International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 5, No. 1, January – March 2014, pp. 1- 23

Characterizations and Edge Partitions of the Boolean Graphs $BG_2(G)$, $BG_3(G)$ and their Complements

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Abstract: *Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G).* $B_{G,\, INC,\, \bar{L}(G)}(G)$ *is a graph with vertex set* $V(G) \cup E(G)$ *and two vertices are adjacent if and only if they correspond to two adjacent vertices of G, to a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by* $BG_2(G)$*, Boolean graph of G-second kind. Similarly,* $B\,\bar{K}_p$*, INC,* $\bar{L}(G)(G)$ *is a graph with the same vertex set and two vertices are adjacent if and only if they correspond to a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by* $BG_3(G)$*, Boolean graph of G-third kind. In this paper, characterizations of* $BG_2(G)$*,* $BG_3(G)$ *and partitions of edges of* $BG_2(G)$*,* $BG_3(G)$ *and their complements are studied.*

1. Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [5], Buckley and Harary [4].

Let G be a connected graph and u be a vertex of G. The *eccentricity* e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max {d(u, v) : u $\in$ V}. The *radius* r(G) is the minimum eccentricity of the vertices, whereas the *diameter* diam(G) is the maximum eccentricity. For any connected graph G, r(G) $\leq$ diam(G) $\leq$ 2r(G). v is a central vertex if e(v) = r(G). The *center* *C*(G) is the set of all central vertices. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a *peripheral vertex* if e(v) = diam(G). The *periphery P*(G) is the set of all such vertices. For a vertex v, each vertex at distance e(v) from v is an eccentric node of v.

A graph is *self-centered* if every vertex is in the center. Thus, in a self-centered graph G all nodes have the same eccentricity, so r(G) = diam(G).

A graph G is called *Hamiltonian* if it has a spanning cycle. Any spanning cycle of G is called *Hamilton cycle*. A *Hamiltonian path* in G is a path, which contains every vertex of G. Clearly, every Hamiltonian graph is 2-connected.

Received: 12 December, 2013; Revised: 02 February, 2014; Accepted: 07 March, 2014

A graph G is said to be *pancyclic* if G contains a cycle of length l, for each l satisfying $3 \leq l \leq p$. In particular, every pancyclic graph is Hamiltonian.

A *decomposition* of a graph G is a collection of subgraphs of G, whose edge sets partition the edge set of G. The subgraphs of the decomposition are called the *parts of the decomposition*.

A graph G is said to be *F-decomposable or F-packable* if G has a decomposition in which all of its parts are isomorphic to the graph F. A graph G can be decomposed into *Hamilton cycles* (*paths*) if the edge set of G can be partitioned into Hamilton cycles (paths).

Theorem 1.1 [2](Bermond)

(i) If p is even, K_p can be decomposed into p/2 Hamiltonian paths.

(ii) If p is odd, K_p can be decomposed into $(p-1)/2$ Hamiltonian cycles.

If p is even, K_p can be decomposed into $(p-2)/2$ cycles of length n and $(p/2)K_2'$s.

A *path* (*cycle*) *partition* of a graph G is a collection of paths (cycles) in G such that every edge of G lies in exactly one path (cycle).

<u>**Motivation:**</u> The Line graphs, Middle graphs, Total graphs and Quasi-total graphs are very much useful in computer networks. In analogous to line graph, total graph, middle graph and quasi-total graph, thirty-two graphs can be defined using different adjacency relations. Out of these operations, eight were already studied. Among the remaining twenty-four graph operations, four are defined and analyzed in [3], [6], [7], [8] and [9]. All the others have been defined and studied thoroughly and will be submitted elsewhere. This is illustrated below.

$G/ \overline{G}/K_p/ \overline{K}_p$	Incident (INC)/ Non-incident (NINC)	$L(G)/ \overline{L(G)}/K_q/ \overline{K}_q$

Defining a new graph from a given graph by using the adjacency relation between two vertices or two edges and incident relationship between vertices and edges is known as Boolean operation. It defines new structure from the given graph and adds extra information of the original graph.

In Management and in social networks, the incident and non-incident relations of vertices and edges are used to define various networks. So these are very much applicable in socio-economical problems. In some cases, it is possible to retrieve the original graph from the Boolean graphs in polynomial time. So these graph operations may be used in graph coding or coding of some grouped signal. Also, it is possible to study the structure of original graphs through these Boolean graph operations. This motivates the study for the exploration of various Boolean operations and study of their structural properties.

In [6],[7] and [8], Boolean graphs are defined as follows:

Let G be a (p, q) simple, undirected graph with vertex set V(G) and edge set E(G). The Boolean graph BG, INC, $\overline{L(G)}$(G) has vertex set V(G) ∪ E(G) and two vertices in BG, INC, $\overline{L(G)}$(G) are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge incident to it in G or two non- adjacent edges of G. For simplicity, denote this graph by $BG_2(G)$, Boolean graph of G-second kind. The vertices of $BG_2(G)$, which are in V(G) are called point vertices and those in E(G) are called line vertices of $BG_2(G)$.

