International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 5, No. 1, January – March 2014, pp. 23 - 36

Restrained Eccentric Domination in Graphs

M. Bhanumathi, J. John Flavia and **M. Kavitha**

Government Arts College for Women,Pudukkottai-622001, TN, India

Email: *bhanu_ksp@yahoo.com, flavimaths@gmail.com, kavisdev@gmail.com*

Abstract: *A subset D of the vertex set V(G) of a graph G is said to be a dominating set, if every vertex not in D is adjacent to at least one vertex in D. A subset D of V(G) is a restrained dominating set, if every vertex not in D is adjacent to a vertex in D and to a vertex in V–D. A subset D of V(G) is a restrained eccentric dominating set, if D is a restrained dominating set of G and for every v ∈ V–D, there exists at least one eccentric point of v in D. The minimum of the cardinalities of the restrained eccentric dominating set of G is called the restrained eccentric domination number of G and it is denoted by $\gamma_{red}(G)$. In this paper, bounds for γ_{red} and its exact value for some particular classes of graphs are found.*

Keywords: *Eccentric dominating set, Restrained dominating set, Restrained eccentric dominating set.*

1. Introduction

Let G be a finite, simple, undirected graph on n vertices with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [5] Buckley and Harary [1].

Definition 1.1: Let G be a connected graph and u be a vertex of G. The **eccentricity** e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max{d(u, v) : u ∈ V}.The **radius** r(G) is the minimum eccentricity of the vertices, whereas the **diameter** diam(G) = d(G) is the maximum eccentricity. For any connected graph G, r(G) ≤ diam(G) ≤ 2r(G). v is a **central vertex** if e(v) = r(G). The **center** C(G) is the set of all central vertices. The central sub graph < C(G) > of a graph G is the sub graph induced by the center. v is a peripheral vertex if e(v) = d(G). The periphery P(G) is the set of all peripheral vertices.

For a vertex v, each vertex at a distance e(v) from v is an **eccentric vertex of v** . **Eccentric set of a vertex** v is defined as E(v) = {u ∈ V(G) / d(u, v) = e(v)}.

Definition 1.2: The **open neighborhood** N(u) of a vertex v is the set of all vertices adjacent to v in G. N[v] = N(v)∪{v} is called the **closed neighborhood** of v. For a vertex v ∈ V(G), $N_i(u)$ = {u ∈ V(G)/d(u, v) = i} is defined to be the i^{th} neighborhood of v in G.

Received: 08 January, 2014; Revised: 23 February, 2014; Accepted: 21 March, 2014

Definition 1.3: A **bigraph or bipartite graph** G is a graph whose point set V can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. If further G contains every line joining points of V_1 to points of V_2 then G is called a **complete bigraph**. If V_1 contains m points and V_2 contains n points then the complete bigraph G is denoted by $K_{m,n}$.

Definition 1.4: A **star** is a complete bi graph $K_{1,n}$.

Definition 1.5 [3, 12]: A set $D \subseteq V(G)$ is said to be a **dominating set** in G, if every vertex in $V-D$ is adjacent to some vertex in D. The cardinality of minimum dominating set is called the **domination number** and is denoted by $\gamma(G)$.

Definition 1.6 [6]: A set $D \subseteq V(G)$ is a **restrained dominating set** if every vertex not in D is adjacent to a vertex in D and to a vertex in $V-D$. The cardinality of minimum restrained dominating set is called the **restrained domination number** and is denoted by $\gamma_r(G)$.

Definition 1.7 [7]: A set $D \subseteq V(G)$ is an **eccentric dominating set** if D is a dominating set of G and for every $v \in V-D$, there exists at least one eccentric point of v in D. The cardinality of minimum eccentric dominating set is called the **eccentric domination number** and is denoted by $\gamma_{ed}(G)$.

If D is an eccentric dominating set, then every superset $D' \supseteq D$ is also an eccentric dominating set. But $D'' \subseteq D$ is not necessarily an eccentric dominating set.

An eccentric dominating set D is a **minimal eccentric dominating set** if no proper subset $D'' \subseteq D$ is an eccentric dominating set.

We need the following results to prove certain results in restrained Eccentric domination.

