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Connectivity and traversability of the Boolean graph $BG_1(G)$

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Abstract: *Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G).* $B_{G,\, NINC,\, \bar{K}_q}(G)$ *is a graph with vertex set* $V(G) \cup E(G)$ *and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge not incident to it in G. For simplicity, denote this graph by* $BG_1(G)$*, Boolean graph of G-first kind. In this paper, connectivity and traversability properties of* $BG_1(G)$ *and its complement are studied.*

Key words: *Boolean graph* $BG_1(G)$*.*

1. Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [9], Buckley and Harary [8]. We need the following definitions and theorems.

The *connectivity* $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. The *line connectivity or edge connectivity* $\lambda = \lambda(G)$ of a graph G is the minimum number of edges whose removal results in a disconnected graph.

A graph G is *n-connected* if $\kappa(G) \geq n$ and *n-edge connected* if $\lambda(G) \geq n$.

A regular graph with $\kappa = \delta$ for which the only minimum disconnecting sets of vertices consists of the neighborhoods of single vertex is called a *super-*κ*-graph*. Similarly, a regular graph with $\lambda = \delta$ for which each minimum sized disconnecting sets of edges isolates a single vertex is called a *super-* λ *-graph*.

Theorem 1.1 [8] For any graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

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Theorem 1.2 [8] (i) If G is a graph of diameter two, then $\lambda(G) = \delta(G)$.

(ii) If G has p vertices and $\delta(G) \geq \lfloor p/2 \rfloor$, then $\lambda(G) = \delta(G)$.

Theorem 1.3 [8] Among all graphs with p points and q lines, the maximum connectivity is zero when $q < p-1$ and is $\lfloor 2q/p \rfloor$, when $q \geq p-1$.

Theorem 1.4 [8] A graph is n-connected if and only if at least n point disjoint paths join every pair of points.

In a graph G any closed trial containing all vertices and edges of G is called an *Eulerian trail.* A graph is said to be *Eulerian* if it has an Eulerian trail.

Theorem 1.5 [8] A connected graph G is an Eulerian graph, if and only if degree of each vertex of G is even.

A graph G is called *Hamiltonian* if it has a spanning cycle. Any spanning cycle of G is called *Hamilton cycle*. A *Hamiltonian path* in G is a path, which contains every vertex of G. Clearly, every Hamiltonian graph is 2-connected.

Theorem 1.6 (Ore's Theorem) [7] Let G be a graph of order p ($\geq$3). For each pair of non-adjacent vertices v and w in G, if deg v+deg w $\geq$ p, then G is Hamiltonian.

Theorem 1.7 (Dirac's theorem) [7] Let G be a graph of order p ($\geq$ 3). If deg v $\geq$ p/2 for each vertex v in G, then G is Hamiltonian.

The *closure cl(G)* of a graph G is the smallest graph H such that (1) G is a spanning subgraph of H, and (2) $\deg_H v + \deg_H w \leq p$ for every pair of non-adjacent vertices v and w in H.

The closure cl(G) can be obtained from G by the recursive procedure of joining two vertices, whenever the sum of their degrees is at least p.

Theorem 1.8 (Bondy and Chavatal) [7] G is Hamiltonian if and only if cl(G) is Hamiltonian.

Theorem 1.9 [7] If cl(G) is a complete graph, then G is Hamiltonian.

Motivation: The Line graphs [6], Middle graphs [1,2], Total graphs [4] and Quasi-total graphs [23] are very much useful in computer networks. In analogous to line graph, total graph, middle graph and quasi-total graph, thirty-two graphs can be defined using different adjacency relations. Out of these operations, eight were already studied. Among the remaining twenty-four graph operations, two are defined and analyzed in [10,11,12], six are defined and studied in [3]. All the others have been defined and studied thoroughly. This is illustrated below.

