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Equi-eccentric point set graph, Glue graph and Complement Glue graph of a graph

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Abstract: *For any graph G, define the Equi-eccentric point set graph G_{ee}, on the same set of vertices, by joining two vertices in G_{ee} if and only if they correspond to two vertices of G with equal eccentricities. Also, define the Glue graph G_g on the same set of vertices by joining two vertices in G_g if and only if they correspond to two adjacent vertices of G or two adjacent vertices of G_{ee} and define the complement Glue graph G_{cg} on the same set of vertices by joining two vertices in G_{cg} if and only if they correspond to two adjacent vertices of $\overline{G}$ or two adjacent vertices of G_{ee}. In this paper, the concept of Equi-eccentric point set graph, Glue graph and complement Glue graph of a graph are introduced and some general and structural properties of G_{ee}, G_g and G_{cg} are presented.*

Key words: *Equi-eccentric point set graph, Glue graph, Complement Glue graph.*

1.Introduction

We consider only finite undirected graphs without loops and multiple edges and follow Buckley and Harary [2] for definitions.

Let G be a connected (p, q) graph with vertex set V = V(G) and edge set E = E(G). Let E_k denote the set of vertices of G with eccentricity k and $|E_k| = c_k$, the cardinality of E_k. We have $c_r \geq 1$ and $c_i > 1$, i = r+1, …,d.[2]

Definition 1.1 [1] The Equi-eccentric point set graph G_{ee} is a graph with vertex set V(G) and two vertices are adjacent if and only if they correspond to two vertices of G with equal eccentricities.

Definition 1.2 [1] The Glue graph G_g of a graph G is a graph with the same vertex set V(G) and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or two adjacent vertices of G_{ee}; that is $E(G_g) = E(G) \cup E(G_{ee})$.

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Definition 1.3 [1] The complement glue graph G_{cg} of a graph G is a graph with the same vertex set V(G) and two vertices are adjacent in G_{cg} if and only if they correspond to two adjacent vertices of $\overline{G}$ or two adjacent vertices of G_{ee}, that is $E(G_{cg}) = E(\overline{G}) \cup E(G_{ee})$.

It is to be noted that for a given G, the graphs G_{ee}, G_g and G_{cg} are uniquely defined, but for a given G_{ee}, G_g, G_{cg} there may exist more than one graph H such that $G_{ee} = H_{ee}$ and $G_g = H_g$, $G_{cg} = H_{cg}$.

The importance of perfect graphs is both theoretical and practical because of their application to perfect channels in communication theory, problems in operations research, optimizing municipal services etc. The Glue graph G_g and the Complement Glue graph G_{cg} are Hamiltonian and perfect. Also, G is a spanning subgraph of G_g and $\overline{G}$ is a spanning subgraph of G_{cg} and connectivity of G_g increases and diameter of G_g decreases as that of G. Hence, these graphs will be useful in communication theory.

Throughout this paper, if not mentioned, G means a connected (p, q) graph with diameter d and radius r.

2. Properties of G_{ee}, G_g

Some observations:

2.1. G_{ee} is disconnected if $d \neq r$, with $n = d-r+1$ components $C_1, C_2, \ldots, C_n$, where each C_i is a complete graph on c_{r+i-1}.

2.2. $G = G_{ee}$ if and only if G is complete.

2.3. G is self-centered if and only if G_{ee} is complete.

2.4. G_{ee} has K_1 as a component if and only if G has only one central vertex.

2.5. $G_{ee} = \overline{G}$ if and only if G is a star (or G is totally disconnected).

2.6. If G is not self-centered, $\overline{G_{ee}}$ is a complete multi-partite graph.

2.7. If G has an unique central vertex v, $\overline{G_{ee}}$ is a connected graph with the same central vertex v with radius one.

2.8. $\overline{G_{ee}}$ is a star graph if and only if G is bi-eccentric with exactly one central vertex.

2.9. If G has more than one central vertex such that G is not self-centered, $\overline{G_{ee}}$ is self-centered with diameter two.

