there exists at least one eccentric vertex of v in S and another eccentric vertex of v in $V-S$. The minimum cardinality of a co-eccentric eccentric dominating set is called the co-eccentric eccentric domination number and is denoted by $\gamma_{cee}(G)$. In this paper we present several bounds on the co-eccentric eccentric domination number, characterize graphs for which $\gamma_{cee}(G) = 2$ and find exact values of $\gamma_{cee}(G)$ for some particular classes of graphs.*

Keywords: *Domination, Eccentric Domination, Co-eccentric eccentric domination.*

2010 Subject Classification 05C6 9,05C12

1. Introduction

Graphs discussed in this paper are finite, undirected and simple. Unless otherwise stated the graphs which we consider are connected. For a graph G, let V(G) and E(G) denote its vertex and edge set respectively. The *degree of a vertex* v in a graph G is denoted by $deg_G(v)$. The minimum and maximum degrees in a graph G is denoted by $\delta(G)$ and $\Delta(G)$ respectively. For graph theoretic terminology refer to Harary [5], Buckley and Harary [3].

The concept of distance in graphs plays a dominant role in the study of structural properties of graphs in various angles using related concept of eccentricity of vertices in graphs. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. For a connected graph G, the *eccentricity* e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$. The *radius* r(G) is the minimum eccentricity of the vertices, whereas the *diameter* diam(G) = d(G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. v is a central vertex if e(v) = r(G). When r(G) = d(G), the graph G is called self-centered graph. The *center* C(G) is the set of all central vertices. For a vertex v, each vertex at a distance e(v) from v is an *eccentric vertex of v*. *Eccentric set* of a vertex v is defined as $E(v) = \{u \in V(G) / d(u, v) = e(v)\}$. G is a unique eccentric vertex (point) graph if no vertex of G has more than one eccentric vertex. The *open neighborhood* N(v) of a vertex v is the set of all vertices adjacent to v in G. $N[v] = N(v) \cup \{v\}$ is called the *closed neighborhood* of v.

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A *bigraph or bipartite graph* G is a graph whose point set V can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. If further G contains every line joining the points of V_1 to the points of V_2 then G is called a *complete bigraph*. If V_1 contains m points and V_2 contains n points then the complete bigraph G is denoted by $K_{m,n}$. A *star* is a complete bi graph $K_{1,n.}$

The concept of domination in graphs was introduced by Ore in his famous book 'Theory of Graphs' [8] in 1962. In 1977, Cockayne and Hedetniemi unfolded its diverse aspects, by surveying all the available results, bringing to light new ideas and citing its application potential in a variety of scientific areas, in their paper 'Towards a theory of domination in graphs' [4]. A set $D \subseteq V$ is said to be a *dominating set* of G, if every vertex in $V-D$ is adjacent to some vertex in D. D is said to be a minimal *dominating set* of G, if no proper subset of D is a dominating set. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is known as *minimum dominating set or a* γ-set. For a detailed study on domination refer to Kulli [7] and Haynes et.al [9].

Janakiraman, Bhanumathi and Muthammai [6] introduced and studied the concept of eccentric dominating sets in 2010. In [1] and [2], they have studied the Eccentric domination in Trees and derived bounds for $\gamma_{ed}(G)$. A set $D \subseteq V(G)$ is an *eccentric dominating set* if D is a dominating set of G and for every $v \in V-D$, there exists at least one eccentric vertex of v in D. If D is an eccentric dominating set, then every superset $D' \supseteq D$ is also an eccentric dominating set. But $D'' \subseteq D$ is not necessarily an eccentric dominating set. An eccentric dominating set D is a *minimal eccentric dominating set* if no proper subset of D is an eccentric dominating set. The minimum cardinality of an eccentric dominating set is called the *eccentric domination number* and is denoted by $\gamma_{ed}(G)$. An eccentric dominating set with cardinality $\gamma_{ed}(G)$ is known as *minimum eccentric dominating set or a* γ_{ed}-set.

In this paper, we introduce a new dominating set called *co-eccentric eccentric dominating set* of a graph through which we study the properties of the graph. A set $S \subseteq V(G)$ is *co-eccentric eccentric dominating set* if S is a dominating set and for every vertex $v \in V-S$, there exist at least one eccentric vertex in S and another eccentric vertex in $V-S$. The co-eccentric eccentric dominating set S is a *minimal co-eccentric eccentric dominating set* if no proper subset of S is a co-eccentric eccentric dominating set. The minimum cardinality of a co-eccentric eccentric dominating set is called the co-eccentric *eccentric domination number* and is denoted by $\gamma_{cee}(G)$. We find upper and lower bounds for this new domination number in terms of various already known parameters.

2. Prior Results

We need the following results to prove certain results in co-eccentric eccentric domination. Let G be a graph on n vertices.

Theorem 1.1: [9] For any graph G, $\lceil n/(1+\Delta(G)) \rceil \leq \gamma(G) \leq n-\Delta(G)$.

Theorem 1.2: [6] $\gamma_{ed}(K_n) = 1$

Theorem 1.3: [6] $\gamma_{ed}(K_{m,n}) = 2$.

Theorem 1.4: [6] $\gamma_{ed}(W_3) = 1$, $\gamma_{ed}(W_4) = 2$, $\gamma_{ed}(W_n) = 3$ for $n \geq 7$.

