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Perfect, Split and Non-split domination of the Boolean graph $BG_2(G)$ and its complement

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Abstract: *Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). BG, INC, $\bar{L}(G)$ is a graph with vertex set V(G) ∪ E(G) and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by $BG_2(G)$, Boolean graph of G-second kind. In this paper, perfect domination, Split and Non-Split domination of $BG_2(G)$ and its complement are studied.*

Key words: *Boolean graph $BG_2(G)$, perfect domination, Split and Non-Split domination.*

1. Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [5], Buckley and Harary [2].

Definition 1.1 [6]: A set $S \subseteq V$ is said to be a *dominating set* in G, if every vertex in V−S is adjacent to some vertex in S. A dominating set D is an *independent dominating set*, if no two vertices in D are adjacent that is D is an independent set. A dominating set D is a *connected dominating set*, if < D > is a connected subgraph of G. A dominating set D is a *perfect dominating set*, if for every vertex u ∈ V(G)−D, |N(u) ∩ D|= 1. A dominating set D is a *total dominating set*, if < D > has no isolated vertices. A set $D \subseteq V(G)$ is a *global dominating set*, if D is a dominating set in G and $\bar{G}$.

[15] A dominating set D is a *Split dominating set*, if the induced subgraph < V−D > is disconnected. A dominating set D is a Non*Split dominating set*, if the induced subgraph < V−D > is connected.

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Definition 1.2 [6]: The *domination number* γ of G is defined to be the minimum cardinality of a dominating set in G. Similarly, one can define the perfect domination number γ_p, connected domination number γ_c, total domination number γ_t, independent domination number γ_i, global domination number γ_g , Split domination number γ_s and NonSplit domination number γ_{ns}.

__Motivation:__ The Line graphs, Middle graphs, Total graphs and Quasi-total graphs are very much useful in computer networks. In analogous to line graph, total graph, middle graph and quasi-total graph, thirty-two graphs can be defined using different adjacency relations. Out of these operations, eight were already studied. Among the remaining twenty-four graph operations, two are defined and analyzed in [1]. All the others have been defined and studied thoroughly and submitted. This is illustrated below.

G/ $\overline{G}$/K_p/ $\overline{K_p}$	Incident (INC)/ Non-incident (NINC)	$L(G)$/ $\overline{L(G)}$/K_q/ $\overline{K_q}$

Defining a new graph from a given graph by using the adjacency relation between two vertices or two edges and incident relationship between vertices and edges is known as Boolean operation. It defines new structure from the given graph and adds extra information of the original graph. In Management and in social networks, the incident and non-incident relations of vertices and edges are used to define various networks. So these are very much applicable in socio-economical problems. Also, these graph operations may be used in graph coding or coding of some grouped signal. This motivates the study for the exploration of various Boolean operations and study of their structural properties.

Definition 1.3: A vertex (point) and an edge are said to *cover* each other, if they are incident. A set of vertices, which cover all the edges of a graph G is called a (*vertex*) *point cover of* G, while a set of lines (edges), which covers all the vertices is a *line cover*. The smallest number of points in any point cover for G is called its *point covering number* and is denoted by $\alpha_o(G)$ or α_o. Similarly, $\alpha_1(G)$ or α_1 is the smallest number of lines in any line cover of G and is called its *line covering number*. Clearly, A point cover (line cover) is called *minimum*, if it contains α_o (respectively α_1) elements.

Definition 1.4: A set of points in G is *independent*, if no two of them are adjacent. The largest number of points in such a set is called the *point independence number* of G and is denoted by $\beta_o(G)$ or β_o. Analogously, an independent set of lines (matching) of G has no two of its lines adjacent and the maximum cardinality of such a set is the *line independence number* $\beta_1(G)$ or β_1, A set of independent edges covering all the vertices of a graph G is called a *1-factor* or a *perfect matching* of G.

Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). $B_{G,\ INC,\ \overline{L}(G)}$ [1, 11,12,13] is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by $BG_2(G)$, Boolean graph of G-second kind. In this paper, perfect domination, Split and Non-Split domination parameters of $BG_2(G)$ and its complement are studied.

Theorem 1.2 [1, 11, 13]: $\gamma(BG_2(G) = 1$ if and only if $G = K_n$.