$V(BG_2(G)) = V(G) \cup E(G)$ and $E(BG_2(G)) = [E(T(G))-E(L(G))] \cup E(\overline{L(G)})$. With an immediate consequence of the definition of $BG_2(G)$, if G is a (p, q) graph, whose vertices are $v_1, v_2, ..., v_p$ having degrees d_i, and edges e_{ij}, note that $BG_2(G)$ has p+q vertices and $(q^2+7q-\Sigma d_i^2)/2$ edges with deg $v_i = 2d_i$; deg $e_{ij} = q+3-(d_i+d_j)$. Also, G and $\overline{L(G)}$ are induced subgraphs of $BG_2(G)$.

Let G be a (p, q) graph with vertex set V(G) and edge set E(G). The Boolean graph B $\overline{K_p}$, INC, $\overline{L(G)}$(G) is a graph with vertex set V(G) ∪ E(G) and two vertices are adjacent if and only if they correspond to two non-adjacent edges of G or to a vertex and an edge incident to it in G. For simplicity, denote this graph by $BG_3(G)$, Boolean graph of G-third kind. The vertices of $BG_3(G)$, which are in V(G) are called point vertices and vertices in E(G) are called line vertices of $BG_3(G)$. $V(BG_3(G)) = V(G) \cup E(G)$ and $E(BG_3(G)) = (E(T(G))-(E(G) \cup E(L(G))) \cup E(\overline{L(G)})$. $BG_3(G)$ has p+q vertices, p-point vertices and q-line vertices. $BG_3(G)$ is a spanning subgraph of $BG_2(G)$. $\overline{K_p}$ and $\overline{L(G)}$ are induced subgraphs of $BG_3(G)$.

Let G be a (p, q) graph with vertices $v_1, v_2, ..., v_p$. Let d_i, i = 1, 2, ..., p denote the degree of the vertices $v_1, v_2, ..., v_p$ in G. Then it follows from the definition that $BG_3(G)$ has $(q(q+5)/2)-(1/2)\Sigma d_i^2$ edges with degree of v_i in $BG_3(G) = deg_G v_i = d_i$ and degree of a line vertex $e_{ij} = v_iv_j \in E(G)$ in $BG_3(G)$ is $q-d_i-d_j+3$.

Theorem 1.2 [10]: Eccentricity of every line vertex is two in $BG_2(G)$ if G is not K_2. Eccentricity of any point vertex is in $BG_2(G)$ is 1, 2 or 3.

Theorem 1.3 [10]:
(1) Radius of $BG_3(G) = 1$ if and only if $G = K_2$.
(2) Eccentricity of every point vertices of $BG_3(G)$ is two if and only if $G = K_2$.
(3) $BG_3(G)$ is self-centered with diameter three if and only if $G = K_3$.
(4) In $BG_3(G)$, eccentricity of point vertex is three and eccentricity of line vertex is two if and only if G satisfies the following conditions.

(i) $r(G) > 1$. (ii) For $u, v \in V(G)$ either $uv \in E(G)$ or there exists non-adjacent edges e_u and e_v such that e_u is incident with u and e_v is incident with v.

(iii) Each $e \in E(G)$ is not adjacent with at least one edge in G, incident with other vertices.

In this paper, characterizations of $BG_2(G)$, $BG_3(G)$ and partitions of edges of $BG_2(G)$, $BG_3(G)$ and their complements are studied.

2.Characterizations of $BG_2(G)$ and $\overline{BG_2(G)}$

A graph H is said to be a Boolean graph of second kind if it is isomorphic to the Boolean graph $BG_2(G)$ for some G. For example, K_3 is a Boolean graph of second kind but $K_{1,2}$ is not.

Theorem 2.1: A graph H is a Boolean graph of second kind if and only if there exists an induced subgraph G of H such that $BG_2(G) = H$.
Proof: Proof is clear from the definition and properties of $BG_2(G)$.

Remark 2.1: In this, $\overline{L(G)} = \langle V(H)-V(G) \rangle$ in H, and each vertex in $V(H)-V(G)$ is adjacent to exactly two adjacent vertices of G.

Following theorems give the characterization of $BG_2(G)$ for some special graphs.

Theorem 2.2: $H = BG_2(G)$, where $G = K_{1,n}$ if and only if H is a friendship (or wind mill) graph.
Proof: If $G = K_{1,n}$, clearly $BG_2(G)$ is a friendship graph. On the other hand, let H be a friendship graph. Then H contains 2n+1, odd number of vertices with a vertex v of degree 2n and all other vertices of degree two. Take one vertex from each triangle with degree two. Let it be D. Take $V = D \cup \{v\}$. $\langle V \rangle = K_{1,n}$ and $H = Q(\langle V \rangle)$. Hence the theorem is proved.

