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On the Boolean Function Graph $B(K_p, INC, \overline{K_q})$ of a Graph

T. N. Janakiraman[1] and S. Muthammai[2]

[1]*National Institute of Technology, Tiruchirappalli – 620 015, India.*
E. Mail:*janaki@nitt.edu, tnjraman2000@yahoo.com*
[2]*Government Arts College for Women, Pudukkottai- 622 001, India.*
E. Mail: *muthammai.sivakami@gmail.com*

Abstract: *For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph* $B(K_p, INC, \overline{K_q})$ *of G is a graph with vertex set* $V(G) \cup E(G)$ *and two vertices in* $B(K_p, INC, \overline{K_q})$ *are adjacent if and only if they correspond to two adjacent vertices of G, two nonadjacent vertices of G or to a vertex and an edge incident to it in G, For brevity, this graph is denoted by* $B_4(G)$*. In this paper, structural properties of* $B_4(G)$ *including traversability and eccentricity properties are studied. Further, decomposition of* $B_4(G)$ *for some known graphs are given.*

Key Word: *Boolean Function Graph*

1. Introduction

Graphs discussed in this paper are undirected and simple. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. *Eccentricity* of a vertex $u \in V(G)$ is defined as $ecc_G(u) = \max \{d_G(u, v): v \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. We denote the eccentricity of vertex v in G as e(v) and the distance between two vertices u, v in G as d(u, v). The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called a *self-centered* graph with radius r, equivalently G is r-self-centered. A vertex u is said to be an eccentric point of v in a graph G, if d(u, v) = e(v). In general, u is called an *eccentric point*, if it is an eccentric point of some vertex. The i^{th} neighborhood of v is denoted as $N_i(v) = \{u \in V(G) : d_G(u, v) = i\}$ the cardinality of the set H is denoted as $|H|$. If $|N_{e(v)}(v)|$ is m, for each point $v \in V(G)$, then G is called an *m- eccentric point graph*. If m = 2, we call the graph G as *bi-eccentric point graph*. A connected graph G is said to be *geodetic*, if a unique shortest path joins any two of its vertices. The graph G with p vertices and q edges is denoted by G(p, q).

Whitney[22] introduced the concept of the line graph L(G) of a given graph G in 1932. The first characterization of line graphs is due to Krausz. The Middle graph M(G) of

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a graph G was introduced by Hamada and Yoshimura[5]. Chikkodimath and Sampathkumar[3] also studied it independently and they called it, the semi-total graph $T_1(G)$ of a graph G. Characterizations were presented for middle graphs of any graph, trees and complete graphs in [1]. The concept of total graphs was introduced by Behzad[2] in 1966. Sastry and Raju[21] introduced the concept of quasi-total graphs and they solved the graph equations for line graphs, middle graphs, total graphs and quasi-total graphs. These graphs are very much useful in the construction of various related networks from the underlying graphs of networks. This motivates us to define and study other graph operations. Janakiraman et al., introduced the concepts of Boolean and Boolean function graphs [6 - 20].

The points and edges of a graph are called its elements. Two elements of a graph are neighbors, if they are either incident or adjacent. The *Total graph* T(G) of G has vertex set $V(G) \cup E(G)$ and vertices of T(G) are adjacent, whenever they are neighbors in G. The *Quasi- total graph* P(G) of G is a graph with vertex set as that of T(G) and two vertices are adjacent if and only if they correspond to two nonadjacent vertices of G or to two adjacent edges of G or to a vertex and an edge incident to it in G. The *Middle graph* M(G) of G is one whose vertex set is as that of T(G) and two vertices are adjacent in M(G), whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge of G incident with it. Clearly, E(M(G)) = E(T(G)) − E(G).

The *Boolean function graph* $B(K_p, INC, \overline{K_q})$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(K_p, INC, \overline{K_q})$ are adjacent if and only if they correspond to two adjacent vertices of G, two nonadjacent vertices of G or to a vertex and an edge incident to it in G. For brevity, this graph is denoted by $B_4(G)$. In this paper, the properties of the Boolean function graph $B_4(G)$ are studied. Also decompositions of $B_4(G)$ for some known graphs are given. For graph theoretic terminology, Harary [4] is referred.

