International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 6, No. 2, April – June 2015, pp. 41 - 61

Necessary Conditions for One Modulo Three Mean Labeling of Graphs

B. Gayathri and V. Prakash

PG and Research Department of Mathematics,

Periyar E.V.R. College, Tiruchirappalli-620 023, India.

E-mail: *maduraigayathri@gmail.com, prakashmaths86@gmail.com*

Abstract: *The concept of mean labeling was introduced by Somasundaram and Ponraj [25]. Different kinds of mean labeling are further studied by Gayathri and Gopi [9-17]. Swaminathan and Sekar [27] introduced the concept of one modulo three graceful labeling. As an analogue Jayanthi and Maheswari [21] introduced one modulo three mean labeling and proved that some standard graphs are one modulo three mean graphs. Motivated by the work of these authors, in this paper, we obtained some necessary conditions and properties for one modulo three mean labeling. Also, we prove the converse part of theorems obtained in [21], where only partial results are obtained.*

Key words: *Mean labeling, Mean graphs, One modulo three mean labeling, one modulo three mean graphs.*

1. Introduction

This paper deals with graph labeling. All graphs considered here are simple, finite and undirected. The terms not defined here are used in the sense of Harary [20].

The **graph labeling** is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a **vertex labeling (or an edge labeling**). By a (p,q) graph G, we mean a graph $G=(V,E)$ with $|V|=p$ and $|E|=q$.

Graph labeling was first introduced in the late 1960's. Many studies in graph labeling refer to Rosa's research in 1967 [24]. Rosa introduced a function f from a set of vertices in a graph G to the set of integers {0, 1, 2, 3...}, so that each edge xy is assigned the label $|f(x)-f(y)|$, with all labels distinct, Rosa called this labeling **β-valuation**. Independently, Golomb studied the same type of labeling and called this labeling as **graceful labeling**.

Labeled graphs serve as useful models for a broad range of applications such as x-ray crystallography, radar, coding theory, astronomy, circuit design and communication network addressing. Particularly, interesting applications of graph labeling can be found in [2-5].

Somasundaram and Ponraj [25-26] have introduced the notion of mean labeling of a (p,q) graph. A graph G is said to have a **mean labeling** if there is an injective function f from the vertices of G to {0, 1, 2, 3.., q} such that when each edge uv is labeled with

Received: 15 March, 2015; Revised: 22 May, 2015; Accepted: 10 June, 2015

$\frac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd then the resulting edge labels are distinct. A graph that admits a mean labeling is called a **mean graph**.

Vaidya [30-33] and et al. have investigated several families of mean graphs. Nagarajan [29] and et al. have found some new results on mean graphs. Ponraj, Jayanthi and Ramya extended the notion of mean labeling to super mean labeling in [23]. Gayathri and Tamilselvi [18-19, 28] extended super mean labeling to *k*–super mean, (*k*, *d*)–super mean, *k*–super edge mean and (*k*, *d*)–super edge mean labeling. Manikam and Marudai [22] introduced the concept of odd mean graph. Gayathri and Amuthavalli [1,6-8] extended this concept to *k*–odd mean and (*k*, *d*)–odd mean graphs.

Different kinds of mean labeling are studied by Gayathri and Gopi in [9-17]. Swaminathan and Sekar [27] introduced the concept of one modulo three graceful labeling. As an analogue, Jayanthi and Maheswari [21] introduced one modulo three mean labeling and proved that some standard graphs are one modulo three mean graphs. Motivated by the work of these authors, in this paper, we obtained some necessary conditions and properties for the one modulo three mean labeling. Also, we prove the converse part of theorems obtained in [21], where only partial results are obtained.

2. Properties

Definition 2.1

A graph $G=(p,q)$ is said to be **one modulo three mean graph** if there is a function f from the vertex set of G to the set $\{0,\ 1,\ 3,\ 4,\ 6,\ 7,\ \ldots\ 3q-3,\ 3q-2\}$ with f is one-one and f induces a bijection f^* from the edge set of G to the set $\{1,\ 4,\ 7,\ 10,\ldots,3q-5,\ 3q-2\}$ where $f^*(uv)=\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ and the function f is called as **one modulo three mean labeling** of G. Here, $f^*(uv)\equiv 1(\text{mod}\,3)$ for every edge uv in G.

Observation 2.2

(i) $3q-2 \equiv 1$ (mod 3) for all q

(ii) $3q-2$ is odd if q is odd and
$3q-2$ is even if q is even

(iii) $$3q-2 \equiv \begin{cases} 1(\text{mod}\,6) & \text{if } q \text{ is odd} \\ 4(\text{mod}\,6) & \text{if } q \text{ is even} \end{cases}$$

(iv) $3q-3$ is even if q is odd and
$3q-3$ is odd if q is even

(v) $$3q-3 \equiv \begin{cases} 0(\text{mod}\,6) & \text{if } q \text{ is odd} \\ 3(\text{mod}\,6) & \text{if } q \text{ is even} \end{cases}$$

Property 2.3

If a graph G is a one modulo three mean graph then 0 and 1 are vertex labels.

Proof

The induced edge label 1 can only be obtained by the adjacent vertex label pair (0,1). Therefore 0 and 1 are ought to be vertex labels.

Property 2.4

If a graph $G=(p,q)$ is a one-modulo three mean graph then $3q-3$ and $3q-2$ are ought to be the vertex labels.