G/ $\overline{G}$/K_p/ $\overline{K_p}$	Incident (INC)/ Non-incident (NINC)	$L(G)$/ $\overline{L(G)}$/K_q/ $\overline{K_q}$

Defining a new graph from a given graph by using the adjacency relation between two vertices or two edges and incident relationship between vertices and edges is known as Boolean operation. It defines new structure from the given graph and adds extra information of the original graph.

In Management and in social networks, the incident and non-incident relations of vertices and edges are used to define various networks. So these are very much applicable in socio-economical problems. In some cases, it is possible to retrieve the original graph from the Boolean graphs in polynomial time. So these graph operations may be used in graph coding or coding of some grouped signal. Also, it is possible to study the structure of original graphs through these Boolean graph operations. This motivates the study for the exploration of various Boolean operations and study of their structural properties.

In [3], the Boolean graph $BG_1(G)$ of a graph G is defined as follows. Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). $B_{G,\ NINC,\ \overline{K_q}}(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge not incident to it in G. For simplicity, denote this graph by $BG_1(G)$, Boolean graph of G-first kind.

$BG_1(G)$ has p+q vertices, p point vertices with degree q and q line vertices with degree p−2. $BG_1(G)$ is always bi-regular and is regular if and only if q = p−2; clearly, in this case G is disconnected. It is easy to determine that $BG_1(G)$ has q(p−1) edges and $\overline{BG_1}(G)$ has (q(q+1)/2)+(p(p−1)/2) edges. In this paper, connectivity and traversability properties of $BG_1(G)$ and its complement are studied.

2. Connectivity of $BG_1(G)$ and $\overline{BG_1}(G)$

In this section, connectivity of $BG_1(G)$ and $\overline{BG_1}(G)$ are discussed. Vertex connectivity and edge connectivity of $BG_1(G)$ and $\overline{BG_1}(G)$ are found out.

Proposition 2.1 Let G be a connected non-trivial (p, q) graph with p ≥ 3. Then $BG_1(G)$ is connected.

Proof: p ≥ 3 and G is connected implies G has at least two edges. Hence, in $BG_1(G)$ every line vertex is adjacent to some point vertex. Hence, G is connected implies $BG_1(G)$ is also connected.

Theorem 2.1 $BG_1(G)$ is disconnected if and only if $G = K_2, 2K_2, nK_1$ and $K_2 \cup nK_1$.
Proof: Let $BG_1(G)$ be a disconnected graph. If G has more than two edges, then $BG_1(G)$ is connected. Hence, $BG_1(G)$ is disconnected implies G has at most two edges. Therefore, there are two possibilities. Clearly, for $q = 1$, $G = K_2$ or $K_2 \cup nK_1$ and for $q = 2$, $G = 2K_2$ or $2K_2 \cup nK_1$. If G has at least two edges which are adjacent or if $p > 4$, $q \geq 2$, then $BG_1(G)$ is connected. Therefore, $G = K_2, 2K_2, K_2 \cup nK_1$. Proof of the converse is obvious.

Theorem 2.2 Let G be a connected (p, q) graph such that $BG_1(G)$ is connected. If $BG_1(G)$ has a cut point, then it is a point vertex only.
Proof: Let x be a cut vertex of $BG_1(G)$. Suppose x is not a point vertex, then x is a line vertex of $BG_1(G)$. x is a cut vertex implies that there exists two vertices u and v such that every path connecting u and v must contain x. Let a path connecting u and v be $u = u_1 u_2 \ldots u_{i-1}$ $x\, u_{i+1} \ldots u_n = v$. Since x is a line vertex, u_{i-1}, u_{i+1} must be point vertices and in G, x is not incident with u_{i-1} and u_{i+1}. Since G is connected there is at least one path joining u_{i-1} and u_{i+1} in G. Let it be $u_{i-1} v_1 \ldots v_k u_{i+1}$. Then $u = u_1 u_2 \ldots u_{i-1} v_1 v_2 \ldots v_k u_{i+1} \ldots u_n$ is a path in $BG_1(G)$ not containing the vertex x. This is a contradiction to our assumption. Therefore, x must be a point vertex.

