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Boolean graph $BG_1(G)$ of a graph G

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Abstract: *Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G).* $B_{G, NINC, \bar{K}_q}(G)$ *is a graph with vertex set* $V(G) \cup E(G)$ *and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge not incident to it in G. For simplicity, denote this graph by* $BG_1(G)$*, Boolean graph of G-first kind. In this paper, some properties of* $BG_1(G)$ *including eccentricity properties and covering numbers are studied.*
Keywords: *Boolean graph* $BG_1(G)$*.*

1. Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [11], Buckley and Harary [8].

The *girth* of a graph G, denoted g(G), is the length of a shortest cycle (if any) in G; the *circumference* c(G) is the length of any longest cycle. The distance d(u, v) between two vertices u and v in G is the minimum length of a path joining them if any; otherwise $d(u, v) = \infty$. A shortest u-v path is called a *u-v geodesic*. A graph G is *geodetic*, if for every pair of vertices (u, v) there exists a unique shortest path connecting them in G.

Let G be a connected graph and u be a vertex of G. The *eccentricity* e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max \{d(u, v) : u \in V\}$. The *radius* r(G) is the minimum eccentricity of the vertices, whereas the *diameter* diam(G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. v is a central vertex if e(v) = r(G). The *center* C(G) is the set of all central vertices. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a *peripheral vertex* if e(v) = diam(G). The *periphery* P(G) is the set of all such vertices. For a vertex v, each vertex at distance e(v) from v is an eccentric node of v. A graph is *self-centered* if every vertex is in the center. Thus, in a self-centered graph G all nodes have the same eccentricity, so r(G) = diam(G).

An edge $uv \in E(G)$ is a *dominating edge* of G, if all the vertices of G other than u and v are adjacent to either u or v.

A vertex (point) and an edge are said to *cover* each other, if they are incident. A set of vertices, which cover all the edges of a graph G is called a (*vertex*) *point cover of G*, while a set of lines (edges), which covers all the vertices is a *line cover*. The smallest number of

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points in any point cover for G is called its *point covering number* and is denoted (respectively α_1) elements. A set of points in G is *independent*, if no two of them are adjacent. The largest number of points in such a set is called the *point independence number* of G and is by $\alpha_o(G)$ or α_o. Similarly, $\alpha_1(G)$ or α_1 is the smallest number of lines in any line cover of G and is called its *line covering number*. A point cover (line cover) is called *minimum*, if it contains α_o denoted by $\beta_o(G)$ or β_o. Analogously, an independent set of lines (matching) of G has no two of its lines adjacent and the maximum cardinality of such a set is the *line independence number* $\beta_1(G)$ or β_1. A set of independent edges covering all the vertices of a graph G is called a *1-factor* or a *perfect matching* of G.

A *coloring* of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ is defined to be the minimum n for which G has n coloring.

The minimum number of complete subgraphs of G needed to cover the vertices of G is known as the *clique cover number of G* and is denoted $\theta(G)$. The maximum number of mutually adjacent vertices of G, in other words the size of the largest complete subgraphs of G is known as the *clique number of G* and is denoted $\omega(G)$.

A graph G is *Berge* [25] if it does not contain odd cycles of length at least five or their respective complement as induced subgraphs. A graph is *perfect* if $\beta_o(H) = \theta(H)$ for every induced subgraph H of G. This implies that $\omega(H) = \chi(H)$ for every induced subgraph H. Clearly, every bipartite graph is perfect.

Theorem 1.1 (Gallai) [9]: For any connected graph G, $\alpha_o+\beta_o = p = \alpha_1+\beta_1$.
Theorem 1.2 [9]: $\beta_o(L(G)) = \beta_1(G)$, $\alpha_o(L(G)) = q-\beta_1(G)$ and $\alpha_1(L(G)) = \lceil q/2 \rceil$.
Theorem 1.3 [11]: $\chi(G) \leq 1 + \Delta(G)$.
Theorem 1.4 [9]: For any simple graph G, $\chi(G) = \theta(\overline{G})$ and $\beta_o(G) = \omega(\overline{G})$.
Theorem 1.5 (Hayward) [25]: If G is Berge and if it contains either a cycle of length at least 6 or its complement as an induced subgraph, then G is perfect.

Jin Akiyama and Kiyoshi Ando [3] characterized the graphs G, which are self-centered with diameter two such that $\overline{G}$ is also self-centered with diameter 2.

Lemma 1.1 [3]: Let both G and $\overline{G}$ be connected and v be a point of G. If $e_G(v) \geq 3$, then $e_{\overline{G}}(v) = e_G(v) = 2$.

Corollary: If G is self-centered with diameter $d \geq 3$, then $\overline{G}$ is self-centered with diameter 2.

An edge $uv \in E(G)$ is a *dominating edge* of G, if all the vertices of G other than u and v are adjacent to either u or v.

Theorem 1.6 [3]: The following three statements are equivalent.

(1) Both G and $\overline{G}$ are self-centered with diameter two.

(2) G is self-centered with diameter two having no dominating edge.

(3) Neither G nor $\overline{G}$ contains a dominating edge.