Proof

By definition 2.1, the number $3q-2$ has to be an edge label. In order to have $3q-2$ as an induced edge label, the only possible vertex pair is $(3q-3, 3q-2)$. Hence $3q-3$ and $3q-2$ are vertex labels.

The following theorem is proved in [21, Theorem 2.2].

Property 2.5 [21]

Let $G=(p,q)$ be a one modulo three mean graph with one modulo three mean labeling f. Let t be the number of edges whose one vertex label is even and the other is odd. Then $\sum_{v\in V(G)} d(v)f(v)+t=q(3q-1)$ where $d(v)$ denotes the degree of vertex v.

Corollary 2.6

If $G=(p,q)$ is a one-modulo three mean graph with one modulo three mean labeling f, then $\sum_{v\in V(G)} d(v)f(v)\geq q^2$.

Proof

By Property 2.5

$$\sum_{v\in V(G)} d(v)f(v)+t=q(3q-1)$$

$$\sum_{v\in V(G)} d(v)f(v)=q(3q-1)-t$$

$$\sum_{v\in V(G)} d(v)f(v)\geq 3q^2-q-q \quad (\text{since } t\leq q)$$

$$\sum_{v\in V(G)} d(v)f(v) \geq q(3q-2)$$

$$\sum_{v\in V(G)} d(v)f(v) \geq q \cdot q \quad \text{(since q} \geq 1)$$

$$\sum_{v\in V(G)} d(v)f(v) \geq q^2$$

Property 2.7

Let $G=(p,q)$ be a l regular one modulo three mean graph with l even. Let t be the number of edges whose one vertex label is even and other is odd then t is even.

Proof

By Property 2.5

$$\sum_{v\in V(G)} d(v)f(v)+t = q(3q-1) \quad \text{...(1)}$$

G being l–regular

$$l\sum f(v)+t = q(3q-1)$$

$$t = q(3q-1)-l\sum f(v) \quad \text{...(2)}$$

If q is even then l is even implies right hand side of (2) is even.

If q is odd then $3q$ – 1 is even and l is even implies right hand side of (2) is even.

Hence t is even.

Property 2.8

Let $G=(p,q)$ be a one modulo three mean graph.

(i) If q is odd then 0,1, $3q$–2 cannot be the vertex labels of the cycle C_3 contained in G.

(ii) If q is even then 0,1, $3q$–3 cannot be the vertex labels of the cycle C_3 contained in G.

(iii) If q is odd then 1, $3q$–3, $3q$–2 cannot be the vertex labels of the cycle C_3 contained in G.

(iv) If q is even then 0, $3q$–3, $3q$–2 cannot be the vertex labels of the cycle C_3 contained in G.

Proof

Let G be a one modulo three mean graph. Let a_0, a_1, a_2 be the vertices of a cycle C_3 contained in G (see Figure 1.1).

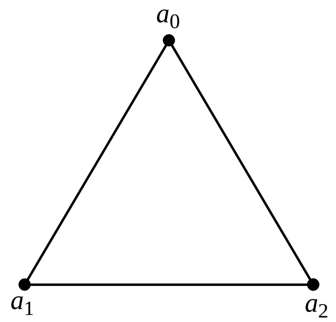

Figure 1.1

(i) Let q be odd then by Observation 2.2(ii), $3q-2$ is odd.

Suppose 0, 1, $3q-2$ be the vertex labels of the cycle C_3 contained in G (See Figure 1.2). Then without loss of generality, assume

$$f(a_0) = 0; \qquad f(a_1) = 1; \qquad f(a_2) = 3q-2$$

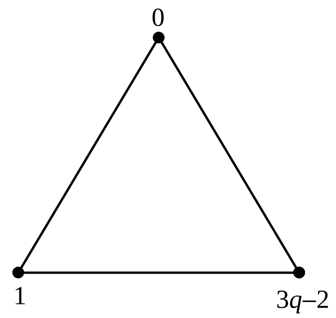

Figure 1.2

In this case, the induced edge labels are

$$f^*(a_0a_1) = 1$$

$$f^*(a_0a_2) = \frac{0+3q-2+1}{2} = \frac{3q-1}{2}$$

$$f^*(a_1a_2) = \frac{1+3q-2}{2} = \frac{3q-1}{2}$$

a contradiction to G is a one modulo three mean graph.

Hence if, q is odd, then 0, 1, $3q-2$ cannot be the vertex labels of the cycle C_3 contained in G.

(ii) Let q be even, then by Observation 2.2(iv), $3q-3$ is odd.

Suppose 0, 1, $3q-3$ be the vertex labels of the cycle C_3 contained in G (See Figure. 1.3). Then without loss of generality, assume

$$f(a_0) = 0; \qquad f(a_1) = 1; \qquad f(a_2) = 3q-3$$

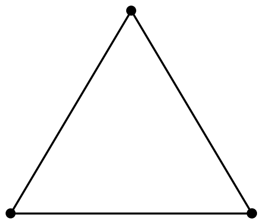

Figure 1.3

In this case the induced edge labels are

$$f^*(a_0a_1) = 1$$

$$f^*(a_1a_2) = \frac{1+3q-3}{2} = \frac{3q-2}{2}$$

$$f^*(a_0a_2) = \frac{0+3q-3+1}{2} = \frac{3q-2}{2}$$

a contradiction to G is a one modulo three mean graph.