Theorem 1.5: [6] $\gamma_{ed}(P_n) = \gamma(P_n)$ or $\gamma(P_n) + 1$.

Theorem 1.6: [6] (i) $\gamma_{ed}(C_n) = n/2$ if n is even.

(ii) $$\gamma_{ed}(C_n) = \begin{cases} n/3 & \text{if } n = 3m \text{ and is odd.} \\ \lceil n/3 \rceil & \text{if } n = 3m+1 \text{ and is odd.} \\ \lceil n/3 \rceil + 1 & \text{if } n = 3m+2 \text{ and is odd.} \end{cases}$$

3. Co-eccentric eccentric dominating sets in graphs

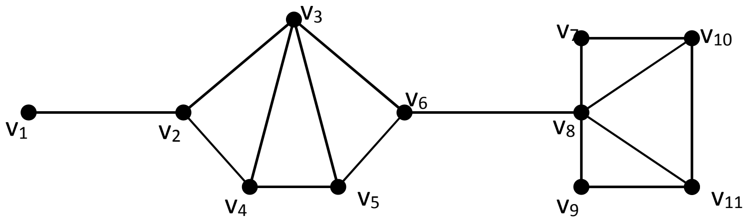
In this paper, we define a new domination parameter namely, co-eccentric eccentric domination as follows.

Definition 3.1: A set $S \subseteq V(G)$ is a co-eccentric eccentric dominating set if S is a dominating set and for every vertex $v \in V-S$, there exist at least one eccentric vertex of v in S and in $V-S$.

The co-eccentric eccentric dominating set S is a minimal co-eccentric eccentric dominating set if no proper subset $S'' \subseteq S$ is the co-eccentric eccentric dominating set.

The minimum cardinality of a co-eccentric eccentric dominating set is called the co-eccentric *eccentric domination number* and is denoted by $\gamma_{cee}(G)$.

Example 3.1:



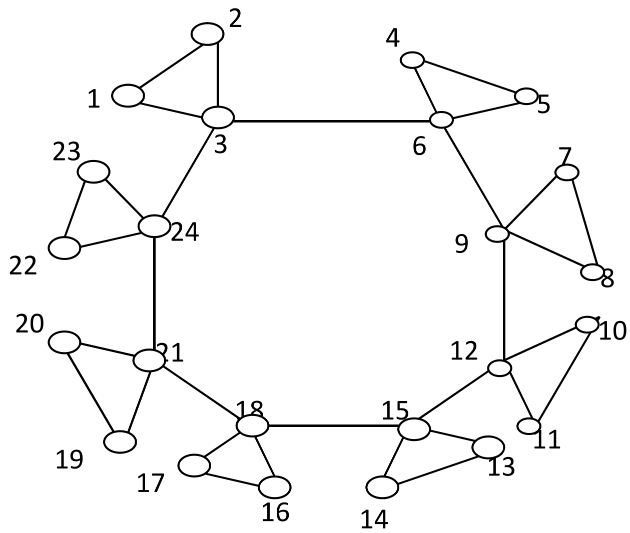
G:

$D_1 = \{v_2, v_6, v_8\}$ is a minimum dominating set. Therefore, $\gamma(G) = 3$.

$D_2 = \{v_1, v_3, v_8, v_{10}\}$ is a minimum eccentric dominating set. Therefore, $\gamma_{ed}(G) = 4$.

$D_3 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\} = V(G)$ is a co-eccentric eccentric dominating set and $\gamma_{cee}(G) = 11$.

Example 3.2:



In this example, $\gamma(G) = \gamma_{ed}(G) = \gamma_{cee}(G) = 8$.

D = {2, 5, 8, 11, 14, 17, 20, 22} is a co-eccentric eccentric dominating set.

Observations:

3.1: Let D be a γ_{cee}-set then $v \in V - D$ has at least two eccentric vertices in V.

3.2: For a graph G with n vertices, $\gamma(G) \leq \gamma_{ed}(G) \leq \gamma_{cee}(G) \leq n$.

3.3: If G is a unique eccentric point graph then D = V(G) is a minimum co-eccentric eccentric dominating set and hence $\gamma_{cee}(G) = n$.

3.4: If $v \in V(G)$ has only one eccentric vertex, then $v \in D$.

3.5: If G has exactly two eccentric vertices and all other vertices are not eccentric to any other vertices then D = V(G) is the minimum co-eccentric eccentric dominating set and $\gamma_{cee}(G) = n$.

3.6: If there exists a vertex $v \in V(G)$ such that v is the only eccentric vertex of elements of a subset $S \subseteq V$, then {v} and S are subsets of every co-eccentric eccentric dominating set.

3.7: If $v \in V(G)$ has exactly two eccentric vertices x, y and they are the only eccentric vertices of G, then v, x, y are in every co-eccentric eccentric dominating set.

Following Theorems give the value of $\gamma_{cee}(G)$ for some standard graphs G.

Theorem 3.1: $\gamma_{cee}(K_n) = 1$ if and only if $G = K_n$, $n > 2$.

