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Total Eccentric domination in Graphs

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Abstract: *A set $S \subseteq V(G)$ is a total eccentric dominating set if S is an eccentric dominating set and also the induced sub graph <S> has no isolated vertices. The cardinality of minimum total eccentric dominating set is called the total eccentric domination number and is denoted by $\gamma_{te}(G)$. In this paper, we present several bounds on the total eccentric domination number and exact values of some particular graphs.*

Keyword: *Total domination, eccentric domination, total eccentric domination.*

1. Introduction

Let G be a finite, simple, undirected graph on n vertices with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [5] Buckley and Harary [4].

Definition 1.1: Let G be a connected graph and u be a vertex of G. The **eccentricity** e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max{d(u, v) : u $\in$ V}.The **radius** r(G) is the minimum eccentricity of the vertices, whereas the **diameter** diam(G) = d(G) is the maximum eccentricity. For any connected graph G, r(G) ≤ diam(G) ≤ 2r(G). v is a central vertex if e(v) = r(G). The **center** C(G) is the set of all central vertices. For a vertex v, each vertex at a distance e(v) from v is an **eccentric vertex of v. Eccentric set of a vertex** v is defined as E(v) = {u $\in$ V(G) / d(u, v) = e(v)}.

Definition 1.2: The **open neighborhood** N(v) of a vertex v is the set of all vertices adjacent to v in G. N[v] = N(v)$\cup${v} is called the **closed neighborhood** of v.

Definition 1.3: A **bigraph or bipartite graph** G is a graph whose point set V can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. If further G contains every line joining the points of V_1 to the points of V_2 then G is called a **complete bigraph**. If V_1 contains m points and V_2 contains n points then the complete bigraph G is denoted by $K_{m,n}$.

Definition 1.4: A **star** is a complete bi graph $K_{1,n}$.

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Definition 1.5 [5, 11]: A set $D \subseteq V$ is said to be a **dominating set** in G, if every vertex in $V-D$ is adjacent to some vertex in D. The cardinality of minimum dominating set is called the **domination number** and is denoted by $\gamma(G)$.

Definition 1.6[9, 11]: A dominating set $D \subseteq V(G)$ is a **total dominating set** if dominating set and also the induced sub graph <D> has no isolated vertices. The cardinality of minimum total dominating set is called the **total domination number** and is denoted by $\gamma_t(G)$.

Definition 1.7 [7]: A set $D \subseteq V(G)$ is an **eccentric dominating set** if D is a dominating set of G and for every $v \in V-D$, there exists at least one eccentric point of v in D. The cardinality of minimum eccentric dominating set is called the **eccentric domination number** and is denoted by $\gamma_{ed}(G)$.

If D is an eccentric dominating set, then every superset $D' \supseteq D$ is also an eccentric dominating set. But $D'' \subseteq D$ is not necessarily an eccentric dominating set.

An eccentric dominating set D is a **minimal eccentric dominating set** if no proper subset $D'' \subseteq D$ is an eccentric dominating set.

We need the following results to prove certain results in total eccentric domination.

Theorem 1.1[5]: For any graph G, $\lceil n/(1+\Delta(G)) \rceil \leq \gamma(G) \leq n-\Delta(G)$.

Theorem 1.2 [7]: $\gamma_{ed}(K_n) = 1$

Theorem 1.3 [7]: $\gamma_{ed}(K_{m,n}) = 2$.

Theorem 1.4 [7]: $\gamma_{ed}(W_3) = 1$, $\gamma_{ed}(W_4) = 2$, $\gamma_{ed}(W_n) = 3$ for $n \geq 7$.

Theorem 1.5 [7]: $\gamma_{ed}(P_n) = \gamma(P_n)$ or $\gamma(P_n) + 1$.

Theorem 1.6[7]: (i) $\gamma_{ed}(C_n) = n/2$ if n is even.

$$\text{(ii) } \gamma_{ed}(C_n) = \begin{cases} n/3 & \text{if } n = 3m \text{ and is odd.} \\ \lceil n/3 \rceil & \text{if } n = 3m+1 \text{ and is odd.} \\ \lceil n/3 \rceil + 1 & \text{if } n = 3m+2 \text{ and is odd.} \end{cases}$$

Theorem 1.7 [9]: If G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \leq 2n/3$.

Observation 1.1[9]: If v is a support vertex of a graph G, then v is in every γ_t(G)-set.

Observation: 1.2[10]: For any connected graph G with diameter at least three, there exists a γ_t(G)-set that contains no leaves of G.

Observation: 1.3[10]: Every tree T of order n $\geq$ 3 and with s support vertices satisfies $\gamma_t(T) \leq (n+s)/2$.

2. Total Eccentric domination

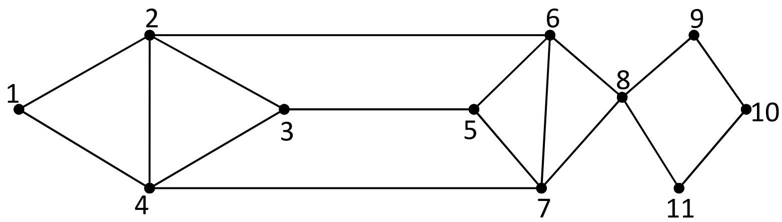
We define total eccentric dominating set of a graph as follows.

A set S $\subseteq$ V(G) is a **total eccentric dominating set** if S is an eccentric dominating set and also the induced sub graph <S> has no isolated vertices. The cardinality of minimum total eccentric dominating set is called the **total eccentric domination number** and is denoted by γ_{te}(G).