Hence if, q is even then 0, 1, $3q$–3 cannot be the vertex labels of the cycle C_3 contained in G.

(iii) Let q be odd then by Observation 2.2 (ii) and (iv), $3q$–2 is odd and $3q$–3 is even

Suppose 1, $3q$–3, $3q$–2 be the vertex labels of the cycle C_3 contained in G (see Figure 1.4). Then without loss of generality, assume

$f(a_0) = 1$; $f(a_1) = 3q - 3$; $f(a_2) = 3q - 2$

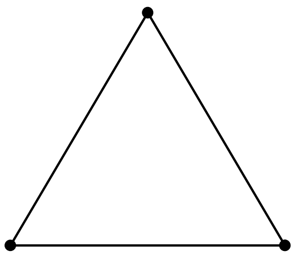

Figure 1.4

In this case, the induced edge labels are

$$f^*(a_1a_2) = \frac{3q-3+3q-2+1}{2} = \frac{6q-4}{2} = 3q-2$$

$$f^*(a_0a_1) = \frac{1+3q-3+1}{2} = \frac{3q-1}{2}$$

$$f^*(a_0a_2) = \frac{1+3q-2}{2} = \frac{3q-1}{2}$$

a contradiction to G is a one modulo three mean graph.

Hence if, q is odd then 1, $3q$–3, $3q$–2 cannot be vertex labels of cycle C_3 contained in G.

(iv) Let q be even then by Observation 2.2(ii) and (iv), $3q$–2 is even and $3q$–3 is odd.

Suppose 0, $3q$–3, $3q$–2 be the vertex labels of the cycle C_3 contained in G (See Figure 1.5). Then without loss of generality, assume

$f(a_0) = 0$; $f(a_1) = 3q - 3$; $f(a_2) = 3q - 2$

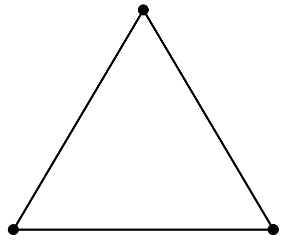

Figure 1.5

In this case, the induced edge labels are

$$f^*(a_1a_2) = \frac{3q-3+3q-2+1}{2} = \frac{6q-4}{2} = 3q-2$$

$$f^*(a_0a_1) = \frac{0+3q-3+1}{2} = \frac{3q-2}{2}$$

$$f^*(a_0a_2) = \frac{0+3q-2}{2} = \frac{3q-2}{2}$$

a contradiction to G is a one modulo three mean graph.

Hence if, q is even then 0, $3q$–3, $3q$–2 cannot be the vertex labels of the cycle C_3 contained in G.

3. Necessary Conditions

Theorem 3.1

Let G be a connected one modulo three mean graph. Let u be a vertex with label $f(u)$. Let v be any vertex adjacent to the vertex u with label $f(v)$

a) If $f(u) \equiv 0 \pmod 6$ then $f(v) \equiv 1 \pmod 6$

b) If $f(u) \equiv 1 \pmod 6$ then $f(v) \equiv 0$ (or) $1 \pmod 6$

c) If $f(u) \equiv 3 \pmod 6$ then $f(v) \equiv 4 \pmod 6$

d) If $f(u) \equiv 4 \pmod 6$ then $f(v) \equiv 3$ (or) $4 \pmod 6$

Proof

a) Let $f(u) \equiv 0 \pmod 6$

By the definition of one modulo three mean labeling, it is clear that for any v,

$$f(v) \not\equiv 2,5 \pmod 6 \qquad \text{... (1)}$$

Claim 1: $f(v) \not\equiv 3 \pmod 6$

On the contrary, suppose $f(v) \equiv 3 \pmod 6$, then $f(v) - 3 = 6k$ for some k, which implies $f(v) = 6k + 3$. Since $f(u) \equiv 0 \pmod 6$, $f(u) = 6k_1$ for some k_1.

$$\text{Therefore, } f^*(uv) = \frac{f(u)+f(v)+1}{2} \quad (\text{since } f(u)+f(v) \text{ is odd})$$

$$= \frac{6k_1+6k+3+1}{2}$$

$$= \frac{6k_1+6k+4}{2}$$

$$= 3k_1 + 3k + 2$$

$$= 3(k_1 + k) + 2$$

$$f^*(uv) \equiv 2 \pmod 3$$

a contradiction to f is a one modulo three mean labeling.

$$\text{Therefore, } f(v) \not\equiv 3 \pmod 6 \qquad \text{... (2)}$$

Claim 2: $f(v) \not\equiv 4 \pmod 6$

On the contrary, suppose $f(v) \equiv 4 \pmod 6$, then $f(v) - 4 \equiv 6k$ for some k, which implies $f(v) = 6k + 4$. Since $f(u) \equiv 0 \pmod 6$, $f(u) = 6k_1$ for some k_1.

$$\text{Therefore , } f^*(uv) = \frac{f(u)+f(v)}{2} \quad (\text{since } f(u)+f(v) \text{ is even})$$

$$= \frac{6k_1+6k+4}{2}$$

$$= 3(k_1+k)+2$$

$$f^*(uv) \equiv 2 \pmod 3$$

a contradiction to f is a one modulo three mean labeling.