Theorem 2.3 Let G and $BG_1(G)$ be connected graphs. Then $BG_1(G)$ has a cut vertex if and only if $G = K_3$ or $K_{1,2}$.
Proof: Assume $BG_1(G)$ has a cut vertex. By the previous proposition, this cut vertex is a point vertex. Suppose G is a connected graph having more than three vertices. Then two vertices in $BG_1(G)$ are connected by more than one path. That is $BG_1(G)$ is two connected. Therefore, G must contain at most three vertices if $BG_1(G)$ has a cut point. Hence, $G = K_3$ or $K_{1,2}$ as G and $BG_1(G)$ are connected.

Remark 2.1 If $q \geq 3$, then $BG_1(G)$ is two connected. If G is connected and $p \geq 4$, then $BG_1(G)$ is two connected. Let G be a graph with at least two adjacent edges, then $BG_1(G)$ is connected. If $G = 2K_2 \cup K_1$, then $BG_1(G)$ has a cut point.

Proposition 2.2 Let G be a (p, q) graph. Then $\kappa(BG_1(G)) \leq \lambda(BG_1(G)) \leq \min\{p-2, q\}$
Proof: By Theorem 1.1, $\kappa(BG_1(G)) \leq \lambda(BG_1(G)) \leq \delta(BG_1(G))$, Hence, $\kappa(BG_1(G)) \leq \lambda(BG_1(G)) \leq \min\{p-2, q\}$.

Lemma 2.1 Let G be a connected graph. Then in $BG_1(G)$ any two point vertices are connected by at least $p-2$ edge disjoint paths.
Proof: Take any two point vertices x and y in $BG_1(G)$.
Case 1: x and y are adjacent in G.

Assume that x and y have k common adjacent vertices in G. Denote them by $v_1, v_2, v_3, \ldots, v_k$. Then there are k+1 paths containing point vertices, namely, x y; x v_i y; i = 1, 2, 3, ..., k. Among the remaining $p-k-2$ point vertices let $u_1, u_2, \ldots, u_m$ be not adjacent to both x and y in G. Then there exists at least $m-1$ distinct edges e_i in G not incident with x and y (since G is connected). Thus, x e_i y, i = 1, 2, ..., $m-1$ is a path in $BG_1(G)$. Now, $BG_1(G)$ has $p-k-2-m$ remaining point vertices. Among these, some may be adjacent to x and others adjacent to y in G. Suppose, u is adjacent to x and not to y in G, then x u e_x y, where $e_x = xv_1 \in E(G)$ is a path in $BG_1(G)$, otherwise y u e_y x, where $e_y = yv_1 \in E(G)$ is a path in $BG_1(G)$. Thus, in this way, there are $p-k-2-m$ edge disjoint paths joining x and y in $BG_1(G)$. Therefore, totally there are at least $k+1+m-1+ p-k-2-m = p-2$ edge disjoint paths joining x and y in $BG_1(G)$.

Case 2: x and y are not adjacent in G.

As in case 1, the result can be proved.

Lemma 2.2 Let G be a connected graph. Then in $BG_1(G)$ any two line vertices are connected by at least $p-2$ edge disjoint paths.

Proof: Let x, y be two line vertices of $BG_1(G)$. In G, there are at least $p-4$ vertices not incident with both the edges x and y. Thus, there are $p-4$ edge disjoint paths x v y in $BG_1(G)$.

Case 1: x and y are adjacent in G.

Let $x = uv_1$ and $y = uv_2 \in E(G)$. Now, x v_2 v_1 y (if v_2 and v_1 are adjacent in G) or x v_2 u v_1 y is a path in $BG_1(G)$. Also, since $p \geq 4$ there exists another vertex v_3 such that x v_3 y is a path in $BG_1(G)$. Thus, totally there are at least $p-2$ edge disjoint paths joining x and y in $BG_1(G)$.