Clearly, (i) For any graph G, $\gamma(G) \leq \gamma_t(G) \leq \gamma_{te}(G)$.

(ii) For any graph G, **.** $\gamma(G) \leq \gamma_{ed}(G) \leq \gamma_{te}(G)$. But γ_t(T) and γ_{ed} (T) are incomparable.

Example 2.1:



G

Figure 2.1

D_1 = {2, 6, 10} is a dominating set, $\gamma(G) = 3$.

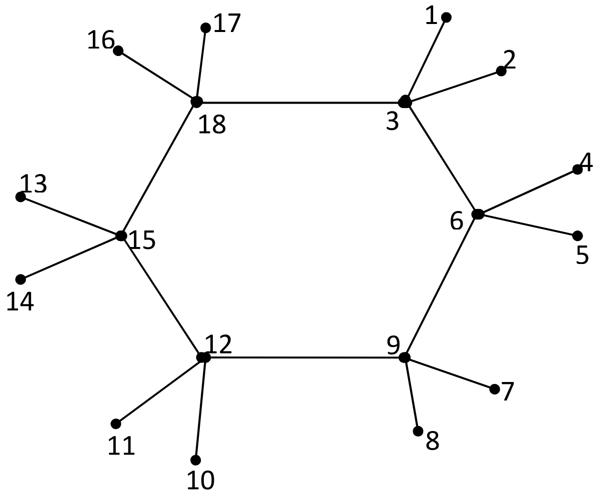
D_2 = {1, 2, 6, 10} is an eccentric dominating set, $\gamma_{ed}(G) = 4$.

D_3 = {2, 6, 9, 10} is a total dominating set $\gamma_t(G) = 4$.

D_4 = {1, 2, 6, 9, 10} is a total eccentric dominating set $\gamma_{te}(G) =5$.

Here, $\gamma(G) < \gamma_t(G) = \gamma_{ed}(G) < \gamma_{te}(G)$

Example 2.2:



G

Figure 2.2

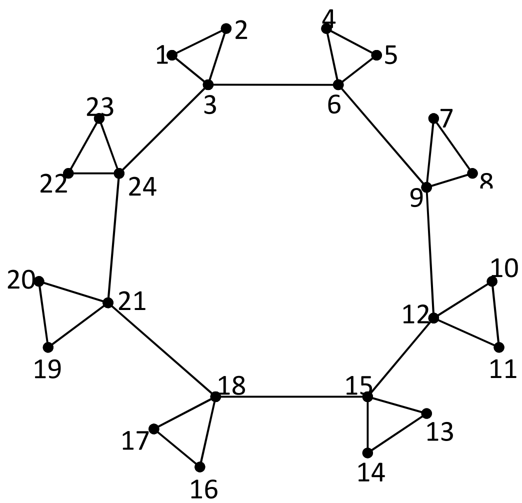
D_1 = {3, 6, 9, 12, 15, 18} is a dominating set, $\gamma(G) = 6$.

D_2 = {1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18} is an eccentric dominating set and also a total eccentric dominating set. $\gamma_{ed}(G) = \gamma_{te}(G) = 12$.

D_3 = {3, 6, 9, 12, 15, 18} is a total dominating set, $\gamma_t(G) = 6$.

$\gamma_{te}(G) = 6 = 2n/3$. Here, $\gamma(G) = \gamma_t(G) = \gamma_{ed}(G)$.

Example 2.3:



G

Figure 2.3

D_1 = {3, 6, 9, 12, 15, 18, 21, 24} is a dominating set, $\gamma(G) = 8$

D_2 = {1, 4, 7, 10, 14, 17, 20, 23} is an eccentric dominating set, $\gamma_{ed}(G) = 8$

D_3 = {3, 6, 9, 12, 15, 18, 21, 24} is a total dominating set, $\gamma_t(G) = 8$.

D_4 = {1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24} is a total eccentric dominating set, $\gamma_{te}(G) = 16$. Here, $\gamma_{ed}(G) < \gamma_{te}(G)$, $\gamma_t(G) < \gamma_{te}(G)$, $\gamma_t(G) = \gamma_{ed}(G)$.

Note: $\gamma_{ed}(G) \leq (n/3) + \gamma(G)$.

Theorem 2.1: $\gamma_{te}(K_n) = 2$.

Proof: When $G = K_n$, radius = diameter r = 1. Hence any vertex $u \in V(G)$ dominates other vertices and is also an eccentric vertex of other vertices. But the induced sub graph has isolated vertex. Hence, any two vertices of G form a total eccentric dominating set, that is $\gamma_{te}(K_n) = 2$.

Theorem 2.2 $\gamma_{te}(K_{1,n}) = 2$, $n \geq 2$.

Proof: When $G = K_{1,n}$. Let S = {u, v}, v central vertex. The central vertex dominates all other vertices in $V - S$ and u is an eccentric vertex of vertices of $V - S$. The induced sub graph <S> has no isolated vertices. Hence $\gamma_{te}(K_{1,n}) = 2$, $n \geq 2$.

Theorem: 2.3: $\gamma_{te}(K_{m,n}) = 2$.

Proof: When $G = K_{m,n}$. $V(G) = V_1 \cup V_2$. $|V_1| = m$ and $|V_2| = n$ such that each element of V_1 is adjacent to every vertex of V_2 and vice versa. Let S = {u, v}, $u \in V_1$ and $v \in V_2$. u dominates all the vertices of V_2 and it is eccentric to elements of $V_1 - \{u\}$. Similarly v dominates all the vertices of V_1 and it is eccentric to elements of $V_2 - \{v\}$. The induced sub graph <S> had no isolated vertices. Hence $\gamma_{te}(K_{m,n}) = 2$.