$$\text{Therefore, } f(v) \not\equiv 4 \pmod 6 \qquad \text{... (3)}$$

Claim 3: $f(v) \not\equiv 0 \pmod 6$

On the contrary, suppose $f(v) \equiv 0 \pmod 6$, then $f(v) = 6k$ for some k, Since $f(u) \equiv 0 \pmod 6$, $f(u) = 6k_1$ for some k_1.

$$\text{Therefore, } f^*(uv) = \frac{f(u)+f(v)}{2} \quad (\text{since } f(u)+f(v) \text{ is even})$$

$$= \frac{6k_1+6k}{2}$$

$$= 3k_1 + 3k$$

$$f^*(uv) = 3(k_1 + k)$$

$$f^*(uv) \equiv 0 \pmod 3$$

a contradiction to f is a one modulo three mean labeling.

$$\text{Therefore, } f(v) \not\equiv 0 \pmod 6 \qquad \text{... (4)}$$

From equations 1 to 4 we have

$$f(v) \equiv 1 \pmod 6 \text{ which proves (a).}$$

b) Let $f(u) \equiv 1 \pmod 6$

Then $f(u) - 1 = 6k_1$ for some k_1, which implies $f(u) = 6k_1 + 1$

By the definition of one modulo three mean labeling, it is clear that for any v,

$$f(v) \not\equiv 2, 5 \pmod 6 \qquad \text{... (1)}$$

Claim 1: $f(v) \not\equiv 3 \pmod 6$

On the contrary, suppose $f(v) \equiv 3$ (mod 6), then $f(v) - 3 = 6k$ for some k, which implies $f(v) = 6k + 3$. Now $f(u) = 6k_1 + 1$

$$\text{Therefore, } f^*(uv) = \frac{f(u)+f(v)}{2} \quad (\text{since } f(u)+f(v) \text{ is even})$$

$$f^*(uv) = \frac{6k_1 + 1 + 6k + 3}{2}$$

$$= \frac{6k_1 + 6k + 4}{2}$$

$$f^*(uv) = 3(k_1 + k) + 2$$

$$f^*(uv) \equiv 2 \pmod 3$$

a contradiction to f is a one modulo three mean labeling.

$$\text{Therefore, } f(v) \not\equiv 3 \pmod 6 \qquad \text{... (2)}$$

Claim 2: $f(v) \not\equiv 4 \pmod 6$

On the contrary, suppose $f(v) \equiv 4$ (mod 6), then $f(v) - 4 = 6k$ for some k, which implies $f(v) = 6k + 4$. Now $f(u) = 6k_1 + 1$.

$$\text{Therefore, } f^*(uv) = \frac{f(u)+f(v)+1}{2} \quad (\text{since } f(u)+f(v) \text{ is odd})$$

$$f^*(uv) = \frac{6k_1 + 1 + 6k + 4 + 1}{2}$$

$$= \frac{6k_1 + 6k + 6}{2}$$

$$= \frac{6(k_1 + k + 1)}{2}$$

$$f^*(uv) = 3(k_1 + k + 1)$$

$$f^*(uv) \equiv 0 \pmod 3$$

a contradiction f is a one modulo three mean labeling.

Therefore, $f(v) \not\equiv 4(\text{mod}\,6)$ ... (3)

From equations (1) to (3) we have, $f(v) \equiv 0$ (or) 1 (mod 6) which proves (b).

c) Let $f(u) \equiv 3$ (mod 6)

Then $f(u) - 3 = 6k_1$ for some k_1, which implies $f(u) = 6k_1 + 3$

By the definition of one modulo three mean labeling, it is clear that for any v,

$f(v) \not\equiv 2,5(\text{mod}\,6)$ ... (1)

Claim 1: $\mathbf{f(v) \not\equiv 0(\text{mod}6)}$

On the contrary, suppose $f(v) \equiv 0(\text{mod}\,6)$, $f(v) = 6k$ for some k. Now $f(u) = 6k_1 + 3$.

$$\text{Therefore, } f^*(uv) = \frac{f(u)+f(v)+1}{2} \quad (\text{since } f(u)+f(v) \text{ is odd})$$

$$f^*(uv) = \frac{6k_1+3+6k+1}{2}$$

$$= 3(k_1 + k) + 2$$

$$f^*(uv) \equiv 2 \text{ (mod 3)}$$

a contradiction to f is a one module three mean labeling.

Therefore, $f(v) \not\equiv 0(\text{mod}\ 6)$ ... (2)

Claim 2: $\mathbf{f(v) \not\equiv 1(\text{mod}\,6)}$

On the contrary, suppose $f(v) \equiv 1(\text{mod}\ 6)$, then $f(v) - 1 = 6k$ for some k, which implies $f(v) = 6k + 1$. Now $f(u) = 6k_1 + 3$.

$$\text{Therefore, } f^*(uv) = \frac{f(u)+f(v)}{2} \quad (\text{since } f(u)+f(v) \text{ is even})$$

$$f^*(uv) = \frac{6k_1+3+6k+1}{2}$$

$$= 3(k_1 + k) + 2$$

$$f^*(uv) \equiv 2\text{(mod 3)}$$

a contradiction to f is a one modulo three mean labeling.