Theorem 1.3 [1, 11, 13]: $\gamma(BG_2(G) \leq \gamma(G)+2$.

Theorem 1.4 [1, 11, 13]: If G is a graph with diam(G) ≤ 2, $\gamma(BG_2(G)) \leq \delta(G)+1$.

Theorem 1.5 [1, 11, 13]: If $G \neq K_{1,n}$, diam(G) ≤ 2 and G has a pendant vertex, then, $\gamma(BG_2(G)) = 2 = \gamma_c(BG_2(G))$.

2. Perfect domination of $BG_2(G)$ and $\overline{BG_2(G)}$

In this sub section, bounds for $\gamma_p(BG_2(G))$ and $\gamma_p(\overline{BG_2(G)})$ are found out. Values of $\gamma_p(BG_2(G))$ and $\gamma_p(\overline{BG_2(G)})$ for some graphs G are also found out.

Lemma 2.1: $\gamma_p(BG_2(G)) = 1$ if and only if $G = K_{1,n}$.
Proof: Follows from Theorem 1.1

Lemma 2.2: D = {u, e} is a perfect dominating set for $BG_2(G)$ if and only if $G = K_2 \cup K_1$.
Proof: Case 1: D = {u, e} is a perfect independent dominating set of $BG_2(G)$.
Therefore, e is not incident with u in G. Suppose e is incident to a vertex v, which is adjacent to u in G. Then D cannot be perfect. Also, both u and e dominates v, which is a contradiction. Suppose u is not isolated and e is an edge not incident to a vertex v, which is adjacent to u in G. Then u and e dominates the edge $e' = uv \in E(G)$. Therefore, u must be isolated and e must be the only edge in G, that is $G = K_2 \cup K_1$.
Case 2: D = {u, e} is connected in $BG_2(G)$.
D cannot be a perfect dominating set. Therefore, this case is not possible.

Converse is obvious.

Lemma 2.3: $D = \{u_1, u_2\} \subseteq V(G)$ is a perfect dominating set for $BG_2(G)$ if and only if $G = K_{1,n} \cup K_{1,m}$ or $K_{1,n} \cup K_1$.
Proof: Similar to the above.

Lemma 2.4: $D = \{e_1, e_2\} \subseteq E(G)$ is a perfect dominating set of $BG_2(G)$ if and only if $G = 2K_2$.

Proof: Since $D = \{e_1, e_2\}$ is a dominating set, all the point vertices are incident with e_1, e_2 only. Also, e_1, e_2 cannot be adjacent, since D is perfect. Also, if there exists any other edge in G, then that cannot be dominated by D in $BG_2(G)$. Hence $G = 2K_2$ only.

Theorem 2.1: $\gamma_p(BG_2(G)) = 2$ if and only if G is any one of the following:

(1) $G = K_2 \cup K_1$. (2) $G = K_{1,n} \cup K_{1,m}$. (3) $G = 2K_2$.

Proof: Follows from the previous Lemmas 2.1, 2.2, 2.3 and 2.4.

Theorem 2.2: Let G be a graph with $p \geq 6$. If G has a perfect matching D, then D is a perfect dominating set of $BG_2(G)$ if and only if $p = 6$ or $G = nK_2$.

Proof: $D \subseteq E(G)$ is a perfect matching. D is a dominating set for $BG_2(G)$. Suppose D is perfect dominating set of $BG_2(G)$. Then every point vertex and line vertex in $V(BG_2(G)) - D$ is adjacent to exactly one element of D. Since D is a perfect matching, in $BG_2(G)$, every point vertex is adjacent to exactly one element of D. Now, consider any line vertex not in D. Let $e \in E(G) \subseteq V(BG_2(G))$ and $e \notin D$. e is adjacent to exactly one element of D in $BG_2(G)$ if and only if the edge $e \in E(G)$ is adjacent to $(p/2) - 1$ edges of G, which are in D. But elements of D are independent edges of G. Therefore, e is adjacent to exactly one element of D if and only if $(p/2) - 1 \leq 2$, that is $p = 6$ (since $p \geq 6$) or there exists no element in $E(G) - D$, that is $G = nK_2$ or $p = 6$. This proves the theorem.

Theorem 2.3: Let D be a perfect dominating set of G such that D is a point cover for G. Then D is a perfect dominating set for $BG_2(G)$ if and only if G is a union of stars.