Motivation: The Line graphs, Middle graphs, Total graphs and Quasi-total graphs are very much useful in computer networks. In analogous to line graph [6], total graph [4], [5], middle graph [1, 2] and quasi-total graph [26], thirty-two graphs can be defined using different adjacency relations. Out of these operations, eight were already studied. Among the remaining twenty-four graph operations, two are defined and analyzed here. All the others have been defined and studied [7, 12] thoroughly and submitted. This is illustrated below.

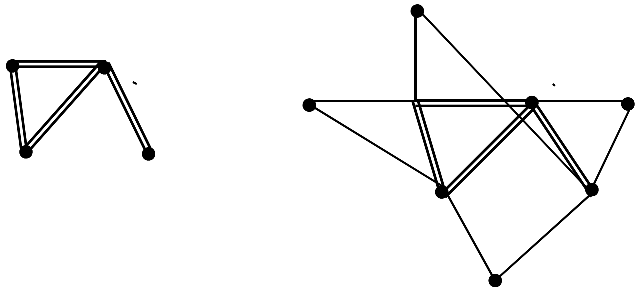
G/ $\overline{G}$/K_p/ $\overline{K_p}$	Incident (INC)/ Non-incident (NINC)	$L(G)$/ $\overline{L(G)}$/K_q/ $\overline{K_q}$

Defining a new graph from a given graph by using the adjacency relation between two vertices or two edges and incident relationship between vertices and edges is known as Boolean operation. It defines new structure from the given graph and adds extra information of the original graph.

In Management and in social networks, the incident and non-incident relations of vertices and edges are used to define various networks. So these are very much applicable in socio-economic problems. In some cases, it is possible to retrieve the original graph from the Boolean graphs in polynomial time. So these graph operations may be used in graph coding or coding of some grouped signal. Also, it is possible to study the structure of original graphs through these Boolean graph operations. This motivates the study for the exploration of various Boolean operations and study of their structural properties.

Now, we define the Boolean graph $BG_1(G)$ of a graph G[7]. Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). $B_{G,\ NINC,\ \overline{K_q}}(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge not incident to it in G. For simplicity, denote this graph by $BG_1(G)$, Boolean graph of G-first kind. The vertices of $BG_1(G)$, which are in V(G) are called point vertices and vertices in E(G) are called line vertices. $V(BG_1(G)) = V(G) \cup E(G)$, $E(BG_1(G)) = [E(\overline{T}(G)) - (E(\overline{L}(G)) \cup E(\overline{G}))] \cup E(G)$.

$BG_1(G)$ has p+q vertices, p point vertices with degree q and q line vertices with degree p-2. $BG_1(G)$ is always bi-regular and is regular if and only if q = p-2; clearly, in this case G is disconnected. It is easy to determine that $BG_1(G)$ has q(p-1) edges and $\overline{\mathbf{BG_1}(G)}$ has (q(q+1)/2)+(p(p-1)/2) edges. Also it is immediate from the definition of $BG_1(G)$ that G and $\overline{K_q}$ are induced subgraphs of $BG_1(G)$. Figure 1.1 depicts the formation of the graph $BG_1(G)$.



G $\qquad$ $BG_1(G)$

Figure 1.1

2. Properties of $BG_1(G)$ and $\overline{BG_1(G)}$

In this section, properties related to girth, geodeticity, induced subgraphs C_n and covering numbers are studied. First properties of $BG_1(G)$ are studied.

Proposition 2.1: Girth of $BG_1(G)$ is three if and only if girth of G is three or G has at least two non-adjacent edges.

Proof: Girth of $BG_1(G)$ is three implies that there is a triangle in $BG_1(G)$. This implies that there is a triangle in G or there is a line vertex which is adjacent to two adjacent point vertices in $BG_1(G)$. Thus, G has at least two non-adjacent edges. Proof of the converse is obvious.

Proposition 2.2: Girth of $BG_1(G)$ is four if and only if $G = K_{1,n}$ or $K_{1,n} \cup mK_1$, $n > 2$.

Proof: By the previous Proposition, girth of $BG_1(G)$ is four implies that G has no triangle or non-adjacent edges. Hence, $G = K_{1,n}$ or $K_{1,n} \cup mK_1$. When $G = K_{1,n}$, when $n = 1$, $BG_1(G)$ is disconnected and when $n = 2$, $BG_1(G) = P_5$. When $G = K_{1,2} \cup K_1$, $BG_1(G) = C_6$, in all other cases $BG_1(G)$ has a C_4. Hence, $G = K_{1,n}$ or $K_{1,n} \cup mK_1$, $n > 2$. Proof of the converse is obvious.

Proposition 2.3: Girth of $BG_1(G)$ is six if and only if $G = K_{1,2} \cup K_1$.

Proof: If $G = K_{1,2} \cup K_1$, then girth of $BG_1(G) = 6$. On the other hand, assume girth of $BG_1(G) = 6$. Therefore, girth of G is greater than or equal to 6 or G is acyclic and G has no two non-adjacent edges. Hence, all the edges of G must be adjacent, that is $G = K_{1,n} \cup mK_1$. But if $n > 2$, and $m > 1$ girth of $BG_1(G)$ is four. Therefore, $G = K_{1,2} \cup K_1$.