Proof: When $G = K_n$, radius of G = diameter of G = 1. Hence any vertex $u \in V(G)$ dominate other vertices and is also an eccentric point of other vertices and every vertex $v \in V-S$, where $S = \{u\}$, has one eccentric point in V and one in $V - S$. Hence, $\gamma_{cee}(K_n) = 1$. Conversely, assume $\gamma_{cee}(K_n) = 1$. The co-eccentric eccentric dominating set has only one vertex u of G. Thus u is adjacent to all other vertices of G and u is also an eccentric vertex of other vertices of G. Hence, G is a complete graph with more than two vertices.

Theorem 3.2: $\gamma_{cee}(K_{1,s}) = 2$, $s > 2$. If $s = 2$, $\gamma_{cee}(K_{1,s}) = 3$. If $s = 1$, $\gamma_{cee}(K_{1,s}) = 2$.

Proof: $G = K_{1,s}$, $s > 2$. Let $S = \{u, v\}$ and let v be a central vertex. The central vertex dominates all vertices of $V-S$ and every vertex $v \in V-S$ has one eccentric point in S and in $V-S$. Hence, $\gamma_{cee}(K_{1,s}) = 2$. By the definition of co-eccentric eccentric dominating set, we have $\gamma_{cee}(K_{1,s}) = 3$, when $s = 2$ and $\gamma_{cee}(K_{1,s}) = 2$, when $s = 1$.

Theorem 3.3: $\gamma_{cee}(K_{m,n}) = 2$, $m, n \geq 3$.

Proof: Let $G = K_{m,n}$ and let $V(G) = V_1 \cup V_2$, $|V_1| = m$, $|V_2| = n$ such that each element of V_1 is adjacent to every vertex of V_2 and vice versa.

Let $S = \{u, v\}$, $u \in V_1$ and $v \in V_2$. u dominates all the vertices of V_2 and it is an eccentric vertex of $V_1 - \{u\}$. Similarly v dominate all the vertices of V_1 and it is eccentric to elements of $V_2 - \{v\}$. Hence, S is a minimum co-eccentric eccentric dominating set. Thus $\gamma_{cee}(K_{m,n}) = 2$.

Theorem 3.4: If n is odd, $\gamma_{cee}(C_n) = \begin{cases} \frac{n}{3} & \text{if } n = 3m, m \text{ is odd} \\ \left\lceil \frac{n}{3} \right\rceil & \text{if } n = 3m+1, m \text{ is even} \\ \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n = 3m+2, m \text{ is odd} \end{cases}$

If n is even, $\gamma_{cee}(C_n) = n$.

Proof:

Case (i): When n is odd.

Let S be a co-eccentric eccentric dominating set of C_n. Each vertex of C_n has exactly two eccentric vertices. If n = 2r+1, where r is the radius of C_n, $v_i \in V(G)$ has v_{i+r}, $v_{i+(r+1)}$ as eccentric vertices. Since C_n is a cycle, if a vertex v_i is not in S implies either v_{i-1} or v_{i+1} is also not in S and if v_i, v_{i+1} are not in S then v_{i-1} and v_{i+2} must be in S.

Subcase(i): If n = 3m = 2r+1, m is odd.

Consider S = {$v_1, v_4, v_7, \ldots, v_r, v_{r+3}, \ldots, v_{2r-1}$}, S is a co-eccentric eccentric dominating set and |S|= n/3. Then $\gamma_{cee}(C_n) \leq n/3$. ------- (1)

We know that $\gamma_{cee}(G) \geq \gamma_{ed}(G)$, and $\gamma_{ed}(C_n) = n/3$ if n = 3m.

Therefore, $\gamma_{cee}(C_n) \geq n/3$. ------- (2)

From (1) and (2) we have $\gamma_{cee}(C_n) = n/3$.

Subcase(ii): If n = 3m + 1 , m is even.

Consider S = {$v_1, v_4, v_7, \ldots, v_{r+1}, v_{r+3}, \ldots, v_{2r-1}$} , S is a co-eccentric eccentric dominating set and $|S|=\lceil n/3 \rceil$, $\gamma_{cee}(C_n) \leq \lceil n/3 \rceil$. ------- (3)

We know that $\gamma_{ed}(C_n) = \lceil n/3 \rceil$ if n = 3m+1, then $\gamma_{cee}(C_n) \geq \lceil n/3 \rceil$ ------- (4)

From (3) and (4) we get $\gamma_{cee}(C_n) = \lceil n/3 \rceil$.

Subcase(iii): If n = 3m+2, m is odd.

Consider S = {$v_1, v_4, \ldots, v_{r-1}, v_r, v_{r+3}, \ldots, v_{2r+1}$}, S is a co-eccentric eccentric dominating set. $|S| = \lceil n/3 \rceil + 1$.

Then $\gamma_{cee}(C_n) \leq \lceil n/3 \rceil + 1$ ------- (5)

We know that $\gamma_{ed}(C_n) = \lceil n/3 \rceil + 1$ if n = 3m+2

$\gamma_{cee}(G) \geq \gamma_{ed}(G)$, $\gamma_{cee}(G) \geq \lceil n/3 \rceil + 1$ ------- (6)

From (1) and (2) we get $\gamma_{cee}(C_n) = \lceil n/3 \rceil + 1$.

Case(ii): If n is even, n = 2r.

In even cycles, every vertex has exactly one eccentric point. Therefore, S contains all the vertices of C_n. Hence, $\gamma_{cee}(C_n) = n$.