Case 2: x and y are not adjacent in G.

Let $x = u_1v_1$ and $y = u_2v_2 \in E(G)$. Now, x $v_2 \ldots u_1$ y (or x $v_2 \ldots v_1$ y or x $u_2 \ldots v_1$y or x $u_2 \ldots u_1$ y) is a path in $BG_1(G)$, where $v_2 \ldots u_1$ is a path in G. Also, since $G \neq 2K_2 \cup mK_1$ there exists at least one more edge e in G. Hence, if $e = u_1v_2$, x u_2 e v_1 y is a path in $BG_1(G)$. Thus, there are at least $p-2$ edge disjoint paths joining x and y in $BG_1(G)$. In all other cases, that is, if e = uv for u, v in V(G) also, the same result is true. This proves the lemma.

Lemma 2.3 Let G be a connected graph and let $u \in V(G)$ and $e \in E(G)$. Then in $BG_1(G)$, e and u are connected by at least $p-2$ edge disjoint paths.

Proof:

Case 1: e is incident with u in G.

Let $e = uv \in E(G)$. In $BG_1(G)$, e and u are not adjacent. Let $\deg_G u = m$ in G. Then there are $m-1$ paths of length two from u to e in $BG_1(G)$ (such as u u_i e). Since G is connected,

$p \geq p-1$. Hence, there are at least $p-m-1$ edges not incident with u, $p-m-1$ vertices not adjacent to u in G. So there are at least $p-m-1$ line vertices adjacent to u in $BG_1(G)$. Hence, there are $p-m-1$ paths such as u e_i v_j e in $BG_1(G)$. Thus, totally there are at least $(m-1)+(p-m-1) = p-2$ paths (edge disjoint) joining u and e in $BG_1(G)$.

Case 2: e is not incident with u in G.

In this case, e u is a path and there are at least $m-2$ paths u u_i e such that there are $m-1$ paths of length one or two joining u and e in $BG_1(G)$. So as in the previous case, the result can be proved. This proves the lemma.

Theorem 2.4 If G is a connected graph, then $\kappa(BG_1(G)) = p-2 = \lambda(BG_1(G))$.

Proof: By Lemmas 2.1, 2.2 and 2.3 $\kappa(BG_1(G)) \geq p-2$. From Proposition 2.2, $\kappa(BG_1(G)) \leq \lambda(BG_1(G)) \leq p-2$. Hence, the theorem is proved.

Theorem 2.5 If $G = G_1 \cup mK_1$, where G_1 is connected and $BG_1(G)$ is connected and $G_1 \neq K_2, 2K_2$, then $\kappa(BG_1(G)) = \min\{q, p-2\}$.

Proof: Proof is similar to the proof of the previous theorem.

Theorem 2.6 Let G be a disconnected graph with more than one non-trivial component. Then $\kappa(BG_1(G)) = \min\{q, p-2\} = \lambda(BG_1(G))$ if and only if $G \neq K_{1,n} \cup K_2, 2K_{1,2}$ and $G \neq 2K_2 \cup nK_1$