Theorem: 1.1[3]: For any graph G, $\lceil n/(1+\Delta(G)) \rceil \leq \gamma(G) \leq n-\Delta(G)$.

Theorem: 1.2 [7]: $\gamma_{ed}(K_n) = 1$

Theorem: 1.3 [7]: $\gamma_{ed}(K_{m,n}) = 2$.

Theorem: 1.4 [7]: $\gamma_{ed}(W_3) = 1$, $\gamma_{ed}(W_4) = 2$, $\gamma_{ed}(W_n) = 3$ for $n \geq 7$.

Theorem: 1.5 [7]: $\gamma_{ed}(P_n) = \gamma(P_n)$ or $\gamma(P_n) + 1$.

Theorem: 1.6 [4]: If $n \neq 2$ is a positive integer, then $\gamma_r(K_n) = 1$.

Theorem: 1.7 [4]: If m and n are integers such that min $\{m, n\} \geq 2$, then $\gamma_r(K_{m,n}) = 2$.

Theorem: 1.8 [4]: Let G be a connected graph of order n. Then $\gamma_r(G) = n$ if and only if G is a star.

Theorem: 1.9 [4]: If D is a minimum restrained dominating set of a tree T, then every pendent vertices of T belongs to D, that is, $\gamma_r(T) \geq e$.

Theorem: 1.10[10]: For any graph G, $p-(2/3)q \leq \gamma_r(G)$.

Theorem: 1.11[7]: (i) $\gamma_{ed}(C_n) = n/2$ if n is even.

$$\text{(ii) } \gamma_{ed}(C_n) = \begin{cases} n/3 & \text{if } n = 3m \text{ and is odd.} \\ \lceil n/3 \rceil & \text{if } n = 3m+1 \text{ and is odd.} \\ \lceil n/3 \rceil + 1 & \text{if } n = 3m+2 \text{ and is odd.} \end{cases}$$

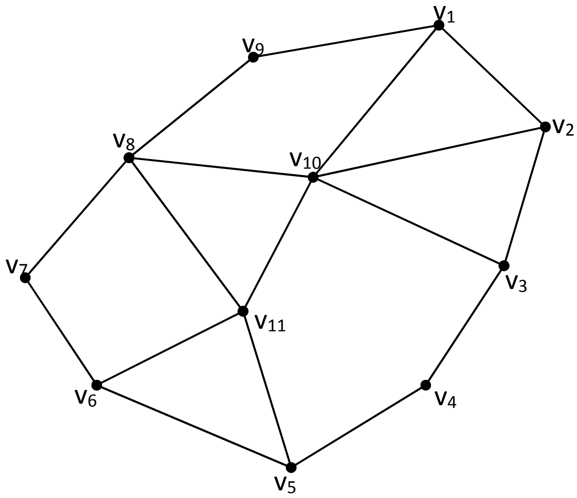
2. Restrained eccentric dominating set

We define restrained eccentric dominating set of a graph as follows.

Definition 2.1: A subset D of V(G) is a **restrained eccentric dominating set**, if D is a restrained dominating set of G and for every $v \in V-D$, there exists at least one eccentric point of v in D. The minimum of the cardinalities of the restrained eccentric dominating set of G is called the **restrained eccentric domination number** of G and it is denoted by $\boldsymbol{\gamma_{red}(G)}$.

Clearly, $\gamma(G) \leq \gamma_{red}(G)$, $\gamma_{ed}(G) \leq \gamma_{red}(G)$ and $\gamma_r(G) \leq \gamma_{red}(G)$. But there is no relation between γ_r and γ_{ed}. Also, for any graph G, $1 \leq \gamma_{red}(G) \leq n$. These lower bound and upper bounds are sharp, since $\gamma_{red}(K_n) = 1$ and $\gamma_{red}(K_{1,\, n-1}) = n$.

Example 2.1:



G

Fig. 2.1

V = {v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}} is the vertex set of G.

D_1 = {v_5, v_8, v_{10}} is a minimum dominating set and also a minimum restrained dominating set.

D_2 = {v_4, v_6, v_9, v_{10}} is a minimum eccentric dominating set.