Theorem 3.5: Let n be an even integer. Let G be obtained from the complete graph K_n by deleting edges of a linear factor. Then $\gamma_{cee}(G) = n$.

Proof: Let u and v be a pair of non adjacent vertices in G. Then u and v are eccentric to each other. Also G is unique eccentric vertex graph. Therefore, $\gamma_{cee}(G) = n$.

Therorem 3.6: $\gamma_{cee}(P_n) = n$.

Proof: A co-eccentric eccentric dominating set of P_n, n > 4 must contain the two end vertices. In path, every non central vertex has exactly one eccentric vertex. Therefore, (n − 1) vertices must be in γ_{cee}-set and the central vertex has two eccentric vertices. (That is the two end vertices). These two end vertices are already in γ_{cee}-set. Therefore, every co-eccentric eccentric dominating set contains all the vertices of P_n. Hence, $\gamma_{cee}(P_n) = n$.

Theorem 3.7: $\gamma_{cee}(W_n) = 3$, $n \geq 5$, $\gamma_{cee}(W_3) = 4$, $\gamma_{cee}(W_4) = 5$.

Proof: When $G = W_n$, $n \geq 6$. Consider S = {u, v, w}, where u and v are any two adjacent non central vertices and w is the central vertex. Then S is the minimum co-eccentric eccentric dominating set of G. Therefore, $\gamma_{cee}(W_n) = 3$, $n \geq 6$. When n = 5, let S = {u, v, w} where u and v are any two adjacent non central vertices and w is the non central vertex, which is not adjacent to both u and v. S is the minimum co-eccentric eccentric dominating set of G. Hence, $\gamma_{cee}(W_5) = 3$. It is easy to verify that $\gamma_{cee}(W_3) = 4$, $\gamma_{cee}(W_4) = 5$.

Theorem 3.8: If $G = \overline{K_m} + K_1 + K_1 + \overline{K_n}$, m, n ≥ 2, $\gamma_{cee}(G) = 4$.

Proof: In the graph G, let $S_1 = V(\overline{K_m})$ and $S_2 = V(\overline{K_n})$. Let u and v be the central vertices, u and v dominate all other vertices of G. Select any one end vertex in S_1 and another one vertex in S_2. $u_1 \in S_1$ is an eccentric vertex of S_2 and $v_1 \in S_2$ is an eccentric vertex of S_1. Now, $S = \{u, v, u_1, v_1\}$ is the co-eccentric eccentric dominating set of G. Thus, $\gamma_{cee}(G) = 4$.

Note 3.1: (i) If $G = K_m + K_1 + K_1 + K_1$ then $\gamma_{cee}(G) = |V(G)| = m+3$.

(ii) $\gamma_{cee}(K_{1,n}) = 2$.

(iii) If $G = K_n + K_1 + K_1 + K_m$, n, m ≥ 2, then $\gamma_{cee}(G) = 2 = \gamma(G) = \gamma_{ed}(G)$.

Theorem 3.9: If G is a spider with more than two legs, then $\gamma_{cee}(G) = \Delta(G) + 1$.

Proof: Let G be a spider. Let u be the vertex of maximum degree $\Delta(G)$. Then |N(u)| vertices dominates other vertices of G. $N(u) \cup \{v\}$, where v is an end vertex of G is a co-eccentric eccentric dominating set. Hence, $\gamma_{cee}(G) = |N(u)| + 1$. Therefore, $\gamma_{cee}(G) = \Delta(G) + 1$.

Theorem 3.10: Let G be a wounded spider having at least three non wounded legs, then $\gamma_{cee}(G) = s(G) + 1$, where s is the number of support vertices of the pendant vertices of G.

Proof: Let G be a wounded spider. Support vertices of pendant vertices dominate all other vertices of G. For the eccentricity property we include one pendant vertex of non wounded leg to the support vertices. This set form a co-eccentric eccentric dominating set. Therefore, $\gamma_{cee}(G) = s(G) + 1$.

Theorem 3.11: (i) $\gamma_{cee}(\overline{C_3}) = 3, \gamma_{cee}(\overline{C_4}) = 2.$

(ii) For n ≥ 5, $\gamma_{cee}(\overline{C_n}) = \begin{cases} \frac{n}{3}, & n=3k \\ \left\lceil \frac{n}{3} \right\rceil, & n=3k+1 \\ \left\lceil \frac{n}{3} \right\rceil + 1, & n=3k+2 \end{cases}$

Proof:

(i) Clearly $\gamma_{cee}(\overline{C_3}) = 3, \gamma_{cee}(\overline{C_4}) = 2.$

(ii) Now, assume that n ≥ 5. Let $v_1, v_2, \ldots, v_n, v_1$ form C_n. Then $\overline{C_n} = K_n - C_n$ and each vertex v_i is adjacent to all other vertices except v_{i-1} and v_{i+1} in $\overline{C_n}$. Hence, eccentric point of v_i in $\overline{C_n}$ is v_{i-1} and v_{i+1} only. Therefore, any co-eccentric eccentric dominating set must contain either v_i or any one of v_{i-1}, v_{i+1}. Hence, $\gamma_{cee}(\overline{C_n}) \geq \lceil n/3 \rceil$.