Theorem 2.4: $\gamma_{te}(W_n) = 3$, $n \geq 5$, $\gamma_{te}(W_3) = 2$, $\gamma_{te}(W_4) = 2$.

Proof: $G = W_3 = K_4$. Hence $\gamma_{te}(W_3) = 2$.

When $G = W_4$. Consider S = {u, v}, where u and v are adjacent non-central vertices. The induced sub graph <S> has no isolated vertices, and S is a minimum total eccentric dominating set. Therefore $\gamma_{te}(W_4) = 2$.

When $G = W_n$. Let S = {u, v, w} where u and v are any two adjacent non-central vertices and w is the central vertex. The induced sub graph <S> has no isolated vertices, S is an eccentric dominating set. Therefore S is a minimum total eccentric dominating set of G.

Hence $\gamma_{te}(W_n) = 3$, $n \geq 5$.

Theorem 2.5: $\gamma_{te}(P_n) = \begin{cases} \frac{n}{2} + 2 \text{ if } n = 4m \\ \left\lceil \frac{n}{2} \right\rceil + 1 \text{ if } n = 4m+1 \\ \frac{n}{2} + 1 \text{ if } n = 4m+2 \\ \left\lceil \frac{n}{2} \right\rceil + 1 \text{ if } n = 4m+3 \text{ for } n > 4 \end{cases}$

$\gamma_{te}(P_3) = 2$, $\gamma_{te}(P_4) = 3$.

Proof: Case (i) n = 4m

When $n \geq 5$ an eccentric dominating set of P_n must contain the two end vertices. Let $v_1, v_2, \ldots, v_{4m}$ represent the path P_n. $S = \{ v_1, v_2, v_5, v_6, \ldots, v_{4m-3}, v_{4m-2}, v_{4m-1}, v_{4m}\}$ is a minimal total eccentric dominating set of G. $|S| = \frac{n}{2} + 2$. Thus $\gamma_{te}(P_n) \leq \frac{n}{2} + 2$…………….. (i)

In this case, P_n has exactly one minimum total dominating set. $\gamma_t(P_n) = n/2$ and the γ_t-set contains no end vertices. Therefore $\gamma_{te}(P_n) \geq \frac{n}{2}+1$. And no eccentric dominating set containing $\frac{n}{2}+1$ vertices is not total dominating set. Hence $\gamma_{te}(P_n) \geq \frac{n}{2} + 2$……………(ii)

From (i) and (ii) $\gamma_{te}(P_n) = \frac{n}{2} + 2.$

Case (ii) n = 4m+1

$S = \{v_1, v_2, v_5, v_6, \ldots, v_{4m}, v_{4m+1}\}$ is a minimal total eccentric dominating set of G. $|S| = \left\lceil \frac{n}{2} \right\rceil + 1$. Thus $\gamma_{te}(P_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1$…………….. (i)

In this case, $\gamma_t(P_n) = \left\lceil \frac{n}{2} \right\rceil + 1$. We have $\gamma_t(P_n) \leq \gamma_{te}(P_n)$. $\gamma_{te}(P_n) \geq \left\lceil \frac{n}{2} \right\rceil + 1$……………(ii)

From (i) and (ii) $\gamma_{te}(P_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$

Case (iii) n= 4m+2

In this case, $S = \{v_1, v_2, v_5, v_6, \ldots, v_{4m+1}, v_{4m+2}\}$ is a minimal total eccentric dominating set of G. $|S| = \frac{n}{2} + 1$. Thus $\gamma_{te}(P_n) \leq \frac{n}{2} + 1$…………….. (i)

We know that, $\gamma_t(P_n) = \frac{n}{2} + 1$. We have $\gamma_t(P_n) \leq \gamma_{te}(P_n)$.

Therefore $\gamma_{te}(P_n) \geq \frac{n}{2} + 1$…………… (ii)

From (i) and (ii) $\gamma_{te}(P_n) = \frac{n}{2} + 1.$

Case (iv) n = 4m+3

$S = \{v_1, v_2, v_5, v_6, \ldots, v_{4m+1}, v_{4m+2}, v_{4m+3}\}$ is a minimal total eccentric dominating set of G. $|S| = \left\lceil \frac{n}{2} \right\rceil + 1$. Thus $\gamma_{te}(P_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1$…………….. (i)

In this case, $\gamma_t(P_n) = \left\lceil \frac{n}{2} \right\rceil$. But no minimum total dominating set of P_n is an eccentric dominating set. $\gamma_t(P_n) < \gamma_{te}(P_n)$. $\gamma_{te}(P_n) > \left\lceil \frac{n}{2} \right\rceil$.............. (ii)

From (i) and (ii) $\left\lceil \frac{n}{2} \right\rceil < \gamma_{te}(P_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1.$

This implies, $\gamma_{te}(P_n) = \left\lceil \frac{n}{2} \right\rceil +1$. When G = P_3. S = $\{v_1, v_2\}$ is a minimum total eccentric dominating set. Hence $\gamma_{te}(P_3) = 2$.

When G = P_4. S = $\{v_1, v_2, v_3\}$ is a minimum total eccentric dominating set. Hence $\gamma_{te}(P_4)$ = 3.