D_3 = {v_3, v_4, v_7, v_9, v_{11}} is a minimum restrained eccentric dominating set.

$\gamma(G) = 3$, $\gamma_r(G) = 3$, $\gamma_{ed}(G) = 4$, $\gamma_{red}(G) = 4$.

Example:2.2

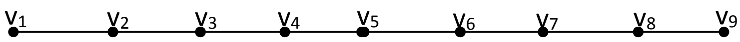


Fig 2.2

D_1 = {v_2, v_5, v_8} is a dominating set. $\gamma(G) = 3$.

D_2 = {v_1, v_4, v_7, v_9} is an eccentric dominating set. $\gamma_{ed}(G) = 4$.

$D_3 = \{v_1, v_4, v_7, v_8, v_9\}$ is a restrained eccentric dominating set. $\gamma_{red}(G) = 5 = \gamma_r(G)$

Theorem: 2.1: For any graph G, $\lceil n/(1+\Delta(G)) \rceil \leq \gamma_{red}(G)$.

Proof: From Theorem: 1.1, $\lceil n/(1+\Delta(G)) \rceil \leq \gamma(G) \leq n-\Delta(G)$ and we have $\gamma(G) \leq \gamma_{red}(G)$. Therefore, $\lceil n/(1+\Delta(G)) \rceil \leq \gamma_{red}(G)$.

This lower bound is sharp, since $\gamma_{red}(P_n) = \lceil n/3 \rceil$ when n = 3k+1.

Theorem: 2.2:

(i) $\gamma_{red}(K_n) = 1$, $n \geq 3$.

(ii) $\gamma_{red}(K_{m,n}) = 2$.

(iii) $\gamma_{red}(W_3) = 1$, $\gamma_{red}(W_4) = 2$, $\gamma_{red}(W_n) = 3$ for $n \geq 4$.

(iv) $\gamma_{red}(K_{1,\, n-1}) = n$.

Proof of (i)

When $G = K_n$. Radius = Diameter = 1. Hence any vertex $u \in V(G)$ dominate other vertices and is also an eccentric vertex of other vertices. The elements of $V - \{u\}$ satisfies the restrained conditions. Hence $\gamma_{red}(K_n) = 1$.

Proof of (ii)

When $G = K_{m,n}$, $V(G) = V_1 \cup V_2$, $|V_1| = m$ and $|V_2| = n$ such that each elements of V_1 is adjacent to every vertex of V_2 and vice versa. $D = \{u, v\}$, $u \in V_1$ and $v \in V_2$ is the restrained dominating set. u dominate all other vertices of V_2 and it is eccentric to elements of $V_1 - \{u\}$. Similarly v dominates all the vertices of V_1 and it is eccentric to elements of $V_2 - \{v\}$. Hence D is a minimum restrained eccentric dominating set. Hence $\gamma_{red}(K_{m,n}) = 2$.

Proof of (iii)

$G = W_3 = K_4$. Hence $\gamma_{red}(W_3) = 1$. When $G = W_4$, consider $D = \{u, v\}$, where u and v are adjacent non central vertices. D is a minimum restrained eccentric dominating set. Therefore, $\gamma_{red}(W_4) = 2$.

When $G = W_n$, $n > 4$. Let $D = \{u, v, w\}$ where u and v are any two adjacent non central vertices and w is the central vertex. Then D is a minimum restrained eccentric dominating set of G. Therefore $\gamma_{red}(W_n) = 3$ for $n > 4$.

Proof of (iv)

$G = K_{1,n-1}$. We know $\gamma_r(K_{1,n-1}) = n-1$ and the end vertices form the minimal restrained dominating set. But it is not an eccentric dominating set. The whole vertex set V is the only restrained eccentric dominating set. Therefore, $\gamma_{red}(K_{1,\,n-1}) = n$.

Theorem: 2.3: $$\gamma_{red}(P_n) = \begin{cases} n/3 + 2, & \text{if } n = 3k \\ \lceil n/3 \rceil, & \text{if } n = 3k+1 \\ \lceil n/3 \rceil + 1, & \text{if } n = 3k+2 \end{cases}$$

Proof: Case (i) n = 3k.