Remark 2.1: Girth of $BG_1(G)$ can not be five.

Theorem 2.1: $BG_1(G)$ has no dominating edge.

Proof: Suppose $BG_1(G)$ has a dominating edge $xy \in E(BG_1(G))$.

Case1: x, y are point vertices of $BG_1(G)$.

In $BG_1(G)$, xy is a dominating edge. Therefore, x, y are adjacent in G. Now, consider $e = xy \in E(G)$, This line vertex e is not adjacent to x or y in $BG_1(G)$. Hence, this case is not possible.

Case2: x is a point vertex and y is a line vertex

xy is a dominating edge in $BG_1(G)$ implies y is not incident with x in G. Therefore, in $BG_1(G)$ the vertices v_1, v_2, where $y = v_1v_2 \in E(G)$ are adjacent to x. Hence, x, v_1, v_2 form a triangle in G. Also in $BG_1(G)$, all other line vertices must be adjacent to x. Hence in G, x is an isolated vertex. Therefore, this case is also not possible.

Case3: x and y are line vertices

This case is also not possible, since any two line vertices are not adjacent in $BG_1(G)$.

This proves the theorem.

Properties related to $\overline{\mathbf{BG}_1}(G)$ are discussed here. $\overline{\mathbf{BG}_1}(G)$ is always connected.

Proposition 2.4: If G is connected and $q \geq 2$, then girth of $\overline{\mathbf{BG}_1}(G)$ is three.

Proof: Since G is connected, there exists $e_1, e_2 \in E(G)$ such that $e_1 = u_1v_1$, $e_2 = u_2v_2$, where $u_1 = u_2$. Then $e_1 e_2 u_1$ is a triangle in $\overline{\mathbf{BG}_1}(G)$. Hence, girth of $\overline{\mathbf{BG}_1}(G)$ is three.

Proposition 2.5: Girth of $\overline{\mathbf{BG}_1}(G)$ is four if and only if $G = 2K_2$ or $K_2 \cup K_1$.

Proof: Girth of $\overline{\mathbf{BG}_1}(G)$ is four implies that $\overline{G}$ has no triangles and G is disconnected. Also, G has less than three edges. Thus, either $G = K_2 \cup K_1$ or $2K_2$. Proof of the converse is obvious.

Theorem 2.2: $\overline{\mathbf{BG}_1}(G)$ is geodetic if and only if $G = K_2$.

Proof: If $G = K_2$, $\overline{\mathbf{BG}_1}(G) = K_{1,2}$, which is geodetic. Now, assume that $\overline{\mathbf{BG}_1}(G)$ is geodetic. Suppose $G \neq K_2$, $p \geq 3$ and G is non trivial implies $\overline{\mathbf{BG}_1}(G)$ contains a C_4 or K_4-x. Hence, $\overline{\mathbf{BG}_1}(G)$ is not geodetic. This proves the theorem.

Theorem 2.3: Let G be a (p, q) graph. Then $\overline{\mathbf{BG}_1}(G)$ has a dominating edge if and only if any one of the following holds: (1) G is connected graph with $p \leq 4$.

(2) $G = K_{1,n} \cup K_{1,m}$. (3) G has at least one pendant vertex.

Proof: If $\overline{\mathbf{BG}_1}(G)$ has a dominating edge xy, then every vertex in $\overline{\mathbf{BG}_1}(G)$ is adjacent to either x or y.

Case 1: Both x and y are line vertices.

Suppose there exists a vertex v, which is not incident with x and y in G, then x and y are not adjacent to v in $\overline{\mathbf{BG}_1}(G)$. Therefore, xy is a dominating edge if and only if $p \leq 4$ and G is connected.

Case 2: x and y are both point vertices.

xy is an edge in $\overline{\mathbf{BG}_1}(G)$ implies x and y are not adjacent in G. Also, xy is a dominating edge implies every line vertex is adjacent to x or y in $\overline{\mathbf{BG}_1}(G)$ and every other point vertices are also adjacent to x or y in $\overline{\mathbf{BG}_1}(G)$. Hence, {x, y} is a dominating set of $\overline{G}$ and are adjacent in $\overline{G}$ and every edge is incident with x or y in G. Combining all these, $G = K_{1,n} \cup K_{1,m}$ and x and y are the central vertices of $K_{1,n}$ and $K_{1,m}$ respectively.

Case 3: x is a line vertex and y is a point vertex.

xy is a dominating edge in $\overline{\mathbf{BG}_1}(G)$ implies x is incident with y in G. Let $x = yy_1 \in E(G)$; $y, y_1 \in V(G)$. In $\overline{\mathbf{BG}_1}(G)$, x is adjacent to all line vertices and adjacent point vertices y and y_1 only. Therefore, if G has more than two vertices, then in $\overline{\mathbf{BG}_1}(G)$ all the other point vertices are adjacent to y (except y_1). That is, y is a pendant vertex in G. Therefore, if G is a graph with at least one pendant vertex y then $\overline{\mathbf{BG}_1}(G)$ has a dominating edge xy, where x is the edge incident with y in G. This proves the theorem.