2.10. The induced sub graphs formed by $E_r, E_{r+1}, \ldots, E_d$ are complete in G_g.

2.11. If $|V(G)| \geq 3$, then G_g has no pendent vertices and has at least two central vertices.

2.12. G is a spanning subgraph of G_g. G_g is complete if and only if G is self-centered or G is of radius one.

2.13. If G is connected such that G has no edge $uv \in E(G)$ with $e(u) = e(v)$, then G_{ee} is a spanning subgraph of $\overline{G}$.

2.14. If $G = P_{2m+1}$, $G_{ee} = mK_2 \cup K_1$ and if $G = P_{2m}$, $G_{ee} = mK_2$.

2.15. $(G_1)_{ee} \cong (G_2)_{ee}$, where G_1, G_2 are connected graphs if and only if for each E_k of G_1; G_2 has some E_k' with the same number of vertices and vice-versa.

2.16. G is self-centered if and only if G_g and G_{cg} are complete and are equal to K_p.

Theorem 2.1: $\overline{G}_{ee}$ is geodetic if and only if G is bi-eccentric with unique central vertex.

Proof: Assume $\overline{G}_{ee}$ is geodetic. We claim that G is bi-eccentric with unique central node. Suppose not, G is n-eccentric, where $n \geq 3$. (G is not self-centered, since if G is self-centered, $\overline{G}_{ee}$ is totally disconnected). $\overline{G}_{ee}$ is a complete n-partite graph $K(c_r, c_{r+1}, ..., c_d)$, where $n = d-r+1$, $c_r \geq 1$, $c_{r+1}, \ldots, c_d \geq 2$. Hence, if $n \geq 3$, $c_{r+1}, c_{r+2} \geq 2$ implies that there is an induced C_4 in $\overline{G}_{ee}$. Therefore $\overline{G}_{ee}$ is not geodetic, which is a contradiction. Therefore, $n = 2$, that is G is bi-eccentric. Now, suppose $c_r > 1$, $\overline{G}_{ee} = K_{m,n}$ where $m = c_r$, $n = c_{r+1}$ contains an induced C_4. Hence, c_r must be one. This proves the claim. Conversely, assume that G is bi-eccentric with unique central vertex. $G_{ee} = K_1 \cup K_n$, $\overline{G}_{ee} = K_{1,n}$ which is a star and hence $\overline{G}_{ee}$ is geodetic.

Theorem 2.2: Let G be a geodetic graph. Then G_g is geodetic if and only if G is one of the following:
(1) G is self-centered. (2) G is a graph with diameter two and radius one.

Proof: Let G be a geodetic graph such that G_g is also geodetic.
Case 1: G_g is complete.
In this case G is self-centered or bi-eccentric with radius one.
Case 2: G_g is not complete.
Since G_g is not complete, G is not self-centered. Therefore, G is n eccentric, where $n = d-r+1$. We claim that $n = 2$. Suppose not, $n \geq 3$. This implies that G_g must have a C_4 or K_4-e. Hence, G_g is not geodetic, which is a contradiction to our assumption. Hence, n must be two. That is, G is bi-eccentric.
Claim G is of radius one.

Suppose not. Let us assume that G is a bi-eccentric graph with radius $r > 1$. There exists no element in E_r, which is adjacent to all the elements of E_{r+1}. If there exists $u \in E_r$ which is adjacent to exactly one element in E_{r+1}, there exists an induced C_4 in G_g, otherwise there is an induced $K_4 - e$ in G_g, which is a contradiction to our assumption. Hence, G is of radius one and diameter two. Proof of the converse is obvious.

Remark: In the above case, G_g is always complete.

Definition 2.1: A graph is said to be *Glue* if it is the glue graph of some graph H. A graph is said to be *complement Glue* if it is the complement glue graph of some graph H.

Definition 2.2 A graph is said to be *unique glue graph* if it is the Glue graph of exactly one graph.

Lemma 2.1: G is a Glue graph if and only if there exists a spanning sub graph H of G such that $H_g = G$.