An eccentric dominating set of P_n must contains the two end vertices.

Let $v_1, v_2, v_3, \ldots, v_{3k}$ represent the path P_n. $D = \{v_2, v_5, v_8, \ldots, v_{3k-1}\}$ is the only $\gamma-$ dominating set of P_n. D is not the restrained eccentric dominating set.

$D' = \{v_1, v_4, v_7, \ldots, v_{3k-2}, v_{3k-1}, v_{3k}\}$ is the restrained eccentric dominating set, and $|D'| = \gamma(P_n)+2 = k+2$. Hence $\gamma_{red}(P_n) = \gamma(P_n)+2$.

Case (ii) n = 3k+1

$D = \{v_1, v_4, v_7, \ldots, v_{3k-3}, v_{3k+1}\}$ is the minimum dominating set in P_n. It contains the two end vertices and hence an eccentric dominating set. Also $\langle V-D \rangle$ has no isolated vertices. Hence it is the minimum restrained eccentric dominating set.

Therefore, $\gamma_{red}(P_n) = \gamma(P_n) = \lceil n/3 \rceil$

Case (iii) n = 3k+2

$D = \{v_2, v_5, v_8, \ldots, v_{3k+2}\}$ is a minimum dominating set. It contains one end vertex v_{3k+2} and it is restrained but not eccentric. Now, $D \cup \{v_1\}$ is a restrained eccentric dominating set with $|D \cup \{v_1\}| = k+1$ and there is no other restrained eccentric dominating set with $|S| < k+1$. Hence $\gamma_{red}(P_n) = \gamma(P_n) + 1 = k+1 = \lceil n/3 \rceil + 1$.

Theorem: 2.4: Let G be a Caterpillar then $\gamma_{red}(G) \leq e + \lceil k/3 \rceil$, where e is the number of pendent vertices of G.

Proof: Let n be the number of vertices and e be the number of pendent vertices of G.

Consider, $k = n-e$. By the definition of caterpillar, after removing the pendent vertices we get P_k. Hence $\gamma_{red}(G) \leq e + \gamma(P_k)$. That is, $\gamma_{red}(G) \leq e + \lceil k/3 \rceil$.

Theorem: 2.5: Let G be a spider then $\gamma_{red}(G) = n- \Delta(G) = \Delta(G)+1$.

Proof: Since G has $\Delta(G)$ pendent vertices, $\gamma_{red}(G) \geq \Delta(G) = n - \Delta(G) - 1$. Let S be the set of all pendent vertices of G, S is not the restrained eccentric dominating set. Hence, $\gamma_{red}(G) \geq n - \Delta(G)$. $D = S \cup \{u\}$, where u is a vertex of G with degree two. D is the restrained eccentric dominating set. Hence $\gamma_{red}(G) = n - \Delta(G)$.

Theorem: 2.6: Let G be a wounded spider which is not a star then $\gamma_{red}(G) = \Delta(G)$.

Proof: Let S be the set of all pendent vertices of G, S is the minimum restrained eccentric dominating set. Hence $\gamma_{red}(G) = \Delta(G)$.

Theorem: 2.7 (i) When n is even, let n = 2r, where r is the radius of C_n then

$$\gamma_{red}(C_n) = \begin{cases} n/2 + 2, \text{ if } r = 4m \\ \frac{n}{2} + 1, \text{if } r = 4m + 1 \text{ or } 4m + 3 \\ n/2, \text{ if } r = 4m + 2 \end{cases}$$

(ii) When n is odd, let n = 2r+1 where r is the radius of C_n, then

$$\gamma_{red}(C_n) = \begin{cases} \lceil n/3 \rceil + 2, \text{ if } r = 3m \\ n/3, \text{ if } r = 3m + 1 \\ \lceil n/3 \rceil + 1, \text{ if } r = 3m + 2 \end{cases}$$

Proof: Let D be minimum restrained eccentric dominating set of C_n. D is a restrained dominating set implies $\langle V-D \rangle$ has no isolated vertices. Since C_n is a cycle, if a vertex i is not in D implies either i−1 or i+1 is also not in D and if i, i+1 are not in D then i−1 and i+2 must be in D, since D is a dominating set. So $\langle V-D \rangle = kK_2$, where k is a positive integer. Thus V−D always contains even number of vertices.