Proof: As in Lemma 2.1 and 2.2 it can be proved that in $BG_1(G)$ any two point vertices are connected by $k = \min\{p-2, q\}$ edge disjoint paths and any point vertex and line vertex are also connected by $k = \min\{p-2, q\}$ edge disjoint paths. Also, if e_1 and e_2 are two edges in the same component of G, then in $BG_1(G)$ they are connected by k disjoint paths. Now, take e_1, e_2 in different components of G. In $BG_1(G)$, e_1 and e_2 are both adjacent to $p-4$ point vertices. Thus, there are $p-4$ edge disjoint paths joining e_1 and e_2 in $BG_1(G)$. Consider $e_1 = u_1v_1$ and $e_2 = u_2v_2$. Any other path joining e_1 and e_2 in $BG_1(G)$, may contain the vertices u_1, v_1, u_2, v_2, e_1, e_2 and no other point vertex. The paths may have the form $e_2\, v_1\, e_3\, v_2\, e_1$; $e_2\, u_1\, e_3\, u_2\, e_1$; $e_2\, u_1\, e_4\, u_2\, e_1$. (Since $G \neq 2K_2$, e_1 and e_2 are always joined by $p-3$ edge disjoint paths in $BG_1(G)$.) Hence, e_1 and e_2 are connected by $p-2$ edge disjoint paths in $BG_1(G)$ only when there is at least four edges in G with the condition that e_3 is not incident with v_1, v_2 and e_4 is not incident with u_2 and u_1 or vice versa.-----I or at least three edges in G such that the third edge is not adjacent to both e_1 and e_2 in G. This is true for any two edges of G. Hence, G must satisfy any one of the following conditions: (1) G must have more than two non-trivial components. (2) G may have two non-trivial components with at least three edges, with the condition that for any two edges there exists another edge not adjacent to both of these edges. (3) G may have two non-trivial components with at least four edges satisfying I. (4) All the edges of G are in the same component of G.

Hence, $\kappa(BG_1(G)) = \min\{q, p-2\}$, only when $q \leq p-3$ or G satisfies the above conditions. That is $G \neq K_{1,n} \cup K_2$, $2K_{1,2}$ or $2K_2 \cup nK_1$.

Remark 2.2 (1) When $G = K_{1,n} \cup K_2$, $BG_1(G)$ is $p-3$ connected. If $G = 2K_2 \cup nK_1$, $BG_1(G)$ is $q-1$ connected.

(2) When $BG_1(G)$ is κ connected and $q = p-2$, $BG_1(G)$ is super-κ-graph and super λ -graph.

(3) If $G = G_1 \cup mK_1$, where G_1 is a connected graph on more than two vertices and $q = p-2$, then $BG_1(G)$ is super-κ-graph and super- λ -graph.

Next, connectivity of $\overline{BG_1(G)}$ can be studied.

Proposition 2.3 $\lambda(\overline{BG_1(G)}) = \min\{p-1, q+1\} \geq \kappa(\overline{BG_1(G)})$.

Proof: From Theorems 1.1, and 1.2, (i) $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

(ii) If G is a graph of diameter two, then $\lambda(G) = \delta(G)$.

(iii) If G has p points and $\delta(G) \geq [p/2]$, then $\lambda(G) = \delta(G)$. Consider $\overline{BG_1(G)}$. In $\overline{BG_1(G)}$, degree of a point vertex is $p-1$, degree of a line vertex is $q+1$ and $\overline{BG_1(G)}$ is always of diameter two. Hence, $\lambda(\overline{BG_1(G)}) = \delta(\overline{BG_1(G)}) = \min\{p-1, q+1\}$. Therefore, $\min\{p-1, q+1\} \geq \kappa(\overline{BG_1(G)})$.

Remark 2.3 When $G = K_2$, $\overline{BG_1(G)} = K_{1,2}$ and hence $\kappa(\overline{BG_1(G)}) = 1$, otherwise $\kappa(\overline{BG_1(G)}) \geq 2$.

Proposition 2.4 $\overline{BG_1(G)}$ has a cut point if and only if $G = K_2$.

Proof: Follows from Remark 2.3.

Theorem 2.7 Let G be a (p, q) graph. Then $\lambda(\overline{BG_1(G)}) = \min\{p-1, q+1\} = \kappa(\overline{BG_1(G)})$.

Proof: As in Theorem 2.4, it can be proved that $\kappa(\overline{BG_1(G)}) \geq \min\{p-1, q+1\}$. Hence, by Proposition 2.3, $\kappa(\overline{BG_1(G)}) = \min\{p-1, q+1\} = \lambda(\overline{BG_1(G)})$.

3. Traversability of $BG_1(G)$ and $\overline{BG_1}(G)$

In this section, traversability properties of the graphs $BG_1(G)$ and $\overline{BG_1}(G)$ are discussed. First theorem gives the characterization for $BG_1(G)$ and $\overline{BG_1}(G)$ to be Eulerian.