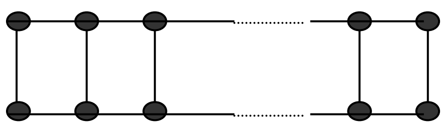
Proof: G is a Glue graph if and only if there exists a graph H such that $H_g = G$ by Definition 2.1. But H is a spanning sub graph of H_g. Hence the lemma is proved.

Theorem 2.3: Let H be a connected graph, then H is unique Glue if and only if $H = G_g$, where $G = P_{2n}$.

Proof: By Lemma 2.1, H is a Glue graph if and only if there exists a spanning sub graph G of H such that $H = G_g$.

Assume $G = P_{2n}$ such that $H = G_g$. Then H = F, where F is given in the following Figure.

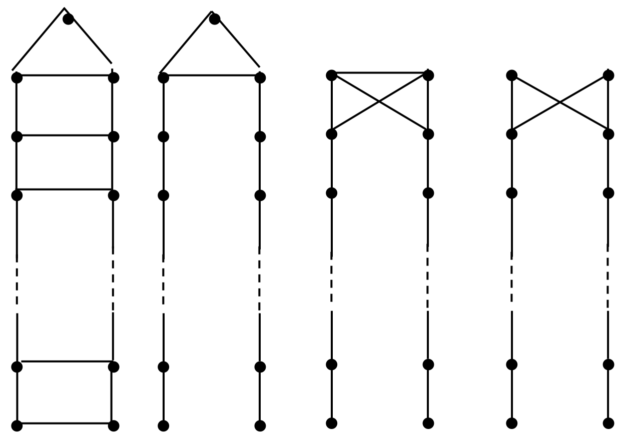
We claim that F is unique Glue. Suppose not, there exists G' ($\neq P_{2n}$) such that $G'_g = F$. Therefore, G' contains 2n vertices and $V(G')$ is partitioned into $V_1, V_2, \ldots, V_n$ each containing exactly two elements with equal eccentricities since G'_g is triangle free. Since $G' \neq P_{2n}$, the central vertices in G' must be adjacent to more than one element of E_{r+1}. This implies that G'_g contains a clique with more than two vertices, which is a contradiction to $G'_g = F$.



Therefore, H is unique Glue. On the other hand, assume H is unique Glue. Since H is Glue, there exists G, a spanning sub graph of H such that $H = G_g$. Now, we claim that $G = P_{2n}$.

Let $E_k = \{u \in V(G):e(u) = k \text{ in } G\}$. If there exists k such that $|E_k| > 2$, for $k \geq r$, we can find more than one graph G such that $H = G_g$, which is a contradiction to our assumption. Therefore, $|E_k|$ is at most 2 for all $k \geq r$. But we know that $|E_k| \geq 2$, for all $k > r$. Hence, $|E_k| = 2$ for all $k > r$, where r is the radius of G. Next, claim that $|E_r| = 2$.

Suppose not, $|E_r| = 1$, that is G is uni-central and $|E_r| = 2$ for all $k > r$. Let v be the central vertex of G. Then v must be adjacent to at least two vertices of eccentricity r+1. But $|E_{r+1}| = 2$. Let $E_{r+1} = \{u_1, u_2\}$. v is adjacent to both u_1 and u_2 in G. Also, the two elements in E_k are not adjacent to each other in G for $k > r+1$. Therefore, $G = P_{2n+1}$. Also, when $G = P_{2n+1}$, $G_g = F_g$, where F is given in the following Figure.



F_g F F_1 F_2

Thus, H is not unique Glue, which is a contradiction. Hence, $|E_r| = 2$. Therefore, $|E_r| = 2$, for all $k = r, r+1, \ldots, d$.

Also, each central vertex is adjacent to exactly one vertex in E_{r+1}. Otherwise, G is of the form F_1 or F_2. This gives $F_{1g} = F_{2g} = F_{3g} = H$, which is again a contradiction to H is unique Glue. Hence, $|E_r| = 2$ and each central vertex is adjacent to exactly one vertex of E_{r+1}. This implies that $G = P_{2n}$ and $H = G_g$. This proves the theorem.

3. Some more properties of G_g

Theorem 3.1: G_g is Hamiltonian.