Proof: $D \subseteq V(G)$ is a perfect dominating set of G and D is a point cover of G. Therefore, D dominates $BG_2(G)$. Since D is a point cover of G, $< V(G) - D >$ is totally disconnected. Since D is a perfect dominating set of G, every element of $V(G) - D$, that is every point vertex of $BG_2(G)$, which is not in D is adjacent to exactly one vertex of D. Now, suppose D is a perfect dominating set for $BG_2(G)$, every line vertex of $BG_2(G)$ is adjacent to exactly one element of D in $BG_2(G)$, and hence in G, $< D >$ must have no edges. Therefore, in G, D is independent, $V(G) - D$ is independent and $\deg_G v = 1$ for all $v \in V(G) - D$. Hence, G is the union of stars. In particular G is bipartite. Converse is obvious.

Theorem 2.4: Let $D \subseteq V(G)$ be a perfect dominating set of G such that D is not independent. Let e be an edge in $< D >$. Then $D \cup \{e\}$ is a perfect dominating set of $BG_2(G)$ if and only if $< D > = K_2$.

Proof: Suppose there exists another e' in $< D >$. Then e' is adjacent to its incident vertices in D. Therefore, $D \cup \{e\}$ cannot be a perfect dominating set. Hence, $< D >$ contains only one edge. Now, suppose $|D| > 2$, there exists $e' \in E(G)$ not adjacent to e in G. Hence in $BG_2(G)$, e' is adjacent to two elements of $D \cup \{e\}$. Hence, $|D| = 2$, that is $< D > = K_2$.

Theorem 2.5: Let $D \subseteq V(G)$ be a perfect dominating set of G such that D is independent. Then $D \cup \{e\}$, $e \in E(G)$ is not a perfect dominating set for $BG_2(G)$.

Proof: Case 1: e is an edge in $< V(G)-D >$.

Let $e = uv \in E(G)$. In $BG_2(G)$, u and v are adjacent to e and some other element in D. Therefore, $D \cup \{e\}$ is not perfect in $BG_2(G)$.

Case 2: $e \in E(G)$, $e = uv$, $u \in D$, $v \notin D$.

In $BG_2(G)$, v is adjacent to both $u \in D$ and e. Hence, $D \cup \{e\}$ is not perfect in $BG_2(G)$.

Theorem 2.6: If G has a dominating edge, which is also a cut edge of G, then $\gamma_p(BG_2(G)) = 3$.

Proof: Let $e \in E(G)$ be a dominating edge of G, which is also a cut edge. Hence, $D = \{u, v\}$, where $e = uv \in E(G)$ is a perfect dominating set of G. Now, consider $D' = \{u, v, e\} \subseteq V(BG_2(G))$. Clearly, D' is a perfect dominating set for $BG_2(G)$. Also, $\gamma_p(BG_2(G)) = 3$. (D is not a perfect dominating set of $BG_2(G)$).

Theorem 2.7: Any line cover for G is a perfect dominating set of $BG_2(G)$, if and only if it is a perfect matching and $p = 6$ or $G = nK_2$.

Proof: Let $D \subseteq E(G)$ be a line cover for G. Suppose D is a perfect dominating set of $BG_2(G)$, every point vertex of $BG_2(G)$ is adjacent to only one element of D. This implies D is a perfect matching. Also, if G has more than three independent edges, $p > 6$ and $G \neq nK_2$, any line vertex not in D is adjacent to more than one element of D. Hence, D is a perfect dominating set if and only if $G = nK_2$ or $p = 6$ and G has a perfect matching.

The following propositions are stated without proof, since they are easy to follow.

Proposition 2.5: Let G be a graph without isolated vertices. If G has a pendant vertex u, $D = \{v, u, e\}$, where $e = uv \in E(G)$ is a perfect dominating set for $\overline{BG_2(G)}$ if and only if $G = K_2$.

Proposition 2.6: $D = \{u, v\} \subseteq V(G)$, where $d_G(u, v) \geq 3$ is a perfect dominating set for $\overline{BG_2(G)}$ if and only if $G = K_{1,n} \cup K_{1,m}$, $2K_2$ or $K_{1,n} \cup K_1$.

Proposition 2.7: $D = \{e_1, e_2\} \subseteq E(G)$ is a perfect dominating set for $\overline{BG_2}(G)$ if and only if $G = 2K_2$.