Case (a): n = 2r.

First let us consider **even n.** When n is even C_n is a unique eccentric point graph with radius r = n/2. If we label the vertices of C_n by 1, 2, 3, ..., n, then eccentric vertex of i is i+r. Let D be a restrained eccentric dominating set. If D is an eccentric dominating set of C_n, i is not in D means i+r must be in D. i, i+1 are not in D implies i−1 and i+2 must be in D, since D is a dominating set, but their eccentric vertices i−r, i+1−r may not be in D. So we can group the vertices in such a way that beginning with vertex 1, first 2 vertices 1, 2 are in D next two vertices 3, 4 are not in D etc., but r+3, r+4 must be in D etc. So we can split V into two parts V_1 containing 1st r vertices and 2nd part V_2 containing 2nd r vertices to find the restrained eccentric dominating set.

Sub case (i) r = 4m, n = 8m.

D = {1, 2, 5, 6, 9, 10, ..., r−3, r−2, r−1, r+2, r+3, r+4, r+7, r+8, ..., 2r−5, 2r−4, 2r−1, 2r} is a restrained eccentric dominating set, and $|D|$ = n/2+2. This implies that $\gamma_{red}(C_n) \leq n/2+2$. By theorem 1.11, $\gamma_{ed}(C_n)$ = n/2 if n is even. But no minimum eccentric dominating set of C_n is restrained. So, $\gamma_{red}(C_n) \geq n/2+1$. If $\gamma_{red}(C_n)$ = n/2+1 = 4m+1, V−D contains 4m−1 vertices, where 4m−1 is odd, which is not possible. Hence, $\gamma_{red}(C_n) \geq n/2+2$. Thus $\gamma_{red}(C_n) = n/2+2$.

Sub case (ii) r = 4m+1, n = 8m+2.

In this case, D = {1, 2, 5, 6, ..., r—4, r—3, r, r+3, r+4, r+7, r+8, ..., 2r—2, 2r—1, 2r} is the restrained eccentric dominating set and D contains 4m+2 = n/2+1 vertices and hence $\gamma_{red}(C_n) \leq n/2+1$. But we have $\gamma_{ed}(C_n) = n/2$ and no minimum eccentric dominating set S is a restrained dominating set since V—S contains odd number of vertices. Therefore, $\gamma_{red}(C_n) \geq n/2+1$. Hence $\gamma_{red}(C_n) = n/2+1$.

Sub case (iii) r = 4m+2, n = 8m+4.

In this case, D = {1, 2, 5, 6, ..., r—1, r, r+3, r+4, r+7, r+8, ..., 2r—3, 2r—2} is an restrained eccentric dominating set. Therefore $\gamma_{red}(C_n) \leq n/2$, and we know that $\gamma_{red}(C_n) \geq \gamma_{ed}(C_n) = n/2$. Hence, $\gamma_{red}(C_n) = n/2$.

Sub case (iv) r = 4m+3, n = 8m+6.

In this case, D = {1, 2, 5, 6, ..., r—2, r—1, r, r+3, r+4, ..., 2r—4, 2r—3, 2r—2} is the restrained eccentric dominating set, and $|D| = n/2+1$. Thus, $\gamma_{red}(C_n) \leq n/2+1$. But we have $\gamma_{ed}(C_n) = n/2$ and no minimum eccentric dominating set S is a restrained dominating set, since V—S contains odd number of vertices. Hence, $\gamma_{ed}(C_n) \geq n/2+1$. This gives $\gamma_{red}(C_n) = n/2+1$.

<u>**Case (b): n = 2r+1.**</u>

When n is odd, we know n = 2r+1 where r is the radius of C_n. In this case C_n is an bi-eccentric point graph. That is every vertex has exactly two eccentric vertices. So, if D is an eccentric dominating set, for a vertex not in D any one of its eccentric vertices must be in D and it must be a restrained dominating set.

Sub case (i) r = 3m, n = 6m+1.