Case (i): n = 3m

$D_1 = \{v_1, v_4, v_7, \ldots, v_{k+1}, v_{k+4}, \ldots , v_{2k-2}\}$, if n is even and $D_2 = \{v_1, v_4, v_7, \ldots, v_k, v_{k+3}, \ldots, v_{2k-1}\}$, if n is odd are co-eccentric eccentric dominating sets and $|D_1| = |D_2| = n/3$.

Hence, $\gamma_{cee}(\overline{C_n}) \leq n/3$. ------- (1)

We know that $\gamma_{ed}(\overline{C_n}) = \lceil n/3 \rceil$ and

$\gamma_{cee}(\overline{C_n}) \geq \gamma_{ed}(\overline{C_n})$. This implies that $\gamma_{cee}(\overline{C_n}) \geq n/3$. ------- (2)

From (1) and (2), we get $\gamma_{cee}(\overline{C_n}) = n/3$.

Case (ii): n = 3m + 1

$D_1 = \{v_1, v_4, v_7, \ldots, v_{k-1}, v_{k+2}, \ldots , v_{2k-1}, v_{2k}\}$ if n is even, $D_2 = \{v_1, v_4, v_7, \ldots, v_{k+1}, v_{k+4}, \ldots, v_{2k-2}, v_{2k+1}\}$ if n is odd are co-eccentric eccentric dominating sets and $|D_1| = |D_2| = \lceil n/3 \rceil$.

Hence, $\gamma_{cee}(\overline{C_n}) \leq \lceil n/3 \rceil$. ------- (3)

We know that $\gamma_{cee}(\overline{C_n}) \geq \gamma_{ed}(\overline{C_n})$. Hence we have $\gamma_{cee}(\overline{C_n}) \geq \lceil n/3 \rceil$. ------- (4)

From (3) and (4), we get $\gamma_{cee}(\overline{C_n}) = \lceil n/3 \rceil$.

Case (iii) n = 3m + 2

$D_1 = \{v_1, v_4, v_7, \ldots, v_k, v_{k+3}, \ldots , v_{2k-1}, v_{2k}\}$ if n is even, $D_2 = \{v_1, v_4, v_7, \ldots, v_{k-1}, v_{k+2}, \ldots , , v_{2k}, v_{2k+1}\}$ if n is odd are co–eccentric eccentric dominating sets with $\lceil n/3 \rceil + 1$ vertices and no eccentric dominating set of $\overline{C_n}$ is a co–eccentric eccentric dominating set of $\overline{C_n}$. Hence, $\gamma_{cee}(\overline{C_n}) = \lceil n/3 \rceil + 1$.

Following theorem characterizes minimal co-eccentric eccentric dominating set of a graph G.

Theorem 3.12: A co-eccentric eccentric dominating set D is a minimal co-eccentric eccentric dominating set if and only if for each vertex u ∈ D, one of the following is true.

(i) u is an isolated vertex of D or u has no eccentric vertex in D or u has no eccentric vertex in V – D.

(ii) There exists some v ∈ V − D such that N(v) ∩ D = {u} or E(v) ∩ D = {u}.

Proof: Assume that D is a minimal co-eccentric eccentric dominating set of G. Then for every vertex u ∈ D, D – {u} is not a co-eccentric eccentric dominating set. That is there exists some vertex v in (V − D) ∪ {u} which is not dominated by any vertex in D − {u} or there exists v in (V − D) ∪ {u} such that v has no eccentric vertex in D − {u} or u has no eccentric vertex in V − D.

Suppose **u** = **v**. Then u is an isolate of D or u has no eccentric vertex in D or in V − D.

Suppose **v ∈ V – D**. If v is not dominated by D – {u} but is dominated by D, then v is adjacent to only u in D. That is N(v) ∩ D = {u}. Suppose v has no eccentric vertex in D – {u} but v has an eccentric vertex in D. Then u is the only eccentric vertex of v in D, that is E(v) ∩ D = {u}. Suppose u has no eccentric vertex in V − D, then E(u) ∩ (V − D) = ϕ.

Conversely, suppose D is a co–eccentric eccentric dominating set and for each u ∈ D one of the conditions holds. We show that D is a minimal co–eccentric eccentric dominating set.

Suppose that D is not a minimal co–eccentric eccentric dominating set, there exists a vertex u ∈ D such that D − {u} is a co–eccentric eccentric dominating set. Hence u is adjacent to at least one vertex v in D − {u} and u has an eccentric vertex in D – {u}. Therefore, condition (i) does not hold. Also if D − {u} is a co–eccentric eccentric dominating set, every vertex v in V − D is adjacent to at least one vertex in D − {u} and v has an eccentric vertex in D − {u} and u has an eccentric vertex in V − D.

Hence, condition (ii) and (iii) does not hold. This is a contradiction to our assumption that for each u ∈ D, one of the conditions holds.

This proves the theorem.

Now, let us study co-eccentric eccentric domination in trees.

If T is a tree of radius one, then G = $K_{1,\,s}$ and by Theorem 3.2, $\gamma_{cee}(K_{1,\,s})$ = 2, s > 2, $\gamma_{cee}(K_{1,\,2})$ = 3 and $\gamma_{cee}(K_{1,\,1})$ = 2. If G = $\overline{K_m} + K_1 + K_1 + \overline{K_n}$, m, n ≥ 2, then $\gamma_{cee}(G)$ = 4 by Theorem 3.8. If G = $\overline{K_m} + K_1 + K_1 + \overline{K_n}$, m ≥ 2 and n = 1, then $\gamma_{cee}(G)$ = 3.