2. Properties

In this section, properties of $B_4(G)$ including traversability, eccentricity properties are studied, where G is a graph with p vertices and q edges.

Observation:

2.1. K_p is an induced subgraph of $B_4(G)$ and the subgraph of $B_4(G)$ induced by q vertices is totally disconnected.

2.2. Number of vertices in $B_4(G)$ is p + q, since $B_4(G)$ contains vertices of both G and the line graph L(G) of G.

2.3. Number of edges in $B_4(G)$ is $\left(\frac{p(p-1)}{2}\right) + 2q$

2.4. For every vertex $v \in V(G)$, $d_{B_4(G)}(v) = p - 1 + d_G(v)$

(a). If G is complete, then $d_{B_4(G)}(v) = 2(p - 1)$.

(b). If G is totally disconnected, then $d_{B_4(G)}(v) = p - 1$.

(c). If G has atleast one edge, then $2 \leq d_{B_4(G)}(v) \leq 2(p - 1)$ and $d_{B_4(G)}(v) = 1$ if and only if $G \cong \mathbf{2K_1}$.

2.5. For an edge $e \in E(G)$, $d_{B_4(G)}(e) = 2$.

2.6. $B_4(G)$ is always connected.

2.7. If G is a graph with atleast three vertices, then each vertex of $B_4(G)$ lies on a triangle and hence girth of $B_4(G)$ is 2.

2.8. If G is a graph with atleast four vertices and atleast one edge, then $B_4(G)$ is bi-regular if and only if G is regular and G is regular if and only if G is totally disconnected.

2.9. If G is a graph with atleast three vertices, then $B_4(G)$ have no cut vertices.

2.10. If G has atleast one edge, then vertex connectivity of $B_4(G)$ = edge connectivity of $B_4(G) = 2$.

2.11. Let G be a (p, q) graph with atleast one edge. If p is odd, then $B_4(G)$ is Eulerian if and only if G is Eulerian.

2.12. If G is r-regular ($r \geq 1$ and is odd), then $B_4(G)$ is Eulerian.

2.13. For any graph G, $B_4(G)$ is geodetic if and only if G is either K_2 or nK_1, $n \geq 2$.

2.14. If G is a graph with atleast four vertices, then $B_4(G)$ is P_4- free.

In the following, a necessary and sufficient condition for $B_4(G)$ to be Hamiltonian is given.

Theorem 2.1.

For any graph G, $B_4(G)$ is Hamiltonian if and only if

(i) G is a path or a cycle on atleast three vertices

(ii) Each component of $B_4(G)$ is K_1, K_2 or $P_m, m \geq 3$

Proof.

Assume $B_4(G)$ is Hamiltonian.

Case (1): G is connected

Since $B_4(G)$ is Hamiltonian, there exists a Hamiltonian cycle C containing all the vertices of both G and L(G) and any two adjacent vertices of C are either vertices of G or a vertex of G and a vertex of L(G). This is possible, if there exists a Hamiltonian path or a Hamiltonian cycle in G containing all the edges of G. Hence, G is a path or a cycle.

Case (2): G is disconnected.

If one of the components of G has either C_3 or $K_{1,3}$ as an induced subgraph, then $B_4(G)$ is non-Hamiltonian. Therefore, each component of $B_4(G)$ is K_1, K_2 or $P_m, m \geq 3$

Conversely, let G be a path P_n on n ($n \geq 3$) vertices. Let { $v_1, v_2, .., v_n$} and {$e_1, e_2, \ldots, e_{n-1}$} be respectively the vertex set and edge set of P_n, where e_i = (v_i, v_{i-1}), i = 1, 2, …, n. Then $V(B_4(G))$ = { $v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}$} and $v_1e_1v_2e_2v_3e_3\ldots v_{n-1}e_{n-1}$ is a Hamiltonian path in $B_4(G)$ and hence $B_4(G)$ is Hamiltonian. Similarly, it can be verified that $B_4(G)$ is Hamiltonian for the remaining graphs given in the Theorem.