Corollary 2.3: If G has more than four vertices and has no pendant vertices, then both $BG_1(G)$ and its complement are self-centered with diameter two.

Proof: Proof follows from Theorems 2.1, 2.3 and Theorem 1.6.

3. Eccentricity properties of $BG_1(G)$ and $\overline{BG_1}(G)$

In this section, radius and diameter of $BG_1(G)$ and $\overline{\mathbf{BG}_1}(G)$ are found out and classification of graphs such as $BG_1(G)$ is self-centered with diameter two or three is also dealt with. Throughout this section, if $v \in V(G)$ and $e \in E(G)$, denote the corresponding vertices of $BG_1(G)$ by v' and e'.

Proposition 3.1: If $BG_1(G)$ is connected, eccentricity of a point vertex is 2, 3 or 4.

Proof: Consider a point vertex v in V(G). If $\deg_G v \geq 1$, then in $BG_1(G)$, v' is not adjacent to the line vertices e_i', where e_i are incident with v in G. If v is isolated in G, v' is not adjacent to other point vertices in $BG_1(G)$. Hence, $e(v') \geq 1$ in $BG_1(G)$. Now, to find $d(v', v_i')$ for $v_i \in V(G)$ in $BG_1(G)$.

If v_i is adjacent to v in G, then $d_G(v, v_i) = 1$, and $d(v', v_i') = 1$ in $BG_1(G)$.

If v_i is not adjacent to v in G, then different cases arises:

Case 1: There exists some edge e not adjacent to both v and v_i.

In this case, $d(v', v_i') = d(v_i', e') = 1$ and hence $d(v', v_i') = 2$ in $BG_1(G)$.

Case 2: $d_G(v, v_i) = 2$.

In this case also, $d(v', v_i') = 2$ in $BG_1(G)$.

Case 3: $d_G(v, v_i) > 2$ and there is no edge which is not adjacent to both v and v_i.

In this case, G must be disconnected and v and v_i must be in different components and there is no edge not adjacent to both v and v_i. Hence, edges in one component are incident with v and in others are all incident with v_i.

Hence, G must be of the form $K_{1,n} \cup K_{1,m} \cup sK_1$. Now, when $G = K_2 \cup K_1, 2K_2$; $BG_1(G)$ is disconnected and if any one of n or m ≥ 2, then $d(v', v_i') = 3$ in $BG_1(G)$, otherwise, that is, when $G = K_2 \cup K_1$, $d(v', v_i') = 4$ in $BG_1(G)$.

Now, to find $d(v', e')$ in $BG_1(G)$, where $e \in E(G)$.

If e is not incident with v in G, then $d(v', e') = 1$ in $BG_1(G)$.

If e is incident with v in G, let $e = vv_1 \in E(G)$.

In $BG_1(G)$, e' is not adjacent to v'. If there exists another vertex v_2, which is adjacent to v in G (that is, if $deg_G v \geq 2$), then v v_2 e is a shortest path in $BG_1(G)$ and hence, $d(v', e') = 2$ in $BG_1(G)$. If there exists no such vertex and if $deg_G v_1 \geq 2$, then $v'\ v_1'\ v_2'\ e'$ (where v_2 is adjacent to v_1) is a shortest path and hence $d(v', e') = 3$ in $BG_1(G)$. Suppose this is also not possible. That is, if $deg_G v_1 = deg_G v = 1$ (that is, K_2 is a component of G) and if there exists another vertex v_3 and an edge e_1, where e_1 is not incident with v and v_3, then $v'\ e_1'\ v_3'\ e'$ is a shortest path. (In this case, $G \neq K_2, 2K_2, K_2 \cup mK_1$). This is not possible means $BG_1(G)$ is disconnected.

Hence, distance from v' to any other vertex is 1, 2, 3 or 4 and $e(v) > 1$ in $BG_1(G)$ implies that the eccentricity is 2, 3 or 4.

Proposition 3.2: If $BG_1(G)$ is connected, then the eccentricity of a line vertex is 2, 3, or 4,

Proof: From the previous result $d(e', v') = 1, 2$ or 3 in $BG_1(G)$.

Now, to find $d(e_1', e_2')$ for $e_1, e_2 \in E(G)$.

Case 1: e_1 and e_2 are not adjacent.

If there exists a vertex v not incident with both e_1 and e_2, then $e_1'\ v'\ e_2'$ is a shortest path and hence $d(e_1', e_2') = 2$ in $BG_1(G)$. If there exists no vertex v, not incident with both e_1 and e_2 and there is no edge adjacent to both e_1 and e_2, then $BG_1(G)$ is disconnected. In this case, $G = 2K_2$. If there exists no vertex v, not incident with both e_1 and e_2 and there exists an edge adjacent to both e_1 and e_2, then $d(e_1', e_2') = 3$ in $BG_1(G)$.

Case 2: e_1 and e_2 are adjacent.