Theorem 3.13: If T is a unicentral tree of radius two having at least three pendant vertices u, v, w at a distance diam(T) from each other and the degree of the support vertex of any one of them is two, then $\gamma_{cee}(G) \leq s(T) + 2$, where s(T) is the number of support vertices of T.

Proof: Let T be a unicentral tree of radius two. The set S of support vertices of the pendant vertices dominate all other vertices of G. Let the degree of the support vertex of w be two. Then $S \cup \{u, v\}$ is an eccentric dominating set for T and w is eccentric to other vertices which are not support vertices. Therefore, s(T) + 2 vertices form a co-eccentric eccentric dominating set. Hence, $\gamma_{cee}(G) \leq s(T) + 2$.

Theorem 3.14: For a tree T with radius > 2, $\gamma_{cee}(T) = \gamma_{ed}(T)$ or n.

Proof: Let T be a tree with n vertices and let d = diam T.

Case (i) Tree has only one pair of peripheral vertices at distance d = diam T > 2.

Let D be a γ_{cee} set. T has one pair of peripheral vertices that are in D. Then every vertex in V – D has eccentric vertex in D and no eccentric vertex in V – D. Therefore, we take all the vertices in D. Hence $\gamma_{cee}(T) = n$.

Case (ii) Tree has more than one pair of peripheral vertices at distance d to each other.

Let D be a γ_{cee} set. If D has exactly one pair of peripheral vertices implies V – D has at least one pair of peripheral vertices at distance d to each other. Therefore, every vertex in V – D has at least one eccentric vertex in V – D. Hence, $\gamma_{cee}(T) = \gamma_{ed}(T)$.

Theorem 3.15: For any connected graph G, $\gamma_{cee}(G) = n$ or $\gamma_{cee}(G) \leq n - 2$.

Proof: By the definition of co-eccentric eccentric dominating set either $V - D = \phi$ or $|V - D| \geq 2$, where D is a co-eccentric eccentric dominating set. Therefore, $\gamma_{cee}(G) \leq n - 2$ or $\gamma_{cee}(G) = n$. Thus $\gamma_{cee}(G) \neq n - 1$.

Next, we study minimal co-eccentric eccentric dominating sets of graphs with radius one and diameter two.

Theorem 3.16: If G is a graph with r(G) = 1, d(G) = 2 and G has more than one central vertex then $\gamma_{cee}(G) \leq n-2$.

Proof: If G has at least two central vertices then the set of all vertices with eccentricity two form a co-eccentric eccentric dominating set. Hence $\gamma_{cee}(G) \leq n-2$. This proves the theorem.

Theorem 3.17: If G is a graph with r(G) = 1, d(G) = 2 and has only one central vertex then $\gamma_{cee}(G) = n$ if and only if each vertex $v \in V(G)$ satisfies any one of the following conditions:

(i) v has only one eccentric vertex.

(ii) $v \in V(G)$ has more than one eccentric vertex but each of these vertices has exactly one eccentric vertex.

Proof: $\gamma_{cee}(G) = n = |V(G)|$ implies that V is the only minimum co-eccentric eccentric dominating set. By the definition of co-eccentric eccentric dominating set, D is a co-eccentric eccentric dominating set implies for $v \in V-D$, v is dominated by some $u \in D$ and there exists $v_1 \in D$ and $v_2 \in V - D$ such that v_1 and v_2 are eccentric vertices of $v \in V$.

Hence, if $v \in V$ has only one eccentric vertex in G then it cannot be in $V - D$. Thus, $v \in D$. Similarly, if v has $v_1, v_2, \ldots, v_k$ as eccentric vertices and if v_i has exactly one eccentric vertex in G, then these v_i's are in D, for $1 \leq i \leq k$. Hence, v has no eccentric vertex in $V-D$. Therefore, v must be in D. This proves the result.

Converse is obvious.

Theorem 3.18: If G is a graph with $p > 3$, $r(G) = 1$, $d(G) = 2$ and G has exactly two peripheral vertices, then $\gamma_{cee}(G) = 2$.

Proof: Let u, v be the only peripheral vertices of G. Then $D = \{u, v\}$ is a minimum co-eccentric eccentric dominating set of G. Also $p > 3$ implies $|V-D| \geq 2$. Hence, $\gamma_{cee}(G) = 2$.

Theorem 3.19: Let G be a graph with $p > 3$ and $r(G) = 1$, $d(G) = 2$. Then $\gamma_{cee}(G) = 2$ if and only if G satisfies any one of the following conditions:

(i) $|E_1(G)| \geq 2$ and $|E_2(G)| = 2$.

(ii) There exist $u, v \in E_2(G)$ such that $G - \{u, v\}$ is an eccentricity preserving subgraph of G and $\{u, v\}$ is a γ-set of $< E_2(G) >$.

(iii) There exists $v \in E_2(G)$ such that $deg(v) = |E_1(G)|$ and $G - v$ is an eccentricity preserving subgraph of G.

Proof: $\gamma_{cee}(G) = 2$. Let $D = \{u, v\}$ be a minimum co-eccentric eccentric dominating set.

Case(i): $D = \{u, v\}$, where $e(u) = e(v) = 1$.