Theorem 2.6: $\gamma_{te}(C_3) = 2$.

$$\gamma_{te}(C_n) = \begin{cases} \frac{n}{2} \text{ if } n = 4m \\ \left\lceil \frac{n}{2} \right\rceil \text{ if } n = 4m+1 \\ \frac{n}{2} + 1 \text{ if } n = 4m+2 \\ \left\lceil \frac{n}{2} \right\rceil + 1 \text{ if } n = 4m+3, \text{where } m \text{ is odd} \end{cases}$$

$$\gamma_{te}(C_n) = \begin{cases} \frac{n}{2} + 2 \text{ if } n = 4m \\ \left\lceil \frac{n}{2} \right\rceil + 1 \text{ if } n = 4m+1 \\ \frac{n}{2} + 1 \text{ if } n = 4m+2 \\ \left\lceil \frac{n}{2} \right\rceil \text{ if } n = 4m+3, \text{where } m \text{ is even} \end{cases}$$

Proof: When G = C_3. S = $\{v_1, v_2\}$ is a minimum total eccentric dominating set. Hence $\gamma_{te}(C_3) = 2$.

Let S be a minimum total eccentric dominating set of C_n. S is a total dominating set implies <S> has no isolated vertices. Since C_n is a cycle if a vertex i is in S implies either $i-1$ or $i+1$ is also in S.

Let r be the radius of C_n. We know that C_n is a self – centered graph. When n is even, radius of C_n = n/2. Therefore n = 2r. In this case denote the vertices of C_n by 1, 2, 3, …, 2r. Hence the eccentric vertex of i is i + r (in mod n).

When n is odd, radius of $C_n = \frac{n-1}{2}$. Therefore, n = 2r + 1. In this case denote the vertices of C_n by 1, 2, 3, …, 2r+1. Hence the eccentric vertices of i is i+r and i+r+1.

When m is odd.

Case (i) n = 4m = 2r.

In this case, S = {1, 2, 5, 6, …, r—1, r, r+3, r+4,…, 2r-3, 2r-2} is a minimum total eccentric dominating set and $|S| = \frac{n}{2}$.

Thus, $\gamma_{te}(C_n) \leq n/2$…………….. (i)

We know that, $\gamma_t(C_n) = n/2$. $\gamma_{te}(C_n) \geq \gamma_t(C_n)$. $\gamma_{te}(C_n) \geq n/2$ …………..(ii)

From (i) and (ii) $\gamma_{te}(C_n) = n/2$.

Case (ii) n = 4m+1 = 2r+1

In this case, S = {1, 2, 5, 6, …, r—1, r, r+3, r+4, …, 2r-3, 2r-2, …, 2r+1} is a minimum total eccentric dominating set and $|S| = \left\lceil \frac{n}{2} \right\rceil$

Thus, $\gamma_{te}(C_n) \leq \left\lceil \frac{n}{2} \right\rceil$ ……………(i)

We know that, $\gamma_t(C_n) = \left\lceil \frac{n}{2} \right\rceil$. $\gamma_{te}(C_n) \geq \gamma_t(C_n)$. $\gamma_{te}(C_n) \geq \left\lceil \frac{n}{2} \right\rceil$………………(ii)

From (i) and (ii) $\gamma_{te}(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Case (iii) n = 4m+2 = 2r

In this case, S = {1, 2, 5, 6, 9, 10, …, r—2, r—1, r+2, r+3, r+6, r+7, …, 2r—1, 2r} is a minimum total eccentric dominating set and $|S| = \frac{n}{2} + 1$.

Thus, $\gamma_{te}(C_n) \leq (n/2)+1$. ………….(i)

We know that, $\gamma_t(C_n) = (n/2)+1$. $\gamma_{te}(C_n) \geq \gamma_t(C_n)$. $\gamma_{te}(C_n) \geq (n/2)+1$…………..(ii)

From (i) and (ii) $\gamma_{te}(C_n) = (n/2)+1$.

Case (iv) n = 4m+3 = 2r+1

In this case, S = {1, 2, 5, 6, 9, 10, …, r—2, r—1, r+2, r+3, r+4, r+7, r+8, …, 2r, 2r+1} is a total eccentric dominating set and $|S| = \left\lceil \frac{n}{2} \right\rceil + 1$.

Thus, $\gamma_{te}(C_n) \leq \left\lceil \frac{n}{2} \right\rceil$+1………….(i)

We know that, $\gamma_t(C_n) = \left\lceil \frac{n}{2} \right\rceil$+1. $\gamma_{te}(C_n) \geq \gamma_t(C_n)$. $\gamma_{te}(C_n) \geq \left\lceil \frac{n}{2} \right\rceil$+1…………..(ii)

From (i) and (ii) $\gamma_{te}(C_n) = \left\lceil \frac{n}{2} \right\rceil$+1.

When m is even

Case (i) n = 4m = 2r

In this case, S = {1, 2, 5, 6 ,…, r—3, r—2, r+1, r+2, r+3, r+7, r+8, …, 2r—1, 2r} is a total eccentric dominating set and

$|S| = \frac{n}{2} + 2$.

Thus, $\gamma_{te}(C_n) \leq (n/2)+2$. We know that, $\gamma_t(C_n) = n/2$. Any total dominating set containing n/2 or n/2+1 vertices is not an eccentric dominating set. Hence $\gamma_{te}(C_n) = (n/2)+2$.

Case (ii) n = 4m+1 = 2r+1

In this case, S = {1, 2, 5, 6, 9, 10, …, r+1, r+2, r+3, r+4, r+7, r+8, …, 2r−1, 2r} is a total eccentric dominating set and $|S| = \left\lceil \frac{n}{2} \right\rceil + 1$.

Thus, $\gamma_{te}(C_n) \leq \left\lceil \frac{n}{2} \right\rceil + 1$ …………….(i)

We know that, $\gamma_t(C_n) = \left\lceil \frac{n}{2} \right\rceil$. Any total dominating set containing n/2 or n/2+1 vertices is not an eccentric dominating set. Hence $\gamma_{te}(C_n) = \left\lceil \frac{n}{2} \right\rceil + 1$.