Proposition 2.8: $D = \{u, v, w\} \subseteq V(G)$ is a perfect dominating set for $\overline{BG_2}(G)$ if and only if $G = K_{1,2}$, K_3 or $K_2 \cup K_1$.

Proposition 2.9: Let G be a graph without isolated vertices. $D = E(G)$ is a perfect dominating set of $\overline{BG_2}(G)$ if and only if $G = 2K_2$.

Theorem 2.8: $D = \{u, v, e\}$, where $u, v \in V(G)$, $e = uv \in E(G)$ is a perfect dominating set for $\overline{BG_2}(G)$ if and only if $G = K_2$.

Proof: Let $D = \{u, v, e\}$, where $e = uv \in E(G)$ be a perfect dominating set for $\overline{BG_2}(G)$. If there exists $w \in V(G)$, $w \neq u, v$, then three different cases arises.

Case 1: w is adjacent to both u and v.

In this case, w is dominated by u and v in $\overline{BG_2}(G)$, which is a contradiction.

Case 2: w is adjacent to any one of u or v say u.

In this case, w is dominated by v and e in $\overline{BG_2}(G)$, which is a contradiction.

Case 3: w is adjacent to both u and v.

In this case, the line vertex corresponding to the edge uw is dominated by v and e and the line vertex corresponding to the edge vw is dominated by u and e in $\overline{BG_2}(G)$, which is again a contradiction. Hence, D is a perfect dominating set if and only if $G = K_2$.

Theorem 2.9: Let G be a graph without isolated vertices. $D = \{u, v, e\}$, $e \neq uv$ is a perfect dominating set for $\overline{BG_2}(G)$ if and only if $G = K_{1,2}$.

Proof: Similar to the proof of Theorem 2.8.

Remark 2.1: (1) If G has no isolated vertices and $p \geq 4$, then $D = V(G)$ is not a perfect dominating set for $\overline{BG_2}(G)$, since any line vertex is dominated by at least two point vertices of D in $BG_2(G)$. (2) $D = V(G)$ is a perfect dominating set of $\overline{BG_2}(G)$ if and only if $G = K_3$, $K_{1,2}$ or $K_3 \cup K_1$.

Theorem 2.10: If $p \geq 5$, $\overline{BG_2}(G)$ has no non-trivial perfect dominating set, where G has no isolated vertices.

Proof: It is already proved that the set of point vertices is not a perfect dominating set of $\overline{BG_2}(G)$ and the set of all line vertices is also not a perfect dominating set of $\overline{BG_2}(G)$ if $p \geq 5$. Similarly, it can be proved that $D \subseteq V(G)$ or $D \subseteq E(G)$ is also not a perfect dominating set, if $p \geq 5$. Now, suppose D contains both point vertices, line vertices and $D \subseteq V(\overline{BG_2(G)})$ is a perfect dominating set of $\overline{BG_2(G)}$. Since $\gamma(\overline{BG_2(G)}) \geq 2$, D contains at least two elements.

Consider the case when $p = 5$.

Let $V(G) = \{u, v, w, x, y\}$ and assume $x \in D$ and $e \in D$, where $e \in E(G)$.

Case 1: e is not incident with x.

Let e = uv. G has no isolated vertices. Therefore, x is adjacent to some vertex say w. e dominates w, therefore w must be adjacent to x in G since D is perfect.

Now, the line vertex corresponding to the edge xw is not dominated by x and e in $\overline{BG_2(G)}$. Hence, $xw = e'$ or y or u or v must be in D. (1) If $e' \in D$, y is dominated by e' and e, which is a contradiction. (2) If $y \in D$, y must be adjacent to w in G, otherwise y and e dominates w. Also, u and v must be adjacent to at least one of x and y. In this case, there is a contradiction, since e and y dominates $e_1 = xu$ or $e_2 = xv$ say e_1; e and x dominates $e_3 = yu$ or $e_4 = yv$ say e_4. Therefore, e_1 and e_4 must be in D. If $e_1, e_4 \in D$, then w is dominated by more than two elements of D, hence w must be in D. This implies that $xy = e_5 \in D$. Proceeding like this, it can be proved that $D = (V(\overline{BG_2(G)})$. When $p > 5$ also, it can be proved in a similar way that $D = V(\overline{BG_2(G)})$. Hence the theorem is proved.