Definition 2.1. An edge e = (u, v) is a **dominating edge** in G, if every vertex of G is adjacent to atleast one of u and v.

In the following, eccentricities of vertices in $B_4(G)$ are found.

Theorem 2.2.

Eccentricity of a vertex in $B_4(G)$ lies between 1 and 3.

Proof:

(1) Eccentricity of a vertex in $V(G) \cap V(B_4(G))$

Let $v \in V(G) \cap V(B_4(G))$.

(i) The subgraph of $B_4(G)$ induced by vertices of V(G) is complete. Therefore, the distance between any two vertices in $V(G) \cap V(B_4(G))$ is 1.

(ii) Let e $\in V(G) \cap V(B_4(G))$ and let e = $(u, v) \in E(G)$. If the edge e is incident with v in G, then $d_{B_4(G)}(v,e) = 1$. If e is not incident with v in G, then vue is a geodetic path in $B_4(G)$ and hence $d_{B_4(G)}(v,e) = 2$.

From (i) and (ii),

$ecc_{B_4(G)}(v) = 1$, if all the edges of G are incident with v in G and $ecc_{B_4(G)}(v) = 2$, if there exists atleast one edge in G not incident with v in G

(2) Eccentricity of a vertex in $V(L(G)) \cap V(B_4(G))$

Let $e \in V(L(G)) \cap V(B_4(G))$. Then $e \in E(G)$.

(i) By (1) (ii), distance between v and e in $B_4(G)$ is 1 or 2.

(ii) Let e, f $\in V(L(G)) \cap V(B_4(G))$ Then e, f $\in$ E(G). Let e = (u, v) and f = (w, x), where u, v, w, $x \in V(G)$.

(a) Let e and f be adjacent edges in G and let v = w. Then evf is a geodesic path in $B_4(G)$ and hence $d_{B_4(G)}(e,f) = 2$.

(b) Let e and f be nonadjacent edges in G. Then euwf is a geodesic path in $B_4(G)$ and hence $d_{B_4(G)}(e,f) = 3$. From (i) and (ii), $ecc_{B_4(G)}(e) = 2$, if all the edges of G are adjacent to e in G. That is, e is a dominating edge of G, and $ecc_{B_4(G)}(v) = 3$, if e is not a dominating edge of G.

Therefore, eccentricity of a vertex in $B_4(G)$ lies between 1 and 3.

Theorem 2.3.

$B_4(G)$ is self-centered with radius 2 if and only if G is either C_3 or $C_3 \cup nK_1, n \geq 1$.

Proof:

Let $B_4(G)$ be self-centered with radius 2. Then the eccentricity of each vertex in $B_4(G)$ is 2. Let $v \in V(G)$. Then $ecc_{B_4(G)}(v) = 2$, if there exists an edge not incident with v in G. That is, for each vertex v in G, there is atleast one edge not incident with v in G. Let $e \in E(G)$. Then $ecc_{B_4(G)}(e) = 2$, if all the edges of G are adjacent to e in G. That is, all the edges of G are adjacent to each other. Both of the above hold, if G is either C_3 or $C_3 \cup nK_1, n \geq 1$.

Conversely, $B_4(C_3)$ and $B_4(C_3 \cup nK_1), (n \geq 1)$ are self-centered with radius 2.

Theorem 2.4.

$B_4(G)$ is bi-eccentric with radius 1 and diameter 2, if and only if G is either $K_{1,n}$ or $K_{1,n} \cup mK_1, n \geq 2, m \geq 1$

Proof.