Proof: In G_g, the induced sub graphs formed by vertices of E_i are complete. Now, consider E_{d-1}. We know at least two elements of E_d are adjacent to two different elements of E_{d-1}.

Let $v_d \in E_d$ be adjacent to $v_{d-1} \in E_{d-1}$ and $v_d{}' \in E_d$ be adjacent to $v_{d-1}{}' \in E_{d-1}$. Since $< E_d >$ is complete in G_g, starting from v_d, traverse all the vertices of E_d and finally come to $v_d{}'$. Join v_d to v_{d-1} and $v_d{}'$ to $v_{d-1}{}'$. Since $< E_{d-1} >$ is complete, we can traverse all the elements in E_{d-1} and there are at least two edges in G, joining distinct elements in E_{d-1} to E_{d-2}. So we can go to E_{d-2}. Proceeding like this, we can traverse all the vertices of G_g and get a Hamiltonian cycle in G_g. This proves the theorem.

Theorem 3.2: Let G be a connected graph with radius r and diameter d.

1) Clique cover number $\theta(G_g) = d-r$ or $d-r+1$
2) Independence number $\beta_o(G_g) = d-r$ or $d-r+1$
3) $\omega(G_g) = \max\{ c_r+c_{r+1}, c_{r+2},\ldots, c_d\}$ or $\max\{ c_r, c_{r+1},\ldots, c_d\}$
4) $\chi(G_g) = \max\{ c_r+c_{r+1}, c_{r+2},\ldots, c_d\}$ or $\max\{ c_r, c_{r+1}, \ldots, c_d\}$

Proof of (1): Let G be a graph with radius r and diameter d. Therefore G is n eccentric where $n = d-r+1$. If every element of E_r is adjacent to every element of E_{r+1}, $< E_r \cup E_{r+1} >$ is complete in G_g, and hence there are $d-r$ complete sub graphs and hence $\theta(G_g) = d-r$. If there exists an element in E_r, which is not adjacent to some elements of E_{r+1}, $\theta(G_g) = d-r+1$.

Proof of (2): If each element of E_r is adjacent to all elements of E_{r+1}, then $< E_r \cup E_{r+1} >$ is complete in G_g. Therefore, taking one element from each clique (which are not adjacent) we can form an independent set, whose cardinality is largest. This proves the result (2).

Proof of (3): If each $v_r \in E_r$ is adjacent to all elements of E_{r+1}, $< E_r \cup E_{r+1} >$ is complete in G_g. Hence, $\omega(G_g) = \max\{ c_r+c_{r+1}, c_{r+2},\ldots, c_d \}$. If there exists $v_r \in E_{r+1}$, not adjacent to some elements of E_{r+1}, $\omega(G_g) = \max\{c_r, c_{r+1},\ldots, c_d\}$. This proves the result (3).

Proof of (4): If each element of E_r is adjacent to all elements of E_{r+1}, $< E_r \cup E_{r+1} >$ is complete in G_g. If $< E_r \cup E_{r+1} >$ is complete, let $c_k = \max\{c_r+c_{r+1}, c_{r+2}, \ldots, c_d\}$ or let $c_k = \max\{ c_r, c_{r+1},\ldots, c_d\}$. Color the vertices in E_k by k colors. Now, an element in E_{k+1} or E_{k-1} is not adjacent to at least one element of E_k, and $c_i \leq c_k$ for $i \neq k$.

Thus, we can color the elements in E_{k+1} or elements in E_{k-1} by at most k colors. Similarly, we can color all the elements of $V(G_g)$ with at most k colors, so that no two adjacent points have the same color. Thus, $\chi(G_g) \leq c_k$. This proves the result (4) and hence the Theorem.

Lemma 3.1: G_g is free from induced C_n's and their compliments for $n \geq 5$.