If there exists no common non-incident vertex, two cases arise, either $G = K_{1,2}$ or $G = K_3$. When $G = K_{1,2}$, $d(e_1', e_2') = 4$ in $BG_1(G)$; when $G = K_3$, $d(e_1', e_2') = 3$ in $BG_1(G)$. Also, in $BG_1(G)$, e' is not adjacent with the vertices v_1' and v_2', if $e = v_1v_2 \in E(G)$. Hence $e(e') > 1$ in $BG_1(G)$. This proves the proposition.

Lemma 3.1: Let G be a (p, q) graph. Eccentricity of every point vertex of $BG_1(G)$ is two if and only if G has more than three vertices and has no pendant vertices.

Proof: Let $u \in V(G)$, $v \in V(G)$ and $\deg_G u \geq 2$ for all $u \in V(G)$.

1. If u and v are adjacent in G, then $d(u', v') = 1$ in $BG_1(G)$.
2. If $d_G(u, v) = 2$, then $d(u', v') = 2$ in $BG_1(G)$.
3. If $d_G(u, v) > 2$ and G connected, then $d(u', v') = 2$ in $BG_1(G)$.
4. If $d_G(u, v) > 2$ and G is not connected, then since G has no pendant vertices, there exists $e \in E(G)$, which is not adjacent to both u and v in G. Hence, $d(u', v') = 2$ in $BG_1(G)$.

To find $d(u', e')$ in $BG_1(G)$:

1. If e is not incident with u in G, $d(u', e') = 1$ in $BG_1(G)$.
2. If e is incident with u in G, $e = uv \in E(G)$, there exists another vertex v_1 adjacent to u in G and e' v_1' u' is a shortest path in $BG_1(G)$.

Therefore, $d(u', e') = 2$ in $BG_1(G)$. Hence, eccentricity of every point vertex is two.

On the other hand, assume eccentricity of every point vertex is two. Hence, $d(u', v') \leq 2$ in $BG_1(G)$ and $d(u', e') \leq 2$ in $BG_1(G)$, for $u, v \in V(G)$ and $e \in E(G)$. Suppose G has a pendant vertex u, $e = uv \in E(G)$, then $d(u', e') > 2$ in $BG_1(G)$, which is a contradiction to the assumption. This proves the theorem.

Lemma 3.2: If G is a non-trivial (p, q) graph with more than four vertices and has no pendant vertices, then eccentricity of every line vertex is two.

Proof: By the previous theorem, $d(u', e') \leq 2$ for any $u \in V(G)$ and $e \in E(G)$. Now, consider $d(e_1', e_2')$ in $BG_1(G)$, where $e_1, e_2 \in E(G)$. Since $p > 4$, for any two e_1 and $e_2 \in E(G)$, there exists a common non-incident vertex $v \in V(G)$. Hence, in $BG_1(G)$, e_1' v' e_2' is a shortest path and hence $d(e_1', e_2') \leq 2$ in $BG_1(G)$. Hence, eccentricity of every line vertex is two.

Theorem 3.1: If G is connected with more than four vertices and has no pendant vertex, then $BG_1(G)$ is self-centered with diameter two.

Proof: The proof of the theorem follows from Lemmas 3.1 and 3.2.

Proposition 3.2: If G is connected with more than three vertices and has pendant vertices, then eccentricity of a pendant vertex u' and pendant edge e' in $BG_1(G)$ is three.

Proof: Let $u \in V(G)$ such that $e = uv \in E(G)$ and $\deg_G u = 1$. Since G is connected, u is adjacent to some other vertex v_1 in G. Hence, in $BG_1(G)$, $e'\ v_1'\ v'\ u'$ is a shortest path. Therefore, $d(u', e') = 3$ in $BG_1(G)$ and distance between u' to any other vertex is at most 2 and distance from e' to any other vertex is also at most 2. Also, non-pendant vertices are of eccentricity two. This proves the proposition.

Theorem 3.2: If G is connected with more than three vertices and has pendant vertices, then $BG_1(G)$ is bi-eccentric with diameter three.

Proof: Proof follows from the previous proposition.

Remark 3.1: If $G = K_{1,2}$, then eccentricity of pendant point vertex is three and eccentricity of pendant edge vertex is 4. Therefore, $BG_1(G)$ is tri-eccentric with radius two and diameter four.

Remark 3.2: If $G = K_3$, then eccentricity of each point vertex is two and eccentricity of each line vertex is three. $BG_1(K_3)$ is bi-eccentric with diameter three.

Theorem 3.3: In $BG_1(G)$, eccentricity of every point vertex is two and line vertex is three if and only if $G = K_3$, K_4 or C_4.