Therefore, $f(v) \not\equiv 1(\text{mod}\ 6)$ ... (3)

Claim 3: $\mathbf{f(v) \not\equiv 3(\text{mod}\,6)}$

On the contrary, suppose $f(v) \equiv 3(\text{mod}\ 6)$, then $f(v) - 3 = 6k$ for some k, which implies $f(v) = 6k + 3$. Now $f(u) = 6k_1 + 3$.

$$\text{Therefore, } f^*(uv) = \frac{f(u)+f(v)}{2} \quad (\text{since } f(u)+f(v) \text{ is even})$$

$$f^*(uv) = \frac{6k_1+3+6k+3}{2}$$

$$= 3(k_1 + k + 1)$$

$$f^*(uv) \equiv 0 \text{ (mod 3)}$$

a contradiction to f is a one modulo three mean labeling.

Therefore, $f(v) \not\equiv 3(\text{mod } 6)$ ... (4)

From Equations 1 to 4 we have

$f(v) \equiv 4$ (mod 6) which proves (c).

d) Let $f(u) \equiv 4$ (mod 6)

Then $f(u) - 4 = 6k_1$ for some k_1, which implies $f(u) = 6k_1 + 4$

By the definition of one modulo three mean labeling, it is clear that for any v,

$f(v) \not\equiv 2,5(\text{mod } 6)$ ... (1)

Claim 1: $f(v) \not\equiv 0(\text{mod } 6)$

On the contrary, suppose $f(v) \equiv 0(\text{mod } 6)$, then $f(v) = 6k$. Now $f(u) = 6k_1 + 4$

$$\text{Therefore, } f^*(uv) = \frac{f(u)+f(v)}{2} \quad (\text{since } f(u)+f(v) \text{ is even})$$

$$= \frac{6k_1+4+6k}{2}$$

$$= 3(k_1 + k) + 2$$

$$f^*(uv) \equiv 2 \text{ (mod 3)}$$

a contradiction to f is a one modulo three mean labeling.

Therefore, $f(v) \not\equiv 0(\text{mod } 6)$ ... (2)

Claim 2: $f(v) \not\equiv 1(\text{mod } 6)$

On the contrary, suppose $f(v) \equiv 1(\text{mod } 6)$, then $f(v) - 1 = 6k$ for some k, which implies $f(v) = 6k + 1$. Now $f(u) = 6k_1 + 4$

$$\text{Therefore, } f^*(uv) = \frac{f(u)+f(v)+1}{2} \quad (\text{since } f(u)+f(v) \text{ is odd})$$

$$f^*(uv) = \frac{6k_1+4+6k+1+1}{2}$$

$$= 3(k_1 + k + 1)$$

$$f^*(uv) \equiv 0 \text{ (mod 3)}$$

a contradiction to f is a one modulo three mean labeling.

Therefore, $f(uv) \not\equiv 1(\text{mod } 6)$ ... (3)

From equations 1 to 3 we have

$f(v) \equiv 3$ (or) 4 (mod 6) which proves (d).

Theorem 3.2

If G is a connected one modulo three mean graph then all its vertices receive labels as either 0 (or) 1 (mod 6).

Proof

Let G be a one modulo three mean graph. Then by Property 2.3, 0 and 1 are vertex labels. Now by Theorem 3.1 (a) and (b) all the vertex labels are of either 0 (or) 1 (mod 6).

Theorem 3.3

If $G = (p,q)$ is a connected one modulo three mean graph then q is odd.

Proof

Let $G = (p,q)$ be a one modulo three mean graph.

Then by Property 2.4, $3q - 2$ is a vertex label.

Claim : q is odd

Suppose q is even, then by Observation 2.2(iii), $3q - 2 \equiv 4 \pmod{6}$. a contradiction to Theorem 3.2.

Corollary 3.4

If G is a connected graph with q even then it is not a one modulo three mean graph.

Proof

Follows from Theorem 3.3.

Theorem 3.5

If $G = (p,q)$ is a tree of odd order then it is not a one modulo three mean tree.

Proof

Suppose $G = (p,q)$ be a one-modulo three mean tree of odd order then p is odd implies $q = p - 1$ is even, a contradiction to Theorem 3.3.

Hence the theorem.

Theorem 3.6

If G is a unicyclic connected graph with q even then it is not a one modulo three mean graph.

Proof

Follows by Corollary 3.4.

Theorem 3.7

If G is a connected one modulo three mean graph of odd size and then $\Delta \leq \frac{q+1}{2}$. Where Δ is the maximum degree of a vertex in G.

Proof

Let G be a connected one modulo three mean graph of odd size then by Theorem 3.3, q is odd and by Theorem 3.2, all the vertex labels are congruent to either 0 (or) 1 (mod 6).

i.e., G contains vertex labels from the set {0, 6, 12, ..., $3q-3$} or {1, 7, 13, ..., $3q-2$}.

Hence,

$$\Delta \leq \frac{3q-2-1}{6}+1 \qquad \text{(or)} \qquad \Delta \leq \frac{3q-3-0}{6}+1$$

$$\Delta \leq \frac{3q-3+6}{6} \qquad\qquad \Delta \leq \frac{3q-3+6}{6}$$

$$\Delta \leq \frac{q+1}{2} \qquad\qquad \Delta \leq \frac{q+1}{2}$$

Thus $\Delta \leq \frac{q+1}{2}$.

4. Necessary and Sufficient condition for Certain family of One Modulo Three Mean Graphs

In [21], one modulo three mean graph is introduced and some standard graphs are verified for one modulo three meanness.

In [21 Theorem 3.1], it has been proved that the path P_n is a one modulo three mean graph if n is even.

In the following theorem, we prove that n is even is also a sufficient condition.