In this case, D = {1, 4, 7, 10, ..., r—2, r+1, r+2, r+3, r+6, r+9, ..., 2r, 2r+1} is a minimum restrained eccentric dominating set, and $|D| = \lceil n/3 \rceil + 2$. Thus, $\gamma_{red}(C_n) \leq \lceil n/3 \rceil + 2$. But, in this case, we have $\gamma_{ed}(C_n) = \lceil n/3 \rceil + 1$ and no minimum eccentric dominating set S is a restrained dominating set, since V—S contains odd number of vertices. Hence, $\gamma_{red}(C_n) \geq \lceil n/3 \rceil + 2$. This gives $\gamma_{red}(C_n) = \lceil n/3 \rceil + 2$.

Sub case (ii) r = 3m+1, n = 6m+3.

In this case n is an odd multiple of 3. D = {1, 4, 7, ..., r, r+3, r+6, ..., 2r—1} is a minimum restrained eccentric dominating set and D contains n/3 vertices. Thus, $\gamma_{ed}(C_n) = \gamma_{red}(C_n) = n/3$.

Sub case (iii) r = 3m+2, n = 6m+5.

In this case, D = {1, 4, 7, 10, ..., r−4, r−1, r, r+3, r+6, r+9, ..., 2r−5, 2r−2, 2r−1} is a minimum restrained eccentric dominating set and $|D| = \lceil n/3 \rceil + 1$. Thus, $\gamma_{ed}(C_n) = \gamma_{red}(C_n) = \lceil n/3 \rceil + 1$.

Hence the theorem is proved.

Now, let us present bounds for restrained eccentric domination number of trees. Each restrained dominating set of a tree is a dominating set containing all the pendent vertices. Therefore restrained dominating set of a tree is always an eccentric dominating set. Hence in trees restrained eccentric domination is same as restrained domination. Therefore, $\gamma_r(T) = \gamma_{red}(T)$.

In [8], Mustapha Chellali has proved that $\lceil (n+2+l-s)/3 \rceil \leq \gamma_r(T) \leq \lfloor (n+2l+s+1)/3 \rfloor$ where l denotes the number of pendent vertices of T and s denotes the number of supports in T. Therefore, we have,

For a tree with l pendent vertices and s support vertices, $\lceil (n+2+l-s)/3 \rceil \leq \gamma_{red}(T) \leq \lfloor (n+2l+s+1)/3 \rfloor$.

Theorem: 2.8: For any graph G, $p - (2/3)\,q \leq \gamma_{red}(G)$.

Proof: From theorem [1.10], For any graph G, $p - (2/3)q \leq \gamma_r(G)$ and we know that $\gamma_r(G) \leq \gamma_{red}(G)$. Therefore, $p - (2/3)\,q \leq \gamma_{red}(G)$.

Theorem: 2.9: If G is a connected graph, $\gamma_{red}(G) \leq \gamma_{ed}(G) + \beta_o(G)$.

Proof: Let D be a γ_{ed} – set of G. Suppose < V−D > has no isolated vertices. Then D is a restrained eccentric dominating set of G. On the other hand, if there exists a set $S \subseteq V-D$ such that each vertex in S is an isolated vertex in < V−D >. Then $D \cup S$ is a restrained eccentric dominating set of G. Hence, $\gamma_{red}(G) \leq |D \cup S|$

$$\leq |D| + |S|$$

$$\gamma_{red}(G) \leq \gamma_{ed}(G) + \beta_o(G).$$

Theorem: 2.10: Let G be a connected graph with $|V(G)| = n$. Then $\gamma_{red}(G \circ K_1) = n$.

Proof: Let V(G) = {v_1, v_2, …, v_n}. Let v_i' be the pendent vertex adjacent to v_i in $G \circ K_1$ for i =1, 2, …, n. Then {v_1', v_2', …, v_n' } is an eccentric dominating set for $G \circ K_1$ and is also a minimum restrained eccentric dominating set for $G \circ K_1$. Hence $\gamma_{red}(G \circ K_1) = n$.

Theorem: 2.11: If G is of radius one and diameter two, then $\gamma_{red}(G) \leq \frac{n-t+2}{2}$, where $t \geq 2$ is the number of vertices with eccentricity one.