Theorem 3.1 Let G be a (p, q) graph. Then (1) $BG_1(G)$ is Eulerian if and only if p and q are even. (2) $\overline{BG_1}(G)$ is Eulerian if and only if p and q are odd.

Proof of (1): If $BG_1(G)$, degree of point vertex is q and degree of a line vertex is p−2. Hence, $BG_1(G)$ is Eulerian if and only if p−2 and q are even that is p and q are even.

Proof of (2): Similar to that of (a).

Now, the properties of $BG_1(G)$ related to Hamiltonian graphs are studied.

Proposition 3.1 If $BG_1(G)$ is Hamiltonian, then $q \leq p$.

Proof: In $BG_1(G)$, every line vertex is adjacent to point vertices only and no two line vertices are adjacent. Hence, if $BG_1(G)$ is Hamiltonian, then $q \leq p$.

Proposition 3.2 (1) $BG_1(K_{1,n})$ is not Hamiltonian.
(2) $BG_1(K_{1,n}+x)$ is not Hamiltonian.

Proof: Let $G = K_{1,n}$, then $BG_1(G)$ has n+1 point vertices and n line vertices and in $BG_1(G)$, the central vertex of G is not adjacent to any line vertices. Similarly, in $BG_1(K_{1,n}+x)$, p = n+1, q = n+1 and the central vertex of G is adjacent to only one line vertex x only. Hence, $BG_1(G)$ is not Hamiltonian.

Proposition 3.3 Let G be a graph with r(G) = 1, then $BG_1(G)$ is not Hamiltonian.

Proof: Let G be a (p, q) graph with r(G) = 1. If $BG_1(G)$ is Hamiltonian, then $q \leq p$. Therefore, G must have p or p−1 edges and since r(G) = 1, $G = K_{1,n}$ or $K_{1,n}+x$. But, by the previous proposition $BG_1(K_{1,n})$ and $BG_1(K_{1,n}+x)$ are not Hamiltonian. This proves the result.

Next theorem gives the necessary and sufficient condition for $BG_1(G)$ to be Hamiltonian.

Theorem 3.2 Let G be a connected (p, q) graph with $p \geq 4$ and r(G) > 1. Then $BG_1(G)$ is Hamiltonian if and only if $q \leq p$, that is, if q = p or p−1.

Proof: Suppose $BG_1(G)$ is Hamiltonian, $q \leq p$ by Proposition 3.1.

On the other hand, Let r(G) >1 and $q \leq p$. Let us prove the theorem by induction on p. When p = 4, since G is connected, q = 3 or 4. When q = 4, $G = C_4$ and clearly $BG_1(C_4)$

is Hamiltonian. When $q = 3$, $G = P_4$ and $BG_1(P_4)$ is also Hamiltonian. For $p > 4$, assume that the result is true for every G, with $p-1$ vertices, $r(G) \geq 2$ and $q \leq p-1$.

Now, consider a graph G, connected with p vertices, $r(G) > 1$ and $q = p$ or $p-1$. Since $q = p$ or $p-1$, any one of the following is true. (1) G is a cycle (2) G is uni-cyclic. (3) G is a tree.

Case 1: G is a cycle.

When G is a cycle C_n, all the vertices of $BG_1(G)$ lie on C_{2n}. Hence, $BG_1(G)$ is Hamiltonian.

Case 2: G is uni-cyclic.

In this case, G has at least one pendant vertex. Remove that pendant vertex $v_p \in V(G)$ such that v_p is adjacent to v_{p-1} and degree v_{p-1} is maximum. Let $e_p = v_{p-1}v_p \in E(G)$. The new graph G_1, obtained has $p-1$ vertices and $p-2$ or $p-1$ edges and $r(G_1) > 1$ or $G_1 = K_{1,n}+x$, $n = p-2$.