3. Split and non-split dominating sets of $BG_2(G)$ and $\overline{BG_2(G)}$

In this sub section, bounds for $\gamma_s(BG_2(G))$, $\gamma_{ns}(BG_2(G))$, $\gamma_s(\overline{BG_2(G)})$ and $\gamma_{ns}(\overline{BG_2(G)})$ are studied. Values of these parameters for some graphs G are also found out.

Proposition 3.1: $\gamma_s(BG_2(G)) = 1$, if and only if $G = K_{1,n}$.

Proof: $\gamma_s(BG_2(G)) = 1$ if and only if $\gamma(BG_2(G)) = 1$ if and only if $G = K_{1,n}$.

Theorem 3.1: Let D be a dominating set of G and let $u \in V(G) - D$ such that $\deg_G u = \delta(G)$. Then $D^1 = D \cup N(u) \cup$ {line vertices which represent the edges incident with u} is a dominating set of $BG_2(G)$ and is a split dominating set, that is $\gamma_s(BG_2(G)) \leq \gamma(G)+2\delta(G)$.

Proof: Let $v \in D$ such that $e = uv \in E(G)$. e dominates all other line vertices in $V(BG_2(G)) - D^1$, which are not dominated by D. Also, in $< V(BG_2(G)) - D^1 >$, u is isolated. Hence, D^1 is a split dominating set of $BG_2(G)$. Therefore, $\gamma_s(BG_2(G)) < \gamma(G)+2\delta(G)$.

Proposition 3.2: Let G be a graph with radius one and let $u \in V(G)$ such that $e(u) = 1$. If G has a pendant vertex v, then $D = \{u\} \cup$ {line vertices corresponding to the edges in $G-u$} is a split dominating set for $BG_2(G)$.

Proof: Let $e = uv \in BG_2(G)$. v is pendant in G. Therefore, in $< V-D >$, $< \{e, u\} >$ is a component. Also, D is a dominating set for $BG_2(G)$. Hence, D is a split dominating set for $BG_2(G)$.

Proposition 3.3: Let $u \in V(G)$ be such that $\deg_G u = \Delta(G)$. Let D = {u, v, line vertices corresponding to edges in $G-u$}, $v \in N(u)$. Then D is a split dominating set for $BG_2(G)$.

Proof: D is a dominating set. Now, consider $e = uv \in E(G)$. In $BG_2(G)$, N(e) is {u, v, edges not incident with u and v}. Hence, in $< V-D >$, e is isolated in $BG_2(G)$. Therefore, D is a split dominating set for $BG_2(G)$.

Theorem 3.2: G has no isolated vertices. Let D be a γ_s-set of G. D is a γ_s-set of $BG_2(G)$, if and only if $G = K_{1,n}$.

Proof: $D \subseteq V(G)$ is a dominating set of $BG_2(G)$. Therefore, D is a point cover for G. Hence, $< V(G)-D >$ is totally disconnected in G. Therefore, two components of $< V(G)-D >$ say G_1, G_2, which contain only point vertices, no edges. In G, consider the vertices u_1 in G_1, u_2 in G_2 such that u_1 is adjacent to e_1 and u_2 is adjacent to e_2. If e_1, e_2 are non-adjacent edges of G, then $u_1\, e_1\, e_2\, u_2$ is a path in $BG_2(G)$ or if there exists an edge e not adjacent to both e_1 and e_2 in G then $u_1\, e_1\, e\, e_2\, u_2$ is a path in $BG_2(G)$. This implies that in $BG_2(G)$, G_1 and G_2 are connected.

If e_1, e_2 are adjacent in G and there exists no e, which is not adjacent to both e_1 and e_2, and if this is true for any two vertices of $V(G)-D$, then all the edges joining elements of $V(G)-D$ to elements of D are adjacent to each other, that is, they are incident at a vertex in D. Let it be u. Since, D is a γ_s-dominating set and all the edges are adjacent, D = {u} and $G = K_{1,n}$ (since, $V(G)-D$ is totally disconnected).

Converse is obvious. Hence, the proposition is proved.

Proposition 3.4: If $G = K_{1,n}$, $\gamma_{ns}(BG_2(G)) = n = \gamma_{ns}(G)$.