This case is not possible, since D has no vertex of eccentricity 2, if cannot be an eccentric dominating set.

Case(ii): $D = \{u, v\}$, where $e(u) = e(v) = 2$.

Subcase (a): $e(w) = 1$ for all $w \in V - D$.

This implies that u and v are the only peripheral vertices. Also, $|V-D| \geq 2$ implies that G has at least two central vertices. Hence, G is a graph with $|E_1(G)| \geq 2$ and $|E_2(G)| = 2$, where $E_i(G) = \{v \in V(G) \text{ with } e(v) = i\}$.

Subcase (b): $e(w) = 1$ for some $w \in V-D$ and $e(w') = 2$ for some $w' \in V-D$.

All vertices of eccentricity one is adjacent to u and v. But D is a dominating set implies vertices with eccentricity two are adjacent to either u or v or both. Also, D is an eccentric dominating set implies vertices with eccentricity two in V−D are eccentric to u or v.

Hence, $w \in V-D$, $e(w) = 2$ implies that w is adjacent to either u or v only (not both).

Also, D is co-eccentric eccentric dominating set implies that $w \in V-D$ with $e(w) = 2$ has eccentric vertices (at distance 2) in V−D. Hence, $< V-D > = G - \{u, v\}$ is a graph having radius one diameter 2, with the condition that $E_1(< V-D >) = E_1(G)$ and $E_2(< V-D >) = E_2(G) - \{u, v\}$. That is the central vertices of < V−D > are exactly the central vertices of G.

That is, $G - \{u, v\}$ is a eccentricity preserving sub graph of G.

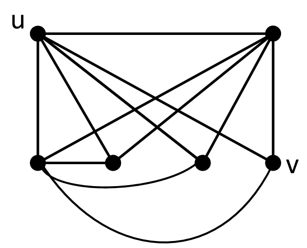
Also, {u, v} dominates all the vertices of eccentricity two implies $\gamma(< E_2(G) >) = 2$.

Case (iii): Suppose $D = \{u, v\}$, where $e(u) = 1$ and $e(v) = 2$.

In this case $w \notin D$ with $e(w) = 2$ is always eccentric to v . Thus $w \notin D$, $e(w) = 2$ is not adjacent to v. Thus only vertices adjacent to v are the vertices of eccentricity one. Hence, $\deg(v) = |E_1(G)|$.

Also, for $w \notin D$ with $e(w) = 2$ there exist another vertex in V−D which is eccentric to w. Hence, $G - v$ is an eccentricity preserving sub graph of G.

Example 3.3:



Theorem 3.20: Let G be a unicentral graph with center x and $r(G) = 1$, $d(G) = 2$. Then $\gamma_{cee}(G) = 2$ if and only if G satisfies any one of the following conditions:

(i) $u, v \in E_2(G)$ such that {u, v} is a γ-set of $< E_2(G) >$ and $G - \{u, v\}$ is an eccentricity preserving sub graph of G.

(ii) There exists a pendent vertex $v \in V(G)$ such that $G - v$ is an eccentricity preserving sub graph of G.

Proof: Follows from the proof of the previous Theorem.

Theorem 3.21: Let G be a unicentral graph with more than four vertices and r(G) = 1, d(G) = 2. Then $\gamma_{cee}(G) = n-2$ if and only if there exists two non adjacent vertices of degree at most n−3 and exactly n−3 vertices of degree at most n−2 in G.

Proof: G has more than two eccentric vertices and there exist at least one vertex which has more than one eccentric vertex.

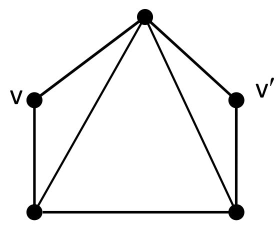
$\gamma_{cee}(G) = n-2$ implies that there exist a $\gamma_{cee}(G)$-set D with cardinality n − 2. Hence $|V - D| = 2$. By the definition of co-eccentric eccentric dominating set, if $v \in V-D$, there exists $v' \in V-D$ such v and v′ are eccentric to each other.

Also, G is an uni-central graph implies, $e(v) = e(v') = 2$. v and v′ are in V−D implies v has some non adjacent vertex in D and v′ has some non adjacent vertex in D. Hence, degree of $v \leq n-3$ and degree of $v' \leq n-3$.

Suppose there exists a vertex x in D which is not adjacent to v and is not adjacent to an element of D, then D − {x} is also a co-eccentric eccentric dominating set, which is a contradiction.

Hence, there exists no vertex in D which is not adjacent to v and not adjacent to a vertex of D. Similar result is true for v′ also. Thus if $x \in D$ with $e(x) = 2$, then degree of $x \leq n - 2$ and if $y \in V - D$, then degree of $y \leq n - 3$.

Example 3.4:



Theorem 3.22: Let G be a graph with p > 4, r(G) = 1, d(G) = 2 and G has two central vertices. Then $\gamma_{cee}(G) = n-2$ if and only if there exists no vertex of eccentricity two such that it is adjacent to a vertex of eccentricity two and not adjacent to at least two vertices of eccentricity two. Otherwise $\gamma_{cee}(G) < n-2$.

Proof: Let D be a γ_{cee}-set.

Case(i): Suppose V − D contains a central vertex.

There exists a vertex $u \in V - D$ such that e(u) = 1 in G. Let V − D = {u, u′} such that e(u′) = 1 or 2.