Let $B_4(G)$ be bi-eccentric with radius 1 and diameter 2. Then there exists a vertex in $B_4(G)$ of eccentricity 1. For each $e \in E(G)$, eccentricity of e in $B_4(G)$ is either 2 or 3. Therefore, there exists a vertex $v \in V(G)$ having eccentricity 1 in $B_4(G)$. This holds, if all the edges of G are adjacent to v in G. If $\beta_0(G) \geq 1$, then there exists an edge e in G such that eccentricity of e in $B_4(G)$ is 3. Hence, $\beta_0(G) = 1$. Therefore, G is a graph with $\beta_0(G) = 1$ and there exists a vertex v in G such that all the edges of G are incident with v. That is, G is either $K_{1,n}$ or $K_{1,n} \cup mK_1, n \geq 2, m \geq 1$.

Conversely, $B_4(K_{1,n})$ and $B_4(K_{1,n} \cup mK_1), n \geq 2$ and $m \geq 1$ are bi-eccentric with radius 1 and diameter 2.

Theorem 2.5.

$B_4(G)$ is bi-eccentric with radius 2 and diameter 3, if and only if $\beta_0(G) \geq 2$.

Proof:

Let $B_4(G)$ be bi-eccentric with radius 2 and diameter 3. If $\beta_0(G) = 1$, then all the edges of G are adjacent to each other. Then eccentricity of each vertex in $B_4(G)$ is 1 or 2. Hence, $\beta_0(G) \geq 2$.

Conversely, assume $\beta_0(G) \geq 2$. Then eccentricity of $v \in V(G)$ in $B_4(G)$ is 2. Also, there exist atleast two independent edges in G. Then, eccentricities of the vertices in $B_4(G)$ corresponding to these edges are each equal to 3. Therefore, $B_4(G)$ is bi-eccentric with radius 2 and diameter 3.

In the following, a necessary and sufficient condition for $B_4(G)$ to have a dominating edge is found.

Theorem 2.6.

Let G be any graph. $B_4(G)$ has a dominating edge if and only if $\alpha_0(G) \leq 2$ or $\beta_1(G) = 1$.

Proof:

Assume $B_4(G)$ has a dominating edge, say x = (x_1, x_2), where x_1, x_2 $\in V(B_4(G))$. Then atleast one of x_1 and x_2 are vertices of G.

Case (1): x_1, x_2 $\in$ V(G)

Since x is a dominating edge in $B_4(G)$, all the vertices in $B_4(G)$ corresponding to the edges in G are adjacent to x_1, x_2 or both. That is, each edge in G is incident with x_1, x_2 or both. Therefore, $\alpha_0(G) \leq 2$.

Case (2): $x_1 \in$ V(G) and x_2 $\in$ E(G)

Then $x_2 \in V(B_4(G))$. Since x_1 and x_2 are adjacent in $B_4(G)$, x_1 is incident with the edge x_2 in G. Also, x is a dominating edge in $B_4(G)$. Therefore, the vertices in $B_4(G)$ corresponding to edges in G are adjacent to the vertex x_1 in $B_4(G)$ .That is, all the edges of G are mutually adjacent. Therefore, $\beta_1(G) = 1$.

Conversely, assume $\alpha_0(G) \leq 2$. Then there exists a point cover D of G with $|D| \leq 2$. If $|D| = 1$, then there exists a vertex v in G such that each edge in G is incident with v in G. Let $u \in V(G), (u \neq v)$. Then $u, v \in V(B_4(G))$ and $(u, v) \in E(B_4(G))$ is a dominating edge of $B_4(G)$. Assume $\beta_1(G) = 1$ Then there exists an edge say, e = (u, v) in G such that all the edges of G are adjacent to e. Let x = (u, e). $x \in E(B_4(G))$ is a dominating edge in $B_4(G)$. Hence, $B_4(G)$ has a dominating edge.

Independence and Covering numbers of $B_4(G)$

In the following, independence number of $B_4(G)$ is found.

Theorem 2.7.