Proof: $G = K_{1,n}$, the peripheral vertices are the minimum γ_{ns} set. Hence, $\gamma_{ns}(G) = n$. Also, $\gamma_{ns}(BG_2(G)) = n$.

Proposition 3.5: Assume G has no isolated vertices. A γ-set of G is a γ_{ns}-set of $BG_2(G)$ if $\gamma(G) = \alpha_o(G) > 2$.

Proof: Let $\gamma(G) > 2$. Let D be a γ set of G. $D \subseteq V(G)$. D is a dominating set of $BG_2(G)$ if D is a point cover for G, that is, $< V(G)-D >$ is totally disconnected in G. Since $\gamma(G) > 2$, every edge in $< D >$ has non-adjacent edges in G and every edge in G has non-adjacent edges in G (not in $< D >$). Therefore, $<V(BG_2(G))-D>$ is a connected subgraph of $BG_2(G)$. Hence, D is a non-split dominating set for $BG_2(G)$ and $\gamma_{ns}(BG_2(G)) = \gamma(G)$.

Proposition 3.6: If $\gamma(G) = 2 = \gamma_c(G)$, $\gamma_{ns}(BG_2(G)) = 3$ or 2.

Proof: Let D be a connected γ-dominating set of G. If $e \in < D >$ has some non- adjacent edges in $< V(G)-D >$, $D^1 = D \cup \{e\} \subseteq V(BG_2(G))$ is a non-split dominating set for $BG_2(G)$. If e in $< D >$ has no non-adjacent edges in $< V(G)-D >$, D is a non-split dominating set of $BG_2(G)$. Hence, $\gamma_{ns}(BG_2(G)) = 2$ or 3.

Proposition 3.7: If there exists a $\gamma(G)$ dominating set, which is not independent and $\gamma(G) \neq \alpha_o(G)$, then $\gamma(BG_2(G)) \leq \gamma(G)+1$, $\gamma_{ns}(BG_2(G)) \leq \gamma(G)+1$.

Proof: Let D be a $\gamma(G)$ dominating set which is not independent. Therefore, there exists at least one edge e in $< D >$. Consider $D^1 = D \cup \{e\} \subseteq V(BG_2(G))$. D^1 is a dominating set of $BG_2(G)$ and $\gamma(BG_2(G)\}) \leq \gamma(G) +1$.

Since $\gamma(G) \neq \alpha_o(G)$, $< V(BG_2(G))-D^1 >$ is a connected subgraph of $BG_2(G)$. Hence, D_1 is a non-split dominating set of $BG_2(G)$. Therefore, $\gamma_{ns}(BG_2(G)) \leq \gamma(G)+1$.

Proposition 3.8: If $\gamma(BG_2(G)) = \gamma(G)+2$, then $\gamma_{ns}(BG_2(G)) = \gamma(BG_2(G)) = \gamma(G)+2$.

Proof: $\gamma(BG_2(G)) = \gamma(G)+2$. Therefore, every $\gamma(G)$ dominating set D of G is independant. $D^1 = D \cup \{v, e\}$, where $u \in D$ and $e = uv \in E(G)$, $v \notin D$ is a dominating set of $BG_2(G)$ and $< V(BG_2(G))-D^1 >$ is connected. Hence, $\gamma_{ns}(BG_2(G)) = \gamma(BG_2(G)) = \gamma(G)+2$.

Remark: (1) Let G be a connected graph. Suppose D is a $\gamma(G)$ dominating set, which is independent, then $V(G)-D$ is a dominating set for $BG_2(G)$.

(2) $\gamma_{ns}(BG_2(G)) \leq p+q-\omega(BG_2(G))+1 = p+q+1-k$, where $k = \omega(BG_2(G)) = \max \{3, \omega(G), \beta_1(G)\}$.

(3) If G is a connected graph, then $D = E(G)$ is a non-split domination set for $BG_2(G)$.

(4) If G is disconnected without isolated vertices, then $D = E(G)$ is a split dominating set for $BG_2(G)$.

(5) If G is disconnected without isolated vertices, then $D = V(G)$ is a non-split dominating set for $BG_2(G)$.

(6) If G is connected and has a dominating edge which is adjacent to every other edges, then D = V(G) is a split dominating set for $BG_2(G)$.