Theorem 4.1

The path P_n is not a one modulo three mean graph if n is odd.

Proof

Let n be odd. On the contrary, assume that P_n is a one modulo three mean graph. Then by Theorem 3.3, $q = |E(P_n)| = n-1$ is odd, which implies n is even, a contradiction to n is odd.

Now, we have the stronger version as given under.

Corollary 4.2

The path P_n is a one modulo three mean graph if and only if n is even.

In [21, Theorem 3.2], it has been proved that the star graph $K_{1,n}$ is a one modulo three mean graph if and only if $n = 1$.

We now establish this theorem with the help of properties obtained in section 2 and necessary conditions of section 3.

Theorem 4.3

The star graph $K_{1,n}$ is a one modulo three mean graph if and only if $n = 1$.

Proof

Necessary part

Let $n \neq 1$.

Case 1: *n* is even

To prove that $K_{1,n}$ is not a one modulo three mean graph.

We know that the number of edges of $K_{1,n} = q = n$. Suppose $K_{1,n}$ is a one modulo three mean graph, then $q = n$ is even.

a contradiction to Theorem 3.3.

Case 2: *n* is odd

By Properties 2.3 and 2.4, any one modulo three mean graph needs atleast 2 non-adjacent edges to receive the edge labels 1 and $3q - 2$ respectively, which contradicts the structure of $K_{1,n}$.

Sufficiency part

When $n = 1$, the result is immediate.

Hence the theorem.

In [21, Theorem 3.3], the following is proved. "The caterpillar obtained by attaching n pendant edges to each vertex of the path P_m is a one modulo three mean graph if $m \equiv 0$ (mod 2)".

In [21], $m \equiv 1$ (mod 2) is left open in the above said theorem.

We now discuss this result in one of the two cases.

Theorem 4.4

The caterpillar G obtained by attaching n pendant edges to each vertex of the path P_m is not a one modulo three mean graph if $m \equiv 1$ (mod 2) and n is even.

Proof

We now observe that G has $m - 1 + mn$ edges. When n is even and m is odd, clearly $q = m - 1 + mn$ is even. Suppose G is a one modulo three mean graph then it contradicts Theorem 3.3.

Theorem 4.5

The caterpillar G obtained by attaching n pendant edges to each vertex of the path P_m is a one modulo three mean graph if m is odd and n = 1.

Proof

We now observe that G has $m - 1 + mn$ edges. When n =1, G is the comb graph P_m^+. So, by Theorem 3.4 of [21], G is a one modulo three mean graph.

Remark 4.6

By using lengthy arguments, we can achieve that the caterpillar G obtained by attaching 3 pendant edges to each vertex of the path P_m is not a one modulo three mean graph whose proof we omit. In general, when $m \equiv 1 \pmod 2$ and n is odd, the one modulo three mean labeling of G of Theorem 4.4 is left open.

In [21, Theorem 3.5], it has been proved that $B_{m,n}$ is a one modulo three mean graph if and only if $m = n$ and the proof is lengthy.

We now give a short proof using necessary conditions.

Theorem 4.7

The Bistar $B_{n,n}$ is a one modulo three mean graph for all n.

Proof

Let $\{u_i, v_i, 1 \leq i \leq n\}$ be the vertices and $\{e_i, 1 \leq i \leq 2n+1\}$ be the edges of $B_{n,n}$.

First we label the vertices as follows:

Define $f : V(G) \rightarrow \{0, 1, 3, 4, ..., 3q-3, 3q-2\}$

$f(u) = 0$

$f(v) = 6n + 1$

$f(u_i) = 6i - 5, \quad$ for $1 \leq i \leq n$

$f(v_i) = 6i, \quad$ for $1 \leq i \leq n$.

Then the induced edge labels are

$f^*(e_i) = 3i - 2, \quad$ for $1 \leq i \leq 2n + 1$.

The above defined function f provides one modulo three mean labeling of the graph $B_{n,n}$. One modulo three mean labeling of $B_{5,5}$ is given in Figure 4.1.

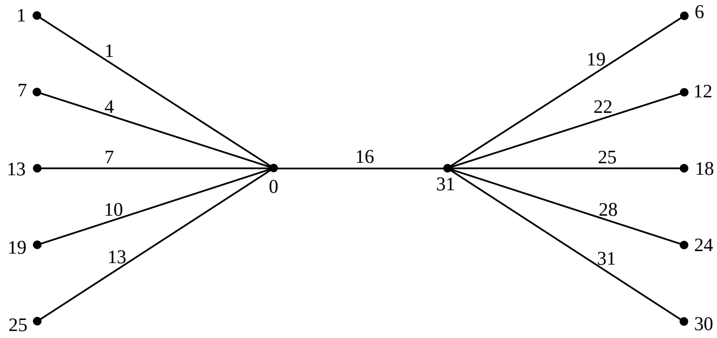

Figure 4.1: OMTML of $B_{5,5}$

Theorem 4.8

If $m \neq n$ then $B_{m,n}$ is not a one modulo three mean graph.

Proof

Suppose on the contrary, $B_{m,n}$ is a one modulo three mean graph. Without loss of generality assume $m > n$. The number of edges in $B_{m,n} = q = m + n + 1$.

Case 1: either *m* (or) *n* is odd

Then q is even, which contradicts Theorem 3.3.