Proof: Since G has no pendant vertices, eccentricity of every point vertex is two and also, the eccentricity of line vertex is three, since for any edge there exists another edge such that they have no common non-incident vertex in G. On the other hand, let eccentricity of every point vertex be 2 and line vertex be 3. Then, $d(u', v') \leq 2$, $d(u', e') \leq 2$ in $BG_1(G)$ and for every line vertex e_1 in E(G), there exists e_2 in E(G) such that $d(e_1', e_2') = 3$ in $BG_1(G)$, for all u and $v \in V(G)$. $d(u', v') \leq 2$ implies $d_G(u, v) \leq 2$ or there exists an edge e not incident with both u and v. But $d(e_1', e_2') = 3$ in $BG_1(G)$ implies, there is no vertex not incident with e_1 and e_2 in G. Hence, $p = 3$ or 4. This e_1 and e_2 may or may not be adjacent. Also e_1 and e_2 are not pendant, otherwise, the eccentricity of the corresponding pendant point vertices in $BG_1(G)$ is three. Hence, the only possibility is $G = K_3$, K_4 or C_4. ($G \neq K_4$-e, since in K_4-e, for a line vertex e, there exists some $v \in V(G)$ non incident with e_1 and $e_2 \in E(G)$ and hence eccentricity of e_1 in $BG_1(G)$ is two). This proves the theorem.

Theorem 3.4: (1) Eccentricity of a line vertex in $BG_1(G)$ is four if and only if $G = K_{1,2}$. (2) Eccentricity of a point vertex is four if and only if $G = 2K_2 \cup nK_1$.

Proof of (1): From Propositions 3.1 and 3.2, $d(e', v') = 1, 2$ or 3 in $BG_1(G)$ and if e_1 and e_2 have some non-incident vertex $d(e_1', e_2') = 2$ in $BG_1(G)$. Hence, eccentricity of line vertex is 4 implies that there exists e_1 such that $d(e_1', e_2') = 4$ in $BG_1(G)$, which in turn implies $p \leq 4$. When $p = 4$, $BG_1(G)$ is bi-eccentric with diameter three. Therefore, $p = 3$ and since $BG_1(G)$ is connected $G \neq K_2$ or $K_2 \cup K_1$ and also $G \neq K_3$ by Theorem 3.3. Therefore, $d(e_1', e_2') = 4$ in $BG_1(G)$ if and only if $G = K_{1,2}$. This proves (1).

Proof of (2): Again from Proposition 3.1, $d(e', v') = 1, 2$, or 3 in $BG_1(G)$. Hence, eccentricity of a point vertex is four implies that, there exists u and $v \in V(G)$ such that $d(u', v') = 4$ in $BG_1(G)$. This gives $d_G(u, v)$ is at least 4. If G is connected and $d_G(u, v)$ is at least 4, then there exists an edge non-incident with both u and v such that $d(u', v') = 2$ in $BG_1(G)$. Hence, $d(u', v') = 4$ in $BG_1(G)$ implies that G is disconnected and u and v are in different components of G such that G has no edge not incident with u and v. Hence, G must have at most two edges (not adjacent since G is disconnected and u, v are in different components). Suppose there are no other point vertices. Then $G = 2K_2$. $BG_1(G)$ is disconnected in this case. Hence, $G = 2K_2 \cup nK_1$. This proves the theorem.

Remark 3.3: Radius of $BG_1(G)$ is 2 and diam $(BG_1(G))$ is 4 if and only if $G = K_{1,2}$, $2K_2 \cup nK_1$.

Theorem 3.5: $BG_1(G)$ is self-centered with diameter three if and only if $G = K_{1,2} \cup K_2$, $K_{1,2} \cup mK_1$, or nK_2, $n \geq 3$.

Proof: Assume $BG_1(G)$ is self-centered with diameter three. Eccentricity of point vertices and line vertices is three in $BG_1(G)$. Hence, G must have pendant edges and hence pendant vertices and $p \geq 4$. Also, G must have more than one edge, otherwise $BG_1(G)$ is disconnected.

Case 1: There exist two adjacent edges in G.

Let e_1, e_2 be two adjacent edges in G. If there exists more than two edges in G, then it must not be adjacent to e_1 or e_2 or both. If one more vertex is in G, then $BG_1(G)$ is not self-centered. Hence, $G = K_{1,2} \cup K_2$. If there is no other edge in G, there must be some isolated vertices, otherwise $BG_1(G)$ is not self-centered. Hence, $G = K_{1,2} \cup mK_1$.

Case 2: There exist two non-adjacent edges.

Let e_1 and e_2 be two non-adjacent edges in G. $G \neq 2K_2$, since if $G = 2K_2$, $BG_1(G)$ is disconnected. Hence, there must be some vertices or edges in G. If there exists another edge adjacent to e_1 or e_2, then by case 1, $G = K_{1,2} \cup K_2$. If there exists another edge not adjacent to e_1 and e_2, then $G = nK_2$, and $BG_1(G)$ is self-centered with diameter three.

This proves the theorem.

Theorem 3.6: $\overline{BG_1}(G)$ is self-centered with diameter two if and any if $G \neq K_2$.
Proof: This can be proved as in Propositions 3.1 and 3.2.

4. Covering numbers of $BG_1(G)$ and $\overline{BG_1}(G)$

Following theorems deal with the chromatic number and the covering numbers of $BG_1(G)$ and $\overline{BG_1}(G)$. For the rest of the section, assume that G is a graph without isolated vertices.