Proof: From the construction of G_g, we can see that G_g is free from induced cycles of length n, $n \geq 5$. Now, we shall prove that $\overline{G_g}$ is free from C_n, $n \geq 5$. Suppose $\overline{G_g}$ contains an induced C_5. Let v_1, v_2, v_3, v_4, v_5 be the vertices of G_g which form an induced C_5. Note that, (1) These five vertices cannot be in same E_i, $i \geq r$. (2) Three or more vertices cannot lie in same E_i. (3) At the most two vertices (which are not adjacent in C_5 that is independent) lie in same E_i. (4) Suppose in $\overline{G_g}$, $v_1 \in E_i$, $v_2 \in E_{i+1}$, …, $v_5 \in E_{i+4}$ then C_5 is not induced, since v_1, v_3 are adjacent in $\overline{G_g}$; v_2, v_4 are adjacent in $\overline{G_g}$ etc. (5) Suppose $v_1 \in E_i$; $v_2, v_5 \in E_{i+1}$; $v_3, v_4 \in E_{i+2}$; then also C_5 is not induced since in $\overline{G_g}$ every element of E_i is adjacent to elements of E_{i+2}, E_{i+3} etc.

Similarly, in all possible cases, we can see that C_5 is not induced in $\overline{G_g}$. Therefore, G_g is free from C_5 and its complement. In a similar way, we can prove that $\overline{G_g}$ is free from C_n, for $n \geq 5$. Hence, G_g is free from C_n's and $\overline{C_n}$'s for $n \geq 5$.

Theorem 3.3: G_g is perfect.

Proof: We can prove this in two ways.

By Lemma 3.1, G_g is free from induced C_n and $\overline{C_n}$ for all $n \geq 5$. Therefore, G_g is Berge and is free from C_n and $\overline{C_n}$ for $n \geq 6$. Therefore, G_g is perfect.

By theorem 3.2 $\chi(G_g) = \omega(G_g)$, $\beta o(G_g) = \theta(G_g)$. Also, as in theorem 3.2, we can prove that $\chi(H) = \omega(H)$ and $\beta o(H) = \theta(H)$ for any induced sub graph H of G_g. Hence, G_g is perfect.

Sridharan and George [6] has defined B-Graph. A graph is a B-graph if every vertex is in a maximum independent set.

Lemma 3.2: If each element of E_{r+1} is adjacent to every element of E_r, then G_g is a B-graph.

Proof: In this case $< E_r \cup E_{r+1} >$ forms a complete sub graph in G_g. Hence, there are $d-r$ complete sub graphs in G_g. We know that an element $v_{r+i} \in E_{r+i}$ is not adjacent to every element of E_{r+i-1}, and is not adjacent to every element of E_{r+i+1}. Therefore, taking one element from each complete sub graph $< E_r \cup E_{r+1} >$, $< E_{r+2} >$, …, $< E_d >$ in G_g (which are not adjacent) we can form an independent set with maximum cardinality $d-r$. Also, in this case every vertex of G_g is in a maximum independent set. Hence, G_g is a B-graph.

Lemma 3.3: If each element of E_{r+1} is not adjacent to all elements of E_r and $c_r > 1$, then G_g is a B-graph.

Proof: By the given conditions there are $d-r+1$ complete sub graphs in G_g, and we can form an independent set with maximum cardinality $d-r+1$, by taking one element from each E_{r+i} (which are not adjacent) and every vertex of G_g is in some maximum independent set. Therefore, G_g is a B-graph.

Lemma 3.4: If G is uni-central with central vertex v_r and v_r is not adjacent to every elements of E_{r+1}, G_g is not a B-graph.

Proof: By the assumption $< E_r \cup E_{r+1} >$ is not complete. Therefore, there are $d-r+1$ complete sub graphs $< E_r >, < E_{r+1} >, \ldots, < E_d >$. Let $D = \{ v_r \} \cup \{v_{r+1}\} \cup \ldots \cup \{v_d\}$, where v_{r+1} is not adjacent to the central vertex v_r and v_{r+i} is not adjacent to v_{r+i-1} and v_{r+i+1}. D is a maximum independent set in G_g, $\beta(G_g) = d-r+1$. But if $v_{r+1}' \in E_{r+1}$ is adjacent to v_r, then v_{r+1}' is not in any maximum independent set of G_g. Therefore, G_g is not a B-graph.