Proof: Let u ∈ V(G) such that e(u) = 1. Let t be the number of vertices with eccentricity one. u dominates all other vertices and for t−1 other vertices u is an eccentric point. Consider the remaining (n − t) vertices of G. They are also dominated by u but their eccentric points are different from u. $(V - E_1(G)) \cup \{u\}$ is a restrained eccentric dominating set of G, where $E_1(G) = \{u \in V(G)/e(u) = 1\}$. Hence

$$\gamma_{red}(G) \leq 1 + \frac{n-t}{2} = \frac{n-t+2}{2}.$$

$$\gamma_{red}(G) \leq \frac{n-t+2}{2}.$$

Theorem: 2.12: Let G be a connected graph with cycles of diameter two with $< N_2(u) >$ has no isolated vertices. Then $\gamma_{red}(G) \leq 1+\delta(G)$.

Proof: diam(G) = 2. Let u ∈ V(G) such that deg u = $\delta(G)$. Consider {u} ∪ N(u) = D. This is a restrained eccentric dominating set for G. Hence $\gamma_{red}(G) \leq 1+\delta(G)$.

Theorem 2.13: A restrained eccentric dominating set D of G is minimal if and only if for each vertex v ∈ D, one of the following conditions holds.

(i) v is an isolated vertex in <D> or v has no eccentric vertex in D.

(ii) there exists a vertex u ∈ V− D such that N(u) ∩ D = {v} or E(u) ∩ D = {v}.

(iii) v is an isolated vertex in < (V − D) ∪ {v} >.

Proof:

Assume that D is a minimal restrained eccentric dominating set of G. Then for every vertex v ∈ D, D − {v} is not a restrained eccentric dominating set. That is there exists some vertex u in (V − D) ∪ {v} which is not dominated by any vertex in D −{v} or there exists u in (V − D) ∪ {v} such that u has no eccentric point in D − {v} or v is an isolated vertex in < (V − D) ∪ {v} >.

Case (i)

Suppose u = v, then v is an isolated vertex in <D> or v has no eccentric vertex in D.

Case (ii)

Suppose $u \in V - D$.

(i) If u is not dominated by $D - \{v\}$, but is dominated by D, then u is adjacent to only v in D, that is $N(u) \cap D = \{v\}$.

(ii) Suppose u has no eccentric point in $D - \{v\}$, but u has an eccentric point in D. Then v is the only eccentric point of u in D, that is $E(u) \cap D = \{v\}$.

Conversely, suppose that D is a restrained eccentric dominating set and for each $v \in D$, one of the above conditions holds, we show that D is a minimal restrained eccentric dominating set.

Suppose that D is not a minimal restrained eccentric dominating set, that is there exist a vertex $v \in D$, $D - \{v\}$ is a restrained eccentric dominating set. Hence, v is adjacent to at least one vertex in $D - \{v\}$, and v has an eccentric point in $D - \{v\}$. Therefore, condition (i) does not hold.

Also, every element x in $V - D$ is adjacent to at least one vertex in $D - \{v\}$ and x has an eccentric point in $D - \{v\}$. Hence condition (ii) does not hold. There exists a vertex $v \in D$ such that v is not isolated in $< (V - D) \cup \{v\} >$. Hence, condition (iii) does not hold. This is a contradiction to our assumption that for each $v \in D$, one of the conditions holds.

Theorem: 2.14: Let n be an even integer $n \geq 4$. Let G be obtained from the complete graph K_n by deleting edges of a linear factor. Then $\gamma_{red}(G) = \frac{n}{2}$.

Proof: Let u and v be a pair of non–adjacent vertices in G. Then u and v are eccentric to each other. Also, G is unique eccentric point graph. Therefore, $\gamma_{red}(G) \geq \frac{n}{2}$............(i).

Consider $D \subseteq V(G)$ such that $\langle D \rangle = K_{n/2}$. D contains n/2 vertices such that each vertex in $V - D$ is adjacent to at least one element in D and each element in $V - D$ has its eccentric vertex in D and also $\langle D \rangle$ has no isolated vertices. Hence $\gamma_{red}(G) \leq \frac{n}{2}$.............(ii).