Case (iii) n = 4m+2 = 2r

In this case, S = {1, 2, 5, 6, 9, 10, …, r, r+1, r+3, r+4, r+7, r+8, …, 2r−2, 2r−1} is a total eccentric dominating set and $|S| = \frac{n}{2} + 1$.

Thus, $\gamma_{te}(C_n) \leq (n/2)+1$…………(i)

We know that, $\gamma_t(C_n) = (n/2)+1$. $\gamma_{te}(C_n) \geq \gamma_t(C_n)$. $\gamma_{te}(C_n) \geq (n/2)+1$…………..(ii)

From (i) and (ii) $\gamma_{te}(C_n) = (n/2)+1$.

Case (iv) n = 4m+3 = 2r+1

In this case, S = {1, 2, 5, 6, 9, 10, …, r, r+1, r+3, r+4, r+7, r+8, …, 2r−2, 2r−1} is a total eccentric dominating set. $|S| = \left\lceil \frac{n}{2} \right\rceil$.

Thus, $\gamma_{te}(C_n) \leq \left\lceil \frac{n}{2} \right\rceil$. ………….(i)

We know that, $\gamma_t(C_n) = \left\lceil \frac{n}{2} \right\rceil$. $\gamma_{te}(C_n) \geq \gamma_t(C_n)$. $\gamma_{te}(C_n) \geq \left\lceil \frac{n}{2} \right\rceil$. …………..(ii)

From (i) and (ii) $\gamma_{te}(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 2.7: If G is of diameter two, $\gamma_{te}(G) \leq 1+\delta(G)$.

Proof: diam(G) = 2. Let $u \in V(G)$ such that deg u = $\delta(G)$. Consider, S = {u}∪N(u). This is a total eccentric dominating set of G. the induced sub graph <S> has no isolated vertices. Therefore, $\gamma_{te}(G) \leq 1+\delta(G)$.

Theorem 2.8: If G is of radius two and diameter three and if G has a pendent vertex v of eccentricity three then $\gamma_{te}(G) \leq 1+\Delta(G)$.

Proof: If G has a pendent vertex v of eccentricity three then its support u is of eccentricity two. In this case, N(u) is an eccentric dominating set. But it is not total eccentric dominating set. N(u)∪{u} form a total eccentric dominating set. Hence $\gamma_{te}(G) \leq 1+\Delta(G)$.

Theorem 2.9: If G is a spider, then $\gamma_{te}(G) = \Delta(G)+2$.

Proof: Let G be a spider, and u be a vertex of maximum degree $\Delta(G)$. u is the central vertex, $|N[u]|$ vertices form a total dominating set. Adding any one end vertex form a total eccentric dominating set and this is the minimum total eccentric dominating set. Hence

$\gamma_{te}(G) = \Delta(G)+2$.

Theorem 2.10: If G is a wounded spider, then $\gamma_{te}(G) = s+2$. where s is the number of support vertices which are adjacent to non-wounded legs.

Proof: Let G be a wounded spider. Let u be the vertex of maximum degree $\Delta(G)$, and S be the set of support vertices which are adjacent to non-wounded legs. The set $|S| \cup \{u\}$ form a total dominating set. But it is not a total eccentric dominating set. Adding any one end vertex form a minimum total eccentric dominating set. Hence $\gamma_{te}(G) = s+2$, where $s = |S|$.

Theorem 2.11: For any connected graph G, $\gamma_{te}(G) \leq 2/3(n+e(G))$.

Proof: Let us assume that G be a graph on n vertices, with eccentric number e(G). Let G′ be a graph on n+e(G) vertices obtained from G as follows:

Let $S = \{ u_1, u_2, \ldots, u_{e(G)}\}$ be the minimum eccentric point set of G. Attach a new vertex u_1' to u_1, u_2' to $u_2, \ldots, u'_{e(G)}$ to $u_{e(G)}$ by edges. Denote the new graph obtained as G′. Then $|V(G')| = n+e(G) = |V(G)| +e(G)$. Now, by the previous observations, there exists a γ_t-–set D containing no u_i' of G′, containing all u_i's and $\gamma_{te}(G') \leq 2/3(n+e(G))$.Therefore, D is a total eccentric dominating set of G. Therefore $\gamma_{te}(G) \leq 2/3(n+e(G))$.

Corollary 2.1: $\gamma_{te}(T) \leq 2/3(n+2)$.

Theorem 2.12: If G is a connected graph with $|V(G)| = n$. Then $\gamma_{te}(G \circ K_1) \leq n+e(G)$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let v_i' be the pendent vertex adjacent to v_i in GoK_1, for i = 1, 2, …, n.

Let k = e(G) be the number of eccentric vertices of G and let them be $\{x_1, x_2, \ldots, x_k\}$. Then $V(G) \cup \{x'_1, x'_2, \ldots, x'_k\}$ is a total eccentric dominating set.

Therefore $\gamma_{te}(G \circ K_1) \leq n+e(G) = n+k$.

Corollary 2.2: $\gamma_{ced}(G) \leq n+e(G)$, where $\gamma_{ced}(G)$ is a connected eccentric dominating set.

Theorem: 2.13: If G is of radius one and diameter two, then $\gamma_{te}(G) \leq (n-t+2) / 2$, where t is the number of vertices with eccentricity one.