Case 2: Both *m*, *n* are odd (or) even

Without loss of generality, let m and n be odd. Then since $m > n$ and both are odd, it follows that $m \geq n + 2$.

In this case $q = m + n + 1$ is odd. Therefore by Theorem 3.7.

$$\Delta(G) \leq \frac{q+1}{2} \qquad \text{... (1)}$$

Here $G = B_{m,n}$; $\Delta = m + 1$; $q = m + n + 1$

from (1) we have

$$m + 1 \leq \frac{m+n+1+1}{2}$$

$$2m + 2 \leq m + n + 2$$

$$2m \leq m + n$$

$$m \leq n$$

which is a contradiction to $m > n$.

Theorem 4.9

$B_{m,n}$ is a one modulo three mean graph if and only if $m = n$.

Proof

Follows from Theorem 4.7 and Theorem 4.8.

In [21, Theorem 3.6], it has been proved the T_p-tree with even number of vertices is a one modulo three mean graph.

In the following theorem, we prove that the T_p-tree with odd number of vertices is not a one modulo three mean graph.

Theorem 4.10

A T_p-tree with odd number of vertices is not a one modulo three mean graph.

Proof

Let the T_p-tree has odd number of vertices, that is p is odd. Then $q = p - 1$ is even. Suppose T_p-tree is a one modulo three mean graph, then it is a contradiction to Theorem 3.3.

Hence we have,

Theorem 4.11

A T_p-tree is a one modulo three mean graph if and only if p is even.

Proof

Follows from Theorem 4.10 and Theorem 3.6 of [21].

In [21, Theorem 4.1], the cycle C_n is proved as one modulo three mean graph when $n \equiv 1 \pmod 4$ and all other modulo's are left open.

In the theorem to follow, we discuss the remaining cases.

Theorem 4.12

If $n \equiv 0$ (or) 2 (mod 4), the cycle C_n is not a one modulo three mean graph.

Proof

If $n \equiv 0$ (or) 2 (mod 4), then $q = \left|E\left(C_n\right)\right| = n$ is even. Suppose C_n is a one modulo three mean graph, then it contradicts Theorem 3.3.

Remark 4.13

With lengthy arguments, we can prove that C_7 is not a one modulo three mean graph whose proof we omit. We observe that C_n, $n \equiv 3 \pmod 4$ is not a one modulo three mean graph, but whose short proof is left open.

In [21, Theorem 4.2], it has been proved that the ladder graph $L_n = P_n \times P_2$ is a one modulo three mean graph if n is odd.

In the following theorem, we prove that the ladder graph $L_n = P_n \times P_2$ is not a one modulo three mean graph if n is even.

Theorem 4.14

The ladder graph $L_n = P_n \times P_2$ is not a one modulo three mean graph if n is even.

Proof

Let n be even. Then $q = \left|E\left(L_n\right)\right| = 3n-2$ is even. Suppose $L_n = P_n \times P_2$ is a one modulo three mean graph, then it contradicts Theorem 3.3.

Hence we have,

Theorem 4.15

The ladder graph $L_n = P_n \times P_2$ is a one modulo three mean graph if and only if n is odd.

Proof

Follows from Theorem 4.14 and Theorem 4.2 of [21].

In [21, Theorem 4.3] it has been proved that $K_{1,n} \times K_2$ is a one modulo three mean graph if n is even.

In the following theorem, we prove that $K_{1,n} \times K_2$ is not a one modulo three mean graph if n is odd.

Theorem 4.16

The graph $K_{1,n} \times K_2$ is not a one modulo three mean graph if n is odd.

Proof

Let n be odd. Then $q = \left|E\left(K_{1,n} \times P_2\right)\right| = 3n+1$ is even. Suppose $K_{1,n} \times K_2$ is a one modulo three mean graph, then it contradicts Theorem 3.3.

In [21, Theorem 4.4] it has been proved that K_n is a one modulo three mean graph if and only if $n \leq 2$. But we provide very short proof in the following theorem.

Theorem 4.17

The graph K_n is a one modulo three mean graph if and only if $n \leq 2$.

Proof

Assume $n \geq 3$. When $n = 3$, Clearly C_3 is not a one modulo three mean graph.

Let $n > 3$.

Claim: K_n is not a one modulo three mean graph

On the contrary, suppose K_n is a one modulo three mean graph.

Case 1: n is even

Then $q = \left|E\left(K_n\right)\right| = \dfrac{n(n-1)}{2} = \text{even}$ which is a contradiction to Theorem 3.3.

Case 2: *n* is odd

By Property 2.8(i), 0, 1 and $3q - 2$ cannot be the vertex labels of any cycle C_3 contained in G. But in K_n ($n > 3$), every edge is an edge of a cycle C_3, which contradicts the Properties 2.3 and 2.4, that 0, 1 and $3q - 2$ are ought to be the vertex labels of a one modulo three mean graph.

Conversely, if $n \leq 2$, then K_1 and K_2 are clearly one modulo three mean graphs.

References:

[1] K. Amuthavalli, Graph labeling and it's Application's some generalizations of odd mean labeling, Ph.D. Thesis, Mother Theresa Women's University, Kodaikanal, July (2010).

[2] G.S. Bloom and S.W. Golomb, Applications of numbered undirected graphs, Proc. IEEE, 65 (1977) 562-570.

[3] G.S. Bloom and S.W. Golomb, Numbered complete graphs, unusual rulers and assorted applications, in Theory and Applications of Graphs, Lecture Notes in Math., 642, Springer-Verlag, New York (1978) 53-65.