From (i) and (ii) $\gamma_{red}(G) = \frac{n}{2}$.

Theorem: 2.15:

$\gamma_{red}(\overline{C}_4) = 4$, $\gamma_{red}(\overline{C}_5) = 3$, and $\gamma_{red}(\overline{C}_n) = \left\lceil \frac{n}{3} \right\rceil$, for $n \geq 6$.

Proof:

Clearly, $\gamma_{red}(\overline{C_4}) = 4$, $\gamma_{red}(\overline{C_5}) = 3$. Now, assume that n $\geq$ 6. Let v_1, v_2, v_3, …, v_n, v_1 form C_n. Then $\overline{C_n} = K_n - C_n$ and each vertex v_i is adjacent to all other vertices except v_{i-1} and v_{i+1} in $\overline{C_n}$. Hence eccentric point of v_i in $\overline{C_n}$ is v_{i-1} and v_{i+1} only. Hence any restrained eccentric dominating set must contain either v_i or any one of

v_{i-1}, v_{i+1}. Thus $\gamma_{red}(\overline{C_n}) \geq \left\lceil \frac{n}{3} \right\rceil$……….(i).

Now, consider a restrained eccentric dominating set as follows.

$\{v_1, v_4, v_7, \ldots, v_{3m-2}\}$ if n = 3m.

$\{v_1, v_4, v_7, \ldots, v_{3m+1}\}$ if n = 3m+1.

$\{v_1, v_4, v_7, \ldots, v_{3m+1}, v_{3m+2}\}$ if n = 3m+2. Hence $\gamma_{red}(\overline{C_n}) \leq \left\lceil \frac{n}{3} \right\rceil$…………(ii)

From (i) and (ii) $\gamma_{red}(\overline{C_n}) = \left\lceil \frac{n}{3} \right\rceil$, for n $\geq$ 6.

Note: If G is a connected graph with n vertices then $\gamma_{red}(G) \leq \frac{2n}{3}$.

References:

[1] Bhanumathi M and Muthammai S, Eccentric domatic number of a Graph, International Journal of Engineering Science, Advanced Computing and Bio-Technology- Volume 1, No. 3, pp 118-128, 2010.

[2] Buckley. F, Harary. F, Distance in graphs, Addison–Wesley, Publishing company (1990).

[3] Cockayne, E.J., Hedetniemi, S.T., Towards a theory of domination in graphs. Net works, 7: 247-261.1977

[4] Gayla S. Domke, Johannes H. Hattingh, Stephen T. Hedetniemi, Renu C.Laskar, Lisa R. Markus, Restrained domination in graphs. Discrete Mathematics 203 (1999) 61-69.

[5] Gayla S. Domke, Johannes H. Hattingh, Stephen T. Hedetniemi, Renu C. Laskar, Lisa R. Markus, Restrained domination in trees. Discrete Mathematics 211 (2000) 1-9.

[6] Harary, F., Graph theory, Addition - Wesley Publishing Company Reading, Mass (1972).

[7] Janakiraman T.N., Bhanumathi M and Muthammai S, Eccentric domination in Graphs, International Journal of Engineering Science, Computing and Bio-Technology, Volume 1, No. 2, pp 1-16, 2010.

[8] T.N. Janakiraman, P.J.A. Alphonse and V. Sangeetha, Distance Closed Domination in Graphs, International Journal of Engineering Science, Computing and Bio-Technology, Vol.1, No. 1-4, 109-117, 2010.

[9] T.N. Janakiraman, P.J.A. Alphonse and V. Sangeetha, Distance Closed Restrained Domination in Graphs, International Journal of Engineering Science, Computing and Bio-Technology, Vol.2, No. 1-4, 57-65, 2011.

[10] Kulli V.R, Theory of domination in graphs,Vishwa International publications (2010).

[11] Mustapha Chellali., On weak and restrained domination in Trees. AKCE International Journal of Graphs and Combinatorics No.1, pp 39-44, 2005.

[12] Teresa W.Haynes, Stephen T.Hedetniemi, Peter J.Slater, Fundamentals of Domination in graphs, Marcel Dekker,New York (1998).