Proof: V(G) can be partitioned into two sets V_1 and V_2 as follows. $V_1 = \{v \in V/e(v) = 1\}$ and $V_2 = \{v \in V/e(v) = 2\}$. Let $u \in V(G)$ such that $e(u) = 1$. Let t be the number of vertices with eccentricity one. u dominates all other vertices and for t−1 other vertices u is an eccentric point. Consider the remaining (n-t) vertices of G. Let S be the subset of V_2 vertices such that vertices in V_2−S have their eccentric vertices in S. Now S∪{u} is a total eccentric dominating set.

Hence $\gamma_{te}(G) \leq 1 + \frac{(n-t)}{2}$ = (n-t+2) / 2.

Theorem 2.14: If G is of radius one and diameter two, then $\gamma_{te}(G) \leq 2 + \left\lceil \frac{\delta(G)-t}{2} \right\rceil$, where t is the number of vertices with eccentricity one.

Proof: V(G) can be partitioned into two sets V_1 and V_2 as follows. $V_1 = \{v \in V/e(v) = 1\}$ and $V_2 = \{v \in V/e(v) = 2\}$. Let u be a vertex in V_2 such that degree of u in H = <V_2> is minimum. Let $deg_H u = \delta_2 = \delta(G) - t$. Then u is eccentric to the remaining $n - t - \delta_2$ vertices. Let S be a subset of $N_H(u)$ such that vertices in $N_H(u)$−S have their eccentric vertices in S. Hence {u, v} ∪ S, where $v \in V_1$ is a total eccentric dominating set of G.

Therefore, $\gamma_{te}(G) \leq 2 + \left\lceil \frac{\delta(G)-t}{2} \right\rceil$.

Theorem 2.15: If G is of radius two and diameter three, then $\gamma_{te}(G) \leq 1 + \frac{(n + \deg_G u - 1)}{2}$.

Proof: Let u be a central vertex with minimum degree. Consider N[u]. N[u] dominates all other vertices of G. Let S be a subset of $N_2(u)$ with minimum cardinality such that vertices in $N_2(u)$−S has their eccentric vertices in S. Then $|S| \leq \frac{|N_2(u)|}{2} = \frac{(n - \deg_G u - 1)}{2}$.

Now N[u]∪S is a total eccentric dominating set of G. Hence, $\gamma_{te}(G) \leq |N[u]| + |S| \leq 1 + \deg_G u + \frac{(n - \deg_G u - 1)}{2} = 1 + \frac{(n + \deg_G u - 1)}{2}$.

Observation 2.1: $1 < \gamma_{te}(G)$

Proof: Every $\gamma_{te}(G)$-set does not contain isolated vertex. Hence $1 < \gamma_{te}(G)$.

Observation: 2.2:

(i) If $G = \overline{K_2} + K_1 + K_1 + \overline{K_2}$, then $\gamma(G) = 2$, $\gamma_{ed}(G) = 4$, $\gamma_t(G) = 2$, $\gamma_{te}(G) = 4$.

(ii) If $G = K_n + K_1 + K_1 + K_n$, n > 2 then, $\gamma(G) = 2$, $\gamma_{ed}(G) = 2$, $\gamma_t(G) = 2$, $\gamma_{te}(G) = 4$.

Theorem 2.16: Let G be a connected graph. Then, $\gamma_{te}(G) = 2$, if and only if G is any one of the following

(i) $G = K_n$.

(ii) $r(G) = 1$, $d(G) = 2$ and $u \in V(G)$ such that $e(u) = 2$ and $d(u, v) = 2$ for all $v \in V(G)$ with $e(v) = 2$.

(iii) G is self – cantered of diameter 2, having a dominating edge which is not in a triangle.

Proof: When G satisfies any one of the above conditions obviously, $\gamma_{te}(G) = 2$. On the other hand, assume that, $\gamma_{te}(G) = 2$. Therefore $\gamma(G) = 1$ or $\gamma(G) = 2$.

Case (i)

$\gamma(G) = 1$ and, $\gamma_{te}(G) = 2$. This implies that G satisfies (i) and (ii).

Case (ii)

$\gamma(G) = 2 = \gamma_{te}(G)$.

Let D be a minimum γ_{te}−dominating set of G. Let $D = \{u, v\} \subseteq V(G)$. Since $\gamma(G) = 2$, $r(G) \geq 2$. Since D is connected u and v are adjacent and the edge uv is dominating edge for G. Therefore $r(G) \geq 2$ and $2 \leq d(G) \leq 3$. Suppose $d(G) = 3$, there exists a vertex x with eccentricity 3 and x is dominated by u or v.

Let $xu \in E(G)$. Now, d is a γ_{te}−set. Hence v must be an eccentric point of x. this implies that $d(x, v) = 3$. But xuv is a path imples $d(x, v) = 2$. Which is a contradiction. Hence, x must be a vertex with eccentricity 2. This implies that $d(G) = 2$, G is a self –cantered with diameter 2. There exists no w, adjacent to both u and v, since in that case, w has no eccentric point in D. Since $r(G) \geq 2$.

Theorem 2.17: If G is bi−eccentric with $\gamma(G) = 2$, that is $r(G) = 2$, $d(G) = 3$, then $\gamma_{te}(G) \leq 4$.

Proof: $\gamma(G) = 2$ implies $D = \{u, v\}$ is a dominating set. Since $d(G) = 3$. Then $d(u, v) = 1, 2$ or 3.

Case (i)

$d(u, v) = 1$.

Since D is connected. u and v are adjacent and the edge uv is a dominating edge for G. let $x, w \in V(G)$ such that $e(x) = e(w) = 3$. w, x are dominated by u or v. Consider $D_1 = \{u, v\} \cup \{w, x\}$ is an eccentric dominating set, and if $y \in V-D$ and $e(y) = 2$, it can't be adjacent to both x and w. Therefore w, x are eccentric to y. If $s \in V-D$ and $e(s) = 3$, it can't be adjacent to w or x. therefore w or x is eccentric to s. $\langle D_1 \rangle$ has no isolated vertices. Thus D_1 is a total eccentric dominating set. Hence $\gamma_{te}(G) \leq 4$.