Let G be a (p, q) graph. Then

$$\beta_0\left(B_4(G)\right)=\begin{cases} q, & \text{if } \delta(G)\geq 1 \\ q+1, & \text{if } \delta(G)=0 \end{cases}$$

Proof:

Let S be the set of vertices of $B_4(G)$ corresponding to the edges of G. Then S is an independent set in $B_4(G)$.

Therefore, $\beta_0(G)\geq |S|$ = q, if $\delta(G)\geq 1$.

Let H be the subgraph of $B_4(G)$ induced by the vertices of G. Then H is complete in $B_4(G)$. Each vertex in H is adjacent to atleast one vertex in S. Therefore, $\beta_0\left(B_4(G)\right)=q$, if $\delta(G)\geq 1$.

Let $\delta(G)=0$ and let $v\in V(G)$ be such that $deg_G(v)$ = 0. Then the set $S\cup\{v\}$ is a maximum independent set in $B_4(G)$ and hence $\beta_0\left(B_4(G)\right)=q+1$, if $\delta(G)=0$.

Remark 2.1.

Since for any graph G with p vertices, $\delta_0(G)+\beta_0(G)=p$, the covering number of $B_4(G)$ is given below. For any (p, q) graph G,

$$\alpha_0\left(B_4(G)\right)=\begin{cases} p, & \text{if } \delta(G)\geq 1 \\ p-1, & \text{if } \delta(G)=0 \end{cases}$$

In the following, edge independence number of $B_4(G)$ is given.

Theorem 2.8.

Let G be any (p, q) graph with $\delta(G)\geq 1$. Then

$$\beta_1\left(B_4(G)\right)=\begin{cases} p, & \text{if } q\geq p \\ q+\left\lfloor \frac{p-q}{2} \right\rfloor, & \text{if } q<p \end{cases}$$

Proof:

Let V(G) = {$v_1, v_2, \ldots, v_p$}. Let $e_1, e_2, \ldots, e_p$ be the edges in G incident with $v_1, v_2, \ldots, v_p$ respectively and let $x_1, x_2, \ldots, x_p$ be the vertices in $B_4(G)$ corresponding to $e_1, e_2, \ldots, e_p$ respectively. Then $\{v_1, v_2, .., v_p, x_1, x_2, .., x_p\}\subseteq V\left(B_4(G)\right)$

Case(1): $q\geq p$

The edges $(v_1, x_1), (v_2, x_2), \ldots, (v_p, x_p)$ are independent in $B_4(G)$. The remaining edges in $B_4(G)$ are adjacent to atleast one of $v_1, v_2, \ldots, v_p$. Also any set of (p + 1) edges in $B_4(G)$ are not independent to each other. Hence, $\beta_1\left(B_4(G)\right)=p$.

Case (2): $q < p$

Let $x_1, x_2, \ldots, x_q$ be the vertices in $B_4(G)$ corresponding to the edges $e_1, e_2, \ldots, e_q$ in G respectively. Then the edges $(v_1, x_1), (v_2, x_2), \ldots, (v_q, e_q)$ are independent in $B_4(G)$. The subgraph of $B_4(G)$ induced by remaining vertices $v_{q-1},\ldots,v_p$ is complete and $\beta_1(K_{p-q}) = \left\lfloor \frac{p-q}{2} \right\rfloor$.

Hence, $\beta_1(B_4(G)) = q + \left\lfloor \frac{p-q}{2} \right\rfloor$

Remark 2.2.

If G has m ($m \geq 1$) isolated vertices, then

$$\beta_1(B_4(G)) = \begin{cases} p + \left\lfloor \frac{m}{2} \right\rfloor, & \text{if } q \geq p \\ q + \left\lfloor \frac{p-q}{2} \right\rfloor, & \text{if } q < p \end{cases}$$

Remark 2.3.

1. If $G \cong H \cup mK_1, (m \geq 1)$, where H is any connected graph, then $\beta_1(B_4(G)) = |V(H)| + \left\lceil \frac{m}{2} \right\rceil$, since the subgraph of $B_4(G)$ induced by vertices of mK_1 is complete.
2. If G is disconnected with ω components $G_1, G_2, \ldots, G_\omega (\omega \geq 2)$, where $\delta(G_i) \geq 1$ and $|G| = p_i$, then $\beta_1(B_4(G)) = \sum_{i=1}^{\omega} p_i$

Remark 2.4.