If $r(G_1) > 1$, then $BG_1(G_1)$ is Hamiltonian by induction. Therefore, $BG_1(G_1)$ has a cycle involving v_1', v_2', …, v_{p-1}', and e_1', e_2', …, e_{q-1}'. Since v_p is a pendant vertex it is not adjacent to v_1, v_2, …, v_{p-2}, and e_1, e_2, …, e_{q-1} are not incident with v_p. Hence, replace the path in the cycle involving $v_s'\ e_a'\ v_t'\ e_b'$ by $v_s'\ e_p'\ v_t'\ e_a'\ v_p'\ e_b'$. Thus, a Hamiltonian cycle in $BG_1(G)$ is obtained.

If $G_1 = K_{1,n}+x$, clearly $BG_1(G)$ is Hamiltonian.

Case 3: G is a tree.

In this case, G has at least two pendant vertices. So as in the previous case, remove a pendant vertex and obtain G_1, which is a tree with $r(G_1) > 1$ or $G_1 = K_{1,p-1}$. If $r(G_1) > 1$, it can be proved that $BG_1(G)$ is Hamiltonian as in the previous case and if $G_1 = K_{1,p-1}$, then $BG_1(G)$ is clearly Hamiltonian. This proves the theorem.

Next theorem proves that $\overline{BG_1(G)}$ is Hamiltonian when $G \neq K_2$.

Theorem 3.3 Let G be a (p, q) graph with $p \geq 3$, then $\overline{BG_1(G)}$ is Hamiltonian.

Proof: Degree of a point vertex is $p-1$ and line vertex is $q+1$ in $\overline{BG_1(G)}$. Consider any two non-adjacent vertices in $\overline{BG_1(G)}$. Suppose these vertices are point vertex u' and another line vertex e'. Then $\deg u' = p-1$ and $\deg e' = q+1$ in $\overline{BG_1(G)}$. Hence, $\deg u' + \deg e' = p+q$, number of vertices in $\overline{BG_1(G)}$. Join these two vertices, again consider a point vertex with degree $p-1$ and line vertex with degree $q+1$ which are not adjacent in $\overline{BG_1(G)}$ and join them. Proceeding like this, a new graph which contains $K_{p,q}$ as a spanning subgraph is obtained.--(A).

In this new graph, consider any two point vertices; they are of degree $\geq q$.

Case 1: $G \neq K_n$

Then there exist point vertices u' and v' not adjacent such that deg u'+deg $v' \geq p+q$. Join u' and v'. Proceeding in this way, a complete graph K_{p+q} is obtained. Hence, $\overline{B}G_1(G)$ is Hamiltonian.

Case 2: $G = K_n$ for $n \geq 3$. Then $q = n(n-1)/2 \geq n$. From step (A), the new graph obtained is Hamiltonian. Hence, when $p \geq 3$, $\overline{B}G_1(G)$ is Hamiltonian by Theorem 1.8.

Conclusion

Maximum connectivity graphs play an important role in the design of reliable networks. A reason for this is its relation to the reliability and vulnerability of large-scale computer and telecommunication networks. When designing a communication network, one not only wants to maximize the connectivity and edge-connectivity, but also to minimize the diameter as well as the number of edges. By minimizing the diameter, transmission times are kept small and the possibility of distortion due to a weak signal is avoided. Minimizing the number of edges will keep down the cost of building the network. In general one cannot simultaneously maximize κ and λ while minimizing $|E(G)|$ and diam(G). Let G be a connected graph with p vertices and q edges. Consider $H = G \cup tK_1$ such that $q = p+t-2$. So, $BG_1(H)$ is q regular and is q connected such that G is an induced subgraph of $BG_1(H)$. Hence these Boolean graphs can be treated as reliable networks. In Management and in social networks, the incident and non-incident relations of vertices and edges are used to define various networks. So these are very much applicable in socio-economical problems.

Other properties such as edge partition and domination parameters of $BG_1(G)$ are studied and submitted.

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