Case (ii)

d(u,v) = 2. Then e(u) = e(v) = 2 and uxv is a shortest path. Let y, w $\in$ V(G) such that e(y) = e(w) = 3. y and w are dominated by u or v. If z $\in$ V$-$D with e(z) = 2, it cannot be adjacent to y. Therefore y is eccentric to z. Then D' = {u, v}$\cup${y, w} is a total dominating set. Hence γ_{te}(G) $\leq$ 4.

Case (iii)

d(u, v) = 3. Then e(u) = e(v) = 3 and uxyw is a shortest path. Eccentric point of y must be u and eccentric point of x must be w. Therefore e(x) = e(y) = 2. Then D'' = {u, v}$\cup${x, y} is a total eccentric dominating set. Hence γ_{te}(G) $\leq$ 4.

Theorem 2.18; If T is a non-trivial tree of order n with l leaves, then, $\frac{n+2-l}{2} \leq \gamma_{te}(G) \leq \frac{n+2+s}{2}$, where s is the number of support vertices of T.

Proof: Let us assume that T be a tree on n vertices and with l leaves. Let T' be a tree on n+2 vertices with attaching 2 vertices x and y with u and v, where u and v are peripheral vertices of T at distance 'd' to each other. From [Observation: 1.3]. Therefore γ_t(T) $\leq \frac{n+2+s}{2}$, from [Observation: 1.2]. Therefore T' is a total eccentric dominating set of T. Therefore γ_{te}(T) $\leq$ (n+2+s)/2. Also, γ_{te}(T) $\geq$ (n+2-l)/2. Therefore γ_{te}(T) $\geq$ γ_t(T) $\geq$ (n+2-l)/2, implies γ_{te}(T) $\geq$ (n+2-l)/2.

Theorem 2.19: For a bi-central tree with radius 2, γ_{te}(T) $\leq$ min {n-Δ(G)+1, 4} **.**

Proof: Let u and v be the central vertices of T, then N[u] and N[v] are total eccentric dominating sets of T. V$-$N(u) $\cup$ {v}, V$-$N(v) $\cup$ {u} are also total eccentric dominating sets of T.

Also deg u + deg v = n. Hence, γ_{te}(T) $\leq$ n $-$ Δ(G) + 1. All the four vertices of a diametral path also form a total dominating set. Hence γ_{te}(T) $\leq$ min {n-Δ(G)+1, 4} **.**

Observation 2.3: For any tree T, $\gamma_t(T) \leq \gamma_{te}(T) \leq \gamma_t(T)+ 2$.

Theorem 2.20 For a tree T, with radius two which is not a path

(i) γ_{te}(T) = γ_t(T) + 1if there exists at least one peripheral vertex with support vertex s, such that deg s = 2.

(ii) γ_{te}(T) = γ_t(T) + 2 if degree of all the support vertices are greater than two.

Proof: Let T be a **bi - central** tree.

In this case, diam(T) = 3 and T = $\overline{K_n} + K_1 + K_1 + \overline{K_m}$ for n, m $\geq$ 1. When n = m = 1, γ_{te}(T) = 3, γ_t(T) = 2. Hence γ_{te}(T) = γ_t(T) +1

When n, m > 1, $\gamma_{te}(T) = \gamma_t(T) + 2 = 4$.

Let T be a **unicentral** tree.

In this case, diam(T) = 4 and let $T \neq P_5$. Let S be the set of all support vertices of T and let v be the central vertex of T.

(a) If there exists only one peripheral vertex with degree of its support = 2. Let x be a peripheral vertex and it is eccentric to all other vertices except its support. Then $(S\cup\{v\})$ is a γ_t-set of T and $(S\cup\{v\} \cup\{x\})$ is a γ_{te}-set of T. Hence $\gamma_{te}(T) = \gamma_t(T) +1$.

(b) If there exists two peripheral vertices with degree of their supports = 2, then the degree of v is greater than two. Let x and y be two peripheral vertices and u, w be their supports such that deg u = deg w = 2. x or y is eccentric to each other vertices. Then $S\cup\{v\}$ is a γ_t-set of T and $(S\cup\{v\}\cup\{x\})$ is a γ_{te}-set of T. Hence $\gamma_{te}(T) = \gamma_t(T) +1$.

(ii) If degree of all the support vertices are greater than two, S is γ_t—set of T and $S\cup\{x,y\}$, where x and y are two peripheral vertices at distance 4 to each other is γ_{te}—set of T. Hence

$\gamma_{te}(T) = \gamma_t(T) + 2$

Corollary 2.3: (i) For a bi central tree T with radius 2, $\gamma_{te}(T) = 3$ if and only if $T = P_4$ or $T = \overline{K_n} + K_1 + K_1 + K_1$

(ii) For a bi central tree T with radius 2, $\gamma_{te}(T) = 4$ if and only if degree of the central vertices are ≥ 3.

Theorem 2.21: Let n be an even integer. Let G be obtained from the complete graph K_n by deleting edges of a linear factor. Then $\gamma_{te}(G) = n/2$.

Proof: Let u and v be a pair of non—adjacent vertices in G. Then u and v are eccentric to each other. Also, G is unique eccentric point graph. Therefore $\gamma_{te}(G) \geq n/2$............(i).