Since $\alpha_1(B_4(G)) + \beta_1(B_4(G)) = p + q$, the following result is obtained.

For any (p, q) graph G with $\delta(G) \geq 1$, the line covering number of $B_4(G)$ is given by

$$\alpha_1(B_4(G)) = \begin{cases} q, & \text{if } q \geq p \\ q + \left\lceil \frac{p-q}{2} \right\rceil, & \text{if } q < p \end{cases}$$

Remark 2.5.

If G has m isolated vertices, then

$$\alpha_1(B_4(G)) = \begin{cases} q + \left\lceil \frac{m}{2} \right\rceil, & \text{if } q \geq p \\ q + \left\lceil \frac{p-q}{2}, \right\rceil, & \text{if } q < p \end{cases}$$

Note 2.1.

1. If $G \cong C_n, n \geq 3$, then $\alpha_1(B_4(G\ )) = n$
2. If $G \cong K_{1,n}$, $n \geq 2$, then $\alpha_1(B_4(G\ )) = n-1$

In the following, chromatic number of $B_4(G)$ is found.

Theorem 2.9.

Let G be any (p, q) graph ($p \geq 3$). Then the chromatic number $\chi(B_4(G))$ of $B_4(G)$ is p.

Proof:

The subgraph of $B_4(G)$ induced by p vertices in G is complete in $B_4(G)$. Colour these p vertices by p colours 1, 2, …, p. Let $e_{ij} = (v_i, v_j) \in E(G)$, $i \neq j$. Then $e_{ij} \in V(B_4(G))$. Colour the vertex e_{ij} by a colour other than i and j. Since no two vertices in $B_4(G)$ corresponding to the edges in G are adjacent, $B_4(G)$ is p-colourable. Hence, $\chi(B_4(G)) = p$.

Note 2.2.

1. If $G \cong K_2$, then $\chi(B_4(G)) = 3$.
2. If $G \cong 2K_1$ then $\chi(B_4(G)) = 2$.

Theorem 2.10.

For any (p, q) graph G, edge chromatic number of $B_4(G)$ is given by $\chi'(B_4(G)) = p + \Delta - 1$ or $p + \Delta$, where $\Delta = \max(\deg_G v)$

Proof:

Maximum degree of $B_4(G)$ = $p + \Delta - 1$ and hence the theorem follows.

In the following, the edge partition of $B_4(G)$ for some known graphs are given.

Theorem 2.11

Edges of $B_4(C_n), (n \geq 3)$ can be partitioned to

(i) $\left(\frac{n-1}{2}\right) C_n$ and C_{2n}, if n is odd

(ii) $\left(\frac{n-2}{2}\right)C_n, \left(\frac{n}{2}\right)K_2$ and C_{2n}, if n is even

Proof:

Let $G \cong C_n$, $n \geq 3$. Then $B_4(C_n)$ has n vertices of degree (n + 1) and n vertices of degree 2.

Let vertices of $B_4(G)$ having degree (n + 1) be $v_1, v_2, \ldots, v_n$ and that of degree n be e_{12}, $e_{23}, \ldots, e_{n-1,n}, e_{n1}$. Edges of $B_4(C_n)$ can be partitioned into K_n and C_{2n}.

Case (1): n is odd

Edges of K_n can be partitioned into $\left(\frac{n-1}{2}\right)$ cycles of length n. Therefore, edges of $B_4(C_n)$ can be partitioned to

$\left(\frac{n-1}{2}\right)C_n$ and C_{2n}, if n is odd

Case (2): n is even

Edges of K_n can be partitioned into $\left(\frac{n-1}{2}\right)$ cycles of length n and $\left(\frac{n}{2}\right)K_2$.