Proposition 3.9: (1) $\gamma_{ns}(\overline{BG_2}(G)) = 1$ if and only if G has an isolated vertex and $G \neq K_{1,n} \cup K_1$. (2) $\gamma_s(\overline{BG_2}(G)) = 1$ if and only if $G = K_1 \cup K_{1,n}$.

Proof: It is proved already that $\gamma(\overline{BG_2}(G) = 1$ if and only if G has an isolated vertex. Therefore, $\gamma_{ns}(\overline{BG_2}(G)) = 1$ implies that G has at least one isolated vertex. If $G = K_{1,n}$, $BG_2(G)$ is disconnected, and so if $G = K_{1,n} \cup K_1$ or $K_3 \cup K_1$, D = {u}, u is isolated in G is a split dominating set. (1) and (2) follows from this.

Proposition 3.10: If G has no isolated vertices and diam(G) $\geq$ 3, then $\gamma_{ns}(\overline{BG_2}(G)) = 2$.

Proof: Consider D = {u, v} $\subseteq$ V(G), where $d_G(u, v) = 3$. In $\overline{BG_2}(G)$, all edges incident with u are adjacent to w and all edges incident with v are adjacent to x, where u x w v is a shortest path from u to v. The line vertex e_1 = (ux) $\in$ E(G) is adjacent to e_2 = (xw) and e_2 is adjacent to the line vertex e_3 = (wv) in $\overline{BG_2}(G)$ and all other line vertices are adjacent to x, w and all other point vertices are adjacent to the line vertex e_3. Therefore, < $V(BG_2(G)) - D$ > is connected and D is a dominating set for $\overline{BG_2}(G)$. Therefore, $\gamma_{ns}(\overline{BG_2}(G)) = 2$ if diam(G) $\geq$ 3.

Proposition 3.11: (1) Let G be a connected graph such that $\gamma_i(L(G)) = 2$, then $\gamma_{ns}(\overline{BG_2}(G)) = 2$. (2) If diam(G) $\leq$ 2 and $\beta_1(G) > 2$, then $\gamma_{ns}(\overline{BG_2}(G)) = 3 = \gamma(\overline{BG_2}(G))$.

Proof of (1): Let e_1, e_2 be two non-adjacent edges of G such that D = {e_1, e_2} dominates L(G). Then D dominates $\overline{BG_2}(G)$ and the vertices in V−D form a connected graph. Therefore, $\gamma_{ns}(\overline{BG_2}(G)) = 2$.

Proof of (2): Consider D = {u, v, e} $\subseteq$ V($\overline{BG_2}(G)$), where u, v $\in$ V(G) and e = uv $\in$ E(G). It is proved that D is a γ-dominating set (Theorem 6.6.12). Since $\beta_1(G) > 2$, there exists at least six vertices. Since diam(G) $\leq$ 2, G is connected. Hence, < V($\overline{BG_2}(G)$)−D > is connected. Therefore, $\gamma_{ns}(\overline{BG_2}(G)) = 3$.

Theorem 3.3: $\gamma_s(\overline{BG_2}(G)) \leq p+q-2\Delta(G)-1$.

Proof: Let u $\in$ V(G) such that $deg_G u = \Delta(G)$, e(u) $\neq$ 1. Then D = {v $\in$ E(G):u is not adjacent to v in G} $\cup$ { e $\in$ E(G): e is not adjacent to u in G} is a dominating set for $\overline{BG_2}(G)$ and < V($\overline{BG_2}(G)$)−D > has u as an isolated vertex. Therefore, D is a split dominating set for $\overline{BG_2}(G)$. Therefore, $\gamma_s(\overline{BG_2}(G)) \leq q-\Delta(G)+p-1-\Delta(G) = p+q-2\Delta(G)-1$.

Proposition 3.12: $\gamma_s(\overline{BG_2}(G)) \leq p-2+\deg u+\deg v$, where and $e = uv$, $G \neq K_n$.

Proof: Consider $D = \{V(G)-\{u, v\}\} \cup \{e^1 \in E(G): e^1$ is adjacent to e in G$\}$ ($G \neq K_n$). Select e such that D is a dominating set for $\overline{BG_2}(G)$. e is isolated in <V(BG$_2$(G))−D>. Hence, D is a split dominating set of $\overline{BG_2}(G)$. Hence the proposition is proved.

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