Theorem 4.1: $\chi(BG_1(G)) = \chi(G)$ or $\chi(G)+1$.
Proof: G and K_q are induced subgraphs of $BG_1(G)$. If a line vertex in $BG_1(G)$ is adjacent to point vertices having $\chi(G)$ colors, then $\chi(BG_1(G)) = \chi(G)+1$, otherwise $\chi(BG_1(G)) = \chi(G)$.

Theorem 4.2: $\omega(BG_1(G)) = \omega(G)+1$ or $\omega(G)$.

Proof: Since G has no isolated vertices $\omega(G) > 1$. Let G_1 be a complete subgraph of G and $|V(G_1)| = \omega(G)$. If there exists an edge e in $< V(G)-V(G_1) >$, then in $BG_1(G)$, e' is adjacent to the point vertices v_i', where $v_i \in V(G_1)$. Hence, $< V(G_1) \cup \{e\} >$ forms a maximum complete subgraph in $BG_1(G)$. (No two line vertices are adjacent in $BG_1(G)$). Hence, $\omega(BG_1(G)) = \omega(G)+1$ or $\omega(G)$.

Theorem 4.3: $\theta(BG_1(G)) = q$ or $q+1$.

Proof: $\theta(G)$ is the minimum number of complete subgraphs of G needed to cover the vertices of G. Let $\theta(G) = k$. Vertices of G can be partitioned into k sets $V_1, V_2, \ldots, V_k$.

Case 1: For each V_i there exists one edge $e \in E(G)$ not incident with elements of V_i in G. In this case, by joining one line vertex to each of the elements of V_i, one can get a complete graph in $BG_1(G)$. Then k complete graphs are obtained, having vertices $|V_1|+1, |V_2|+1, \ldots, |V_k|+1$. The remaining q-k line vertices form K_1's. Therefore, totally $k+(q-k) = q$ complete subgraphs are needed to cover $BG_1(G)$. Therefore, $\theta(BG_1(G)) = q$.

Case 2: There exists some V_1 such that there exists no edge in G such that e is not incident with all the elements of V_1.

In this case, $\theta(BG_1(G)) = k+(q-k+1) = q+1$. (Since at most one such V_1 is possible). Hence the theorem is proved.

Theorem 4.4:

(i) If $G = K_{1,n}$, then $\alpha_o(BG_1(G)) = q$.

(ii) If $G \neq K_{1,n}$, and $q \geq p$, then $\alpha_o(BG_1(G)) = p$.

(iii) If $G \neq K_{1,n}$ and , if $q < p$, then $\alpha_o(BG_1(G)) = \min\{p, q+\alpha_o(G)\}$.

Proof: Case 1: $G = K_{1, n}$.

The central vertex u of G is not adjacent to any line vertices in $BG_1(G)$. All edges incident with the central vertex is covered by other point vertices in $BG_1(G)$. Therefore, $\alpha_o(BG_1(G))=p-1=q$ in $BG_1(G)$.

Case 2: $G \neq K_{1,n}$, $q \geq p$.

In $BG_1(G)$, every point vertex u is adjacent to some line vertices and the set of all line vertices is independent in $BG_1(G)$. Therefore, the set of all point vertices covers all the edges of $BG_1(G)$. (Any proper subset of V(G) cannot cover all the edges of $BG_1(G)$). Hence, $\alpha_o(BG_1(G)) = p$.

Case 3: $q < p$, $G \neq K_{1, n}$.

The set of all line vertices covers all the edges joining point vertices and line vertices. $\alpha_o(G)$ point vertices covers all the edges in G. Therefore, $E(G) \cup D$, where D is a minimum point cover for G covers all the edges of $BG_1(G)$. Therefore, $\alpha_o(BG_1(G)) \leq q+\alpha_o(G)$. Also, set of all point vertices covers all the edges in $BG_1(G)$. Therefore, $\alpha_o(BG_1(G)) = \min\{p, q+ \alpha_o(G)\}$.

Theorem 4.5:

(1) If $G = K_{1,n}$, then $\alpha_1(BG_1(G)) = p$.

(2) If $G \neq K_{1,2}$, then $\alpha_1(BG_1(G)) \leq \max\{p, q\}$.

Proof: If $G = K_{1,n}$, clearly, $\alpha_1(BG_1(G)) = p$. Hence assume that $G \neq K_{1,n}$. Therefore, every point vertex is adjacent to some other line vertices in $BG_1(G)$.

Case 1: $p \geq q$.

The line vertices are independent in $BG_1(G)$ and every point vertex is adjacent to at least one line vertex. Let $D \subseteq E(BG_1(G))$ such that D contains q edges each joining a line vertex to a point vertex (distinct). Consider $D_1 \subseteq E((BG_1(G))$ containing p-q edges, incident with remaining p-q point vertices. $D \cup D_1$ is a line cover for $BG_1(G)$. Therefore, $\alpha_1(BG_1(G)) \leq q+(p-q) = p$.

Case 2: $p \leq q$.