Therefore, edges of $B_4(C_n)$ can be partitioned to $\left(\frac{n-2}{2}\right)C_n, \left(\frac{n}{2}\right)K_2$ and C_{2n}, if n is even.

Theorem 2.12.

Edges of $B_4(K_n), (n \geq 4)$ can be partitioned into

(i) $\left(\frac{n-1}{2}\right)C_n$ and $\left(\frac{n-1}{2}\right)C_{2n}$, if n is odd

(ii) $\left(\frac{n-2}{2}\right)C_n, \left(\frac{n-2}{2}\right)C_{2n}$ and $\left(\frac{n}{2}\right)P_3$, if n is even.

Proof:

$B_4(K_n)$ has $n + \left(\frac{n(n-1)}{2}\right)$ vertices and $B_4(K_n)$ has n vertices of degree 2(n-1) and $\left(\frac{n(n-1)}{2}\right)$ vertices of degree 2 and K_n is an induced subgraph of $B_4(K_n)$.

Case(1): n is odd

Edges of K_n can be partitioned into $\left(\frac{n-1}{2}\right)$ cycles of length n. Consider n vertices in $B_4(K_n)$ corresponding to n edges of a cycle in K_n. These vertices together with n vertices in K_n forms a C_{2n} and a K_n in $B_4(K_n)$. Corresponding to $\left(\frac{n-1}{2}\right)$ cycles of length n in G, there exist $\left(\frac{n-2}{2}\right)C_{2n}$ and K_n in $B_4(K_n)$.

Case (2): n is even

Edges of K_n can be partitioned into $\left(\frac{n-2}{2}\right)$ cycles of length n and $\left(\frac{n}{2}\right)K_2$. As before, corresponding to $\left(\frac{n-2}{2}\right)$ cycles of length n in G, there exist $\left(\frac{n-2}{2}\right)C_{2n}$ and K_n in $B_4(K_n)$. Consider $\left(\frac{n}{2}\right)K_2$ in G. These edges form $\left(\frac{n}{2}\right)$ vertices in $B_4(K_n)$ and each of these $\left(\frac{n}{2}\right)$ vertices is adjacent to two vertices in $B_4(K_n)$. These form $\left(\frac{n}{2}\right)P3$.

Theorem 2.13

Edges of $B_4(K_{1,n}),(n \geq 3)$ can be partitioned into

(i) $\left(\frac{n-2}{2}\right)C_n$ and nC_3, if n is odd

(ii) $nC_3\left(\frac{n-2}{2}\right)C_n, \left(\frac{n}{2}\right)K_2$ and nC_3, if n is even.

Proof:

Let $G \cong K_{1,n}$. $B_4(G)$ has 2n + 1 vertices. Let v, v_1, v_2, … , v_n $\in V(G)$, where v is the central vertex and let e_1, e_2, …, e_n be the n edges in G, where $e_i = (v, v_i)$. Let x_1, x_2, …, x_n be the vertices in $B_4(G)$ corresponding to e_1, e_2, …, e_n. In $B_4(G)$, v is adjacent to x_1, x_2, …, x_n and each v_i is adjacent to x_i, i = 1, 2, …, n. Also the subgraph induced by v, v_1, v_2, …, v_n forms K_{n+1} in $B_4(G)$. Therefore, vv_ix_i (i = 1, 2, …, n) forms n triangles in $B_4(G)$ and the subgraph induced by v_1, v_2, .., v_n forms K_n in $B_4(G)$. Therefore, edges of $B_4(G)$ can be partitioned into K_1 and nC_3. Edges of K_n can be partitioned into $\left(\frac{n-2}{2}\right)C_n$, if n is odd and can be partitioned into $\left(\frac{n-2}{2}\right)C_n$ and $\left(\frac{n}{2}\right)K_2$, if n is even. Hence the Theorem.

Remark 2.6.

If G is a (p,q) graph, then edges of $B_4(G)$ can be partitioned into qC_3 and $E(\overline{G})$.

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