Theorem 3.4: G_g is not a B-graph if and only if G is uni-central and there exists at least one vertex with eccentricity r+1, which is not adjacent to the center.

Proof: Follows from Lemmas 3.2, 3.3 and 3.4.

Theorem 3.5: G_g is pancyclic if $G \neq P_{2n}$.

Proof: Let $G \neq P_{2n}$. Then G_g contains cycles (not induced) of length 3, 4, …, $|V(G)|$. Therefore, G_g is pancyclic. If $G = P_{2n}$, G_g does not contain cycles of length 3, 5, 7, …. Hence, G_g is not pancyclic when $G = P_{2n}$.

4. Properties of the graph G_{cg}

Some observations of graph G_{cg} and $\overline{G}_{cg}$

(1) Diameter and radius G_{cg} is less then or equal to that of G. Diameter and radius of $\overline{G}_{cg}$ is greater than or equal to that of G.

(2) For a connected G, the graph G_{cg} is always connected, but $\overline{G}_{cg}$ may be disconnected.

(3) G_{cg} is complete if and only if G is self-centered. $\overline{G}_{cg}$ is totally disconnected if and only if G is self-centered.

(4) If $r(G) = 1$ and $d(G) = 2$, then $\overline{G}_{cg}$ is complete bipartite.

(5) If $r(G) = 1$ with only one central vertex, then G_{cg} is complete and $\overline{G}_{cg}$ is $K_{1,p-1}$.

(6) $\overline{G}$ is a spanning sub graph of G_{cg} and $\overline{G}_{cg}$ is a spanning sub graph of G.

(7) If $r(G) > 1$ and G is bi-eccentric, then $\overline{G}_{cg}$ is a bipartite graph.

(8) If $\overline{G}_{cg}$ is connected then each vertex of E_i for $i > r$ has at least one predecessor.

The first two propositions are immediate from the definitions.

Proposition 4.1: $G_{cg} = \overline{G}$ if and only if G has no edge $e = uv$, where $e_G(u) = e_G(v)$.

Proposition 4.2: If G is a tree with uni-center, then $\overline{G}_{cg} = G$. If G is a tree with bi-center, then $\overline{G}_{cg}$ is disconnected.

Proposition 4.3: $\overline{G}_{cg}$ has no triangles. In general it has no odd cycles.
Proof: If G has a cycle C_3, then at least two vertices of that C_3 have the same eccentricity in G. Hence, in $\overline{G}_{cg}$, at least one edge of that triangle is deleted. Hence, $\overline{G}_{cg}$ has no triangles. Similarly, it has no odd cycles.

Remark 4.1: $\overline{G}_{cg}$ may have some induced C_{2n}.

Proposition 4.4: $\overline{G}_{cg}$ is connected if and only if for any two vertices u and v of G, there exists a path joining u and v in which end points of each edge have different eccentricities.

Proof: Proof is clear from the definition.

Proposition 4.5: $\overline{G}_{cg}$ is Hamiltonian if and only if G has a Hamilton cycle in which end points of each edge have different eccentricities.

Proof: Proof is clear from the definition. Note that $\overline{G}_{cg}$ is Hamiltonian only when p is even.

Proposition 4.6: $d-1 \leq |E(\overline{G}_{cg})| \leq \min\{q, [p^2/4]\}$.
Proof: $\overline{G}_{cg}$ has no triangles. Hence, $|E(\overline{G}_{cg})| \leq [p^2/4]$ and $\overline{G}_{cg}$ is a sub graph of G. Hence, $|E(\overline{G}_{cg})| \leq \min\{q, [p^2/4]\}$. Since d is the diameter of G, G has at least one diametral path of length d, which has at least two edges joining vertices of different eccentricities (namely d and $d-1$) and corresponding to other vertices there exists at least $d-3$ edges joining vertices of different eccentricities. This proves the proposition.

Remark 4.2: If $G = P_{2n}$, $|E(\overline{G}_{cg})| = d-1$. When $G = P_{2n+1}$, $|E(\overline{G}_{cg})| = d = q$.

Other properties of G_{cg} and $\overline{G}_{cg}$ are studied and published in [4].

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