Consider D, which contains p edges each joining a point vertex to a line vertex (distinct) and D_1, which contains (q-p) edges each joining the remaining (q-p) line vertices to some point vertices. $D \cup D_1$ is a line cover for $BG_1(G)$ and hence $\alpha_1(BG_1(G)) \leq p+(q-p) = q$.

This proves the theorem.

Theorem 4.6:

(1) If $G = K_{1,n}$, then $\beta_1(BG_1(G)) = q = n$.

(2) If $G \neq K_{1,n}$, then $\beta_1(BG_1(G)) \geq \min\{p, q\}$.

Proof: Proof follows from Theorem 4.5 and Theorem 1.1.

Remarks 4.1 Actually, if $G \neq K_{1,n}$, then $\beta_1(BG_1(G)) \geq q$ if $q \leq p$ and p if $p < q$. If $G = K_{1,n}$, then $\beta_1(BG_1(G)) = q$.

Theorem 4.7:

(1) If $G = K_{1,n}$, then $\beta_o(BG_1(G)) = p$.

(2) If $G \neq K_{1,n}$ and $p \geq q$, then $\beta_o(BG_1(G)) = \max\{q, \beta_o(G)\}$.

(3) If $G \neq K_{1,n}$ and $q < p$, then $\beta_o(BG_1(G)) = p$.

Proof: Proof follows from Theorem 4.4 and Theorem 1.1. If $G = K_{1,n}$, then $S = \{u\} \cup E(G)$, where u is the central node of G is the maximal independent set of $BG_1(G)$. If $G \neq K_{1,n}$ and $q \geq p$, the set of all line vertices are maximum independent set in $BG_1(G)$. If $G \neq K_{1,n}$ and $q < p$, set of all line vertices of $BG_1(G)$ is a maximal independent set. Also, any independent subset of G is an independent subset of $BG_1(G)$ and there is no independent set containing point and line vertices having cardinality at least q+1.

Theorem 4.8:

(1) If $G = K_{1,n}$, then $\chi(\overline{BG_1(G)}) = p$.

(2) If $G \neq K_{1,n}$, then, $\chi(\overline{BG_1(G)}) = q$ or $q+1$.

(3) If $G \neq K_{1,n}$, then $\omega(\overline{BG_1(G)}) = \max\{q, \beta_o(G)\}$.

(4) If $G = K_{1,n}$, then $\omega(\overline{BG_1(G)}) = n+1 = p$.

(5) $\theta(\overline{BG_1(G)}) = \chi(G)$ or $\chi(G)+1$.

(6) $\beta_o(\overline{BG_1(G)}) = \omega(G)$ or $\omega(G)+1$.

(7) $\alpha_o(\overline{BG_1(G)}) = q+\alpha_o(G)$ or $(q-1)+\alpha_o(G)$.

Proof of all these results follow from the previous Theorems and Theorem 1.1.

Theorem 4.9:

(1) If $G = K_{1,n}$, then $\alpha_1(\overline{BG_1(G)}) = p$.

(2) If $G \neq K_{1,n}$, then $\alpha_1(\overline{BG_1(G)}) \geq \min\{p, q\}$.

Proof: If $G = K_{1, n}$, then clearly $\alpha_1(BG_1(G)) = p = n+1$. Now, assume that $G \neq K_{1, n}$. Since G has no isolated vertices, the edges joining point vertices and line vertices in $\overline{\mathbf{BG}_1(G)}$ covers all the vertices of $\overline{\mathbf{BG}_1(G)}$. Therefore, $\alpha_1(\overline{\mathbf{BG}_1(G)}) \leq 2q$.

Case 1: $q \leq p$.

Let G_1 be an induced subgraph of $\overline{G}$ containing p-q vertices. If G_1 has no isolated vertices, then consider the smallest line cover D of G_1; otherwise, consider the minimum number of edges in $\overline{\mathbf{BG}_1(G)}$, which cover all the vertices of G_1, as D. Let $k = \min \{|D|\}$, where the minimum is taken over such G_1's. Consider the remaining q point vertices and consider q edges in $\overline{\mathbf{BG}_1(G)}$, which are edges joining these q point vertices to q line vertices. Thus, $\alpha_1(\overline{\mathbf{BG}_1(G)}) = k+q$.

Case 2: $q > p$.

Smallest line cover of $\overline{\mathbf{BG}_1(G)}$ contains p edges joining p point vertices top line vertices and edges in a smallest line cover of K_{q-p}. Therefore, $\alpha_1(\overline{\mathbf{BG}_1(G)}) = p+\alpha_1(K_{q-p}) = p+[(q-p+1)/2]$. Thus, in all cases, $\alpha_1(\overline{\mathbf{BG}_1(G)}) \geq \min \{p, q\}$.

Theorem 4.10:

(1) If $G \neq K_{1,n}$, then $\beta_1(\overline{\mathbf{BG}_1(G)}) \leq \max\{p, q\}$.

(2) If $G = K_{1,n}$, then $\beta_1(\overline{\mathbf{BG}_1(G)}) = q$.

Proof: Proof follows from Theorem 4.9 and Theorem 1.1.

Conclusion: Other properties such as traversability, connectivity, edge partition and domination parameters are studied and submitted.

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