[4] G.S. Bloom and D.F. Hsu, On graceful digraphs and a problem in network addressing, Congressus Numerantium, 35 (1982) 91-103.

[5] J.A. Gallian, A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, 17 (2014) #DS6.

[6] B. Gayathri and K. Amuthavalli, k—odd mean labeling of Crown graphs, International Journal of Mathematics and Computer Science, 2(3) (2007) 253-259.

[7] B. Gayathri and K. Amuthavalli, (k,d)—odd mean labeling of some graphs, Bulletin of Pure and Applied Sciences, 26E(2) (2007) 263-267.

[8] B. Gayathri and K. Amuthavalli, k—odd mean labeling of $\langle K_{1,n} : K_{1,m} \rangle$, Acta Ciencia Indica, 34(2) (2008) 827-834.

[9] B. Gayathri and R. Gopi, k—even mean labeling of $D_{m,n} @ C_n$, International Journal of Engineering Sciences, Advanced Computing and Bio-Technology, Vol. 1, No. 3, July-September 2010, P. No. 137-145.

[10] B. Gayathri and R. Gopi, k—even mean labeling of $D_{m,n}$, Acta Ciencia Indica, Vol. XXXVII, No. 2, 291-300, 2011.

[11] B. Gayathri and R. Gopi, k—even mean labeling of $C_n \cup P_m$, Elixir International Journal of Applied Sciences, No. 36, 2011, P.No. 3102-3105.

[12] B. Gayathri and R. Gopi, k—even mean labeling of $T_{n,m,t}$, International Journal of Engineering Science, Advanced Computing and Bio-Technology, Vol. 2, No. 2, April-June 2011, P. No. 77-87.

[13] B. Gayathri and R. Gopi, International Conference on Mathematics and Computer Science, k—even mean labeling of C_m e $\overline{K}_n$, Loyola College, Chennai, January 7-8, 2011, Proc. P. No. 519-524.

[14] B. Gayathri and R. Gopi, (*k*,*d*)−even mean labeling of P_m e nK_1 , International Journal of Mathematics and Soft Computing, Vol. 1, No. 1, 17-23, August 2011.

[15] B. Gayathri and R. Gopi, *k*−even mean labeling of some graphs, Heber International Conference on Applications of Mathematics and Statistics, Bishop Heber College (Autonomous), January 5-7, 2012 Proc. P. No. 420-425.

[16] B. Gayathri and R. Gopi, *k*−even mean labeling of some trees, International Conference on Mathematics in Engineering and Business Management, Stella Mary's College (Autonomous), Chennai, March 9-10, 2012, Proc. P. No. 99-102.

[17] R. Gopi, A Study on Different kinds of Mean labeling, Ph.D. Thesis, Bharathidasan University, Trichy, February (2013).

[18] B. Gayathri, M. Tamilselvi and M. Duraisamy, (*k*,*d*)−super mean labeling of some graphs, International Journal of Mathematics and Computer Science, 2(3) (2007) 245-252.

[19] B. Gayathri and M. Tamilselvi, *k*−super mean labeling of some trees and cycle related graphs, Bulletin of Pure and Applied Sciences, Volume 26E(2) (2007) 303-311.

[20] F. Harary, Graph Theory, Addison Wesley Reading Massachusetts, 1972.

[21] P. Jeyanthi and A. Maheswari, One Modulo Three Mean Labeling of Graphs, Amer. J. Appl. Math. and Stat., 2(5) (2014) 302-306.

[22] K. Manickam and M. Marudai, Odd mean labelings of graphs, Bulletin of Pure and Applied Sciences, 25E(1) (2006) 149-153.

[23] R. Ponraj, P. Jayanthi and D. Ramya, On super mean graphs of order $\leq$ 5, Bulletin of Pure and Applied Sciences, 25E (2006) 143-148.

[24] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod Paris (1967) 349-355.

[25] S. Somasundaram and R. Ponraj, Mean labelings of graphs, National Academy Science Letter, 26 (2003)10-13.

[26] S. Somasundaram and R. Ponraj, Non-existence of mean labeling for a wheel, Bull. Pure and Appl. Sciences (Mathematics & Statistics), 22E (2003) 103-111.

[27] V. Swaminathan and C. Sekar, Modulo three graceful graphs, Proceed. National Conference on Mathematical and Computational Models, PSG College of Technology, Coimbatore, 2001, 281-286.

[28] M. Tamilselvi, A Study in Graph Theory-Generalization of super mean labeling, Ph.D. Thesis, Vinayaka Mission University, Salem, August (2011).

[29] R. Vasuki and A. Nagarajan, Further results on mean graphs, Scientia Magna, 6(2) (2010) 26-39.

[30] S.K. Vaidya and Lekha Bijukumar, Mean labeling for some new families of graphs, Journal of Pure and Applied Sciences, 1(18) (2010) 115-116.

[31] S.K. Vaidya and Lekha Bijukumar, Some new families of mean graphs, Journal of Mathematics Research, 2(3) (2010) 169-176.

[32] S.K.Vaidya and Lekha Bijukumar, New mean graphs, International J. Math. Combin., 3 (2011) 107-113.

[33] S.K. Vaidya and Kanani, Some new mean graphs, International Journal of Information Science and Computer Mathematics, 1(1) (2010) 73-80.