Consider $D \subseteq V(G)$ such that $\langle D\rangle = K_{n/2}$. D contains n/2 vertices such that each vertex in V—D is adjacent to at least one element in D and each element in V—D has its eccentric vertex in D and also $\langle D\rangle$ has no isolated vertices. Hence $\gamma_{te}(G) \leq n/2$..............(ii).

From (i) and (ii) $\gamma_{te}(G) = n/2$.

Theorem 2.22: $\gamma_{te}(\overline{C_4}) = 4$, $\gamma_{te}(\overline{C_5}) = 3$, and $\gamma_{te}(\overline{C_n}) = \left\lceil \frac{n}{3} \right\rceil$, for $n \geq 6$.

Proof: Clearly $\gamma_{te}(\overline{C_4}) = 4$, $\gamma_{te}(\overline{C_5}) = 3$. Now, assume that $n \geq 6$. Let $v_1, v_2, v_3, \ldots, v_n, v_1$ form C_n. Then $\overline{C_n} = K_n - C_n$ and each vertex v_i is adjacent to all other vertices except v_{i-1} and v_{i+1} in $\overline{C_n}$. Hence eccentric point of v_i in $\overline{C_n}$ is v_{i-1} and v_{i+1} only. Hence any total eccentric dominating set must contain either v_i or any one of v_{i-1}, v_{i+1}. Thus $\gamma_{te}(\overline{C}_n) \geq \left\lceil \frac{n}{3} \right\rceil$..........(i). Now, consider a total eccentric dominating set as follows.

$\{v_1, v_4, v_7, \ldots, v_{3m-2}\}$ if n = 3m.

$\{v_1, v_4, v_7, \ldots, v_{3m+1}\}$ if n = 3m+1.

$\{v_1, v_4, v_7, \ldots, v_{3m+1}, v_{3m+2}\}$ if n = 3m+2. Hence $\gamma_{te}(\overline{C}_n) \leq \left\lceil \frac{n}{3} \right\rceil$............(ii)

From (i) and (ii) $\gamma_{te}(\overline{C}_n) = \left\lceil \frac{n}{3} \right\rceil$, for $n \geq 6$.

Observations 2.4:

(i) $\gamma_{te}(G) + \gamma_t(G) \nleq \frac{4n}{3}$.

Example: G = C_7.

(ii)) $\gamma_{te}(G) + \gamma_{ed}(G) \leq \frac{4n}{3}$.

(iii) $\gamma(G) + \gamma_{te}(G) \nleq \frac{4n}{3}$.

Example: G = C_7.

(iv) i(G) + γ_{te}(T) $\nleq n$.

Example: G = $H \circ 2K_1$, where H = C_6. $\gamma_{te}(G)$ = 2n/3 and i(G) = n/2, where i(G) is the independent dominating number of G.

Theorem 2.23: If D is a minimal total eccentric dominating set of a connected graph G = (V, E), then each vertex u ∈ D, one of the following is true

(i) There exists some v∈V−D such that N(v)∩D = {u} (or) E(v)∩D = {u}.

(ii) <D−{u}> contains an isolated vertex or u has no eccentric vertex in D.

Proof: Assume that D is a minimal total eccentric dominating set of G. then for every vertex u ∈ D, D−{u} is not a total eccentric dominating set. That is there exists some vertex v in (V−D)∪{u} which is not dominated by any vertex in D−{u} or there exists v in (V−D)∪{u} such that v has no eccentric point in D−{u} or there exists a vertex v in D such that N(v)∩(D−{u}) = ϕ.

Case (i)

Suppose v ∈ V−D.

(a) If v is not dominated by D−{u}, but is dominated by D, then v is adjacent only u in D, that is N(v)∩D = {u}.

(b) Suppose v has no eccentric point in D−{u} but v has an eccentric point in D. Then u is the only eccentric point of v in D, that is E(v)∩D = {u}.

Case (ii)

Suppose u = v, then <D−{u}> contains an isolated vertex or u has no eccentric vertex in D.

Conversely, suppose D is a total eccentric dominating set and for each u ∈ D one of the conditions holds, we show that D is a minimal total eccentric dominating set.

Suppose that D is not a minimal total eccentric dominating set. That is, there exists a vertex u ∈ D such that D−{u} is a total eccentric dominating set. Hence every element x in V−D is adjacent to at least one vertex in D−{u} and x has an eccentric point in D−{u}. Hence (i) does not hold.

Also, if D−{u} is a total eccentric dominating set. Hence, <D−{u}> has no isolated vertex and u has an eccentric point in D−{u}. Therefore, condition (ii) does not hold. This is a contradiction to our assumption that for each u ∈ D, one of the conditions holds.

Theorem 2.24: If H is any self−centered unique eccentric point graph with m vertices and G = Ho2K_1, then $\gamma_{te}(G)$ = 2n/3 where n = 3m.

Proof: If H is any self−centered unique eccentric point graph, then every vertex of H is an eccentric vertex. Hence m is even and G has 3m vertices. Let $v_1, v_2, v_3, \ldots, v_m$ represent the vertices of H and $\{v_i', v_i''\}$ for i = 1, 2, 3, …, m be the vertices of m copies of 2K_1. Then in G, v_i', v_i'' are adjacent to v_i and if v_j is the eccentric vertex of v_i in H, then v_i', v_i'' are eccentric vertices of v_j in G and v_j', v_j'' are the eccentric vertices of v_i. It is clear that $\{v_1, v_2, v_3, \ldots, v_m\} \cup \{x_1, x_2, x_3, \ldots, x_m\}$, where $x_i = v_i'$ or v_i'' are minimum total eccentric dominating sets of G. Hence $\gamma_{te}(G)$ = 2n/3.

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