Subcase(a): e(u′) = 1.

Then $D = E_2(G)$ contains only vertices of eccentricity two. So, D is a γ_{cee}-set implies that there exists no proper subset S of D such that for $x \in D - S$, there exists $y, y' \in S$ and $y'' \in D - S$ such that $d(x, y) =1$, $d(x, y') = 2$ and $d(x, y'') = 2$. If this S exists, S is a γ_{cee}-set, which is a contradiction.

Subcase(b): $e(u') = 2$.

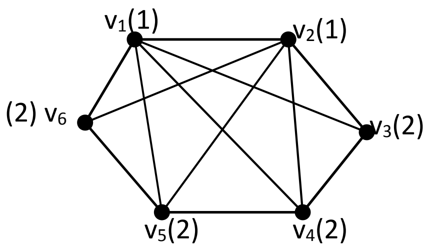
This case is not possible since u' has no eccentric vertex in $V - D$.

Case(ii): $V - D$ contains no central vertex.

Then D contains two central vertices, say x, y. But in this case $D - \{x\}$ or $D - \{y\}$ is also a co-eccentric eccentric dominating set. So this case is also not possible.

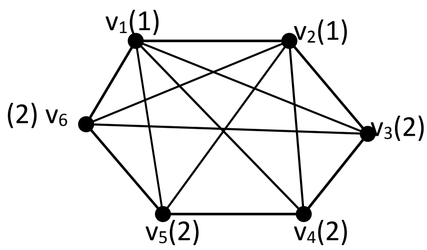
This proves the theorem.

Example 3.5:



In this graph G, $D = \{v_4, v_5\}$ is a co-eccentric eccentric dominating set. $\gamma_{cee}(G) \neq n-2$.

Example 3.6:



In this graph, $D = \{v_3, v_4, v_5, v_6\}$ is a co-eccentric eccentric dominating set, $\gamma_{cee}(G) = 4 = n-2$.

Corollary 3.22: Let G be a graph with $r(G) = 1$, $d(G) = 2$ and $|E_1(G)| > 2$, then $\gamma_{cee}(G) < n - 2$.

Theorem 3.23: Let G be a connected graph with n vertices. Then $\gamma_{cee}(G \circ 2K_1) \leq n + e(G)$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let v_i' and v_i'' be the pendent vertices adjacent to v_i in $G \circ 2K_1$ for $i = 1, 2, \ldots, n$.

Let $k = e(G)$ be the number of eccentric vertices of G and let them be $\{x_1, x_2, \ldots, x_k\}$. Then $V(G) \cup \{x_1', x_2', \ldots, x_k'\}$ is a co–eccentric eccentric dominating set.

Therefore, $\gamma_{cee}(G \circ 2K_1) \leq n + e(G)$.

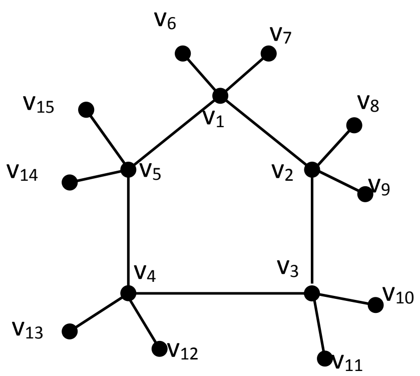
Note 3.2: If $H = G \circ 2K_1$ where G is a unique eccentric point graph, then $\gamma_{cee}(G) \leq 2n/3$ where $n = |V(G)|$.

Theorem 3.24: If H is any self centered unique eccentric point graph with m vertices and $G = H \circ 2K_1$, then $\gamma_{cee}(G) = 2n/3 = 2m$.

Proof: If H is any self-centered unique eccentric point graph then every vertex of H is an eccentric vertex. Hence, m is even and G has 3m vertices. Let $v_1, v_2, \ldots, v_m$ represent the vertices of H and $\{v_i', v_i''\}$ for $i = 1, 2, 3, \ldots, m$ be the vertices of m copies of $2K_1$. Then in G, v_i', v_i'' are adjacent to v_i and if v_j is the eccentric vertex of v_i in H, then v_i', v_i'' are eccentric vertices of v_j in G and v_j', v_j'' are the eccentric vertices of v_i. It is clear that $\{v_1, v_2, \ldots, v_m\} \cup \{v_1', v_2', \ldots, v_m'\}$ and $\{v_1, v_2, \ldots, v_m\} \cup \{v_1'', v_2'', \ldots, v_m''\}$ are minimum co-eccentric eccentric dominating sets of G. Hence, $\gamma_{cee}(G) = 2n/3 = 2m$.

Note 3.3: If H is self centered but not a unique eccentric vertex graph then $\gamma_{cee}(G)$, where $G = H \circ 2K_1$ need not be $2n/3$.

For example, when $G = C_5 \circ 2K_1$, $\gamma_{cee}(G) = 8 < 2n/3 = 10$.



$D = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_{12}\}$ is a co-eccentric eccentric dominating set.

$|D| = 8 < 2n/3 = 10$.

Conclusion: In this paper, we have defined and studied the new domination parameter $\gamma_{cee}(G)$. We have characterized graphs G with radius one and diameter two having $\gamma_{cee}(G) = 2$, n−2 and n.

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