Lemma 2.1: Let G be a bi-central tree with radius two. That is, G is a double star, then $H = BG_2(G)$ has the following properties. (1) H has odd vertices, say $2p-1$. (2) H is bi-eccentric with radius two and has exactly $p-2$ vertices with eccentricity three. (3) Edges of H can be partitioned into $(p-1)$ triangles and a complete bipartite graph $K_{m,n}$, where $m+n = p-2$. (4) H has exactly $p-1$ triangles each having a vertex of degree two and exactly one triangle u v w such that $e(u) = e(v) = e(w) = 2$, deg w = 2, deg u = 2m, deg v = 2n in H, where $m+n = p-2$.

(5) Among the other p—2 triangles, m triangles meet at u and n triangles meet at v such that m+n = p—2.

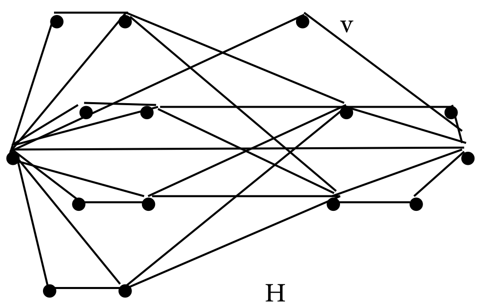
Proof: Let G be a bi-central tree with radius two. Then diam(G) = 3 and G has exactly two vertices with eccentricity two. Let them be u and v, which are the central nodes of G. Let deg_G u = m and deg_G v = n, m+n = p—2. G has exactly p—1 edges. In H, vertices u, v and all line vertices have eccentricity two and point vertices except u and v have eccentricity three in H.

The point vertices u, v and line vertex w = e = uv ∈ E(G) form a triangle such that deg w = 2, deg u = 2m, deg v = 2n and e(u) = e(v) = e(w) = 2 in $BG_2(G)$. Consider all edges incident with u in G. In H, the line vertices corresponding to these edges are adjacent to those line vertices, which correspond to edges incident with v, and vice-versa. All these together prove the proposition.

Theorem 2.3: H = $BG_2(G)$, where G is a bi-central tree with radius two and diameter three if and only if edges of H with $|V(H)|$ = 2m+2n+3 can be partitioned into $K_{m,n}$ and m+n+1 triangle such that there are m triangles with degree n+2, 2 and 2m+2, and n triangles with degree m+2, 2 and 2n+2 and a triangle with degree 2, 2m+2, 2n+2.

Proof: Suppose G is a bi-central tree with radius two and diameter three then H = $BG_2(G)$ satisfies the given conditions from Lemma 6.3.1.

On the other hand, suppose H satisfies the given conditions, structure of H is as follows:



Let v ∈ V(H) such that deg v = 2. Now, take D = {v} ∪ set of all vertices in $K_{m,n}$. Consider S = V(H)—D. S has m+n+2 vertices and < S > is an induced subgraph of H, which is a bi-central tree with radius two and diameter three. Hence the theorem is proved.

Lemma 2.2: Let G be an uni-central tree with radius two, then H = $BG_2(G)$ has the following properties. (1) H has odd number of vertices say 2p—1, each lying on a triangle.

(2) H is bi-eccentric with radius two such that there are exactly p vertices with eccentricity 2 and (p−1) vertices with eccentricity 3.

(3) There exists only one vertex w with eccentricity 2 such that, it lies in a shortest path of length two joining vertices with eccentricity 3. Also, $< N(w) >$ is a friendship graph, having k vertices with eccentricity 2 and k vertices with eccentricity 3. (4) Let $S \subseteq V(H)$ such that S is the set of all vertices with eccentricity 3 in H. $< S \cup \{w\} >$ is a uni-central tree $\cong$ G.

(5) If $x \in V(H)$ such that $e(x) = 3$. Let $y \in V(H)$ be an eccentric node of x in H. Then the shortest path from x to y is any one of the following two types. (i) x u v y, where $e(u) = e(v) = 2$. (ii) Path from x to y contains w and another vertex with eccentricity 3.

Proof: Let G be a uni-central tree with radius two. Then diam(G) = 4. Suppose $|V(G)| = p$, $|E(G)| = p-1$. Hence, $BG_2(G)$ has $2p-1$ vertices. Also, it is known that, in $BG_2(G)$ eccentricity of every point vertex is two and eccentricity of point vertices except the central node w of G is three, $e(w) = 2$ in $BG_2(G)$. Thus, (1), (2), (3) and (4) follow.

(5) Let $x \in V(H)$ with $e(x) = 3$. Then x is a point vertex and its eccentric point is also a point vertex. Hence in G, $d_G(x, y) \geq 3$. So, if u is an edge incident with x and v is an edge incident with y, then u and v are not adjacent in G. So, x u v y is a shortest path, where $e(u) = e(v) = 2$ in H.

Suppose x (or y) is adjacent to w in G. Then x w z y, where x, w, z, y are point vertices of $BG_2(G)$ is also a shortest path from x to y, where $e(w) = 2$ and $e(z) = 3$ in $BG_2(G) = H$. This proves (5).

Theorem 2.4: $H = BG_2(G)$, where G is a uni-central tree with radius two if and only if (1) H is bi-eccentric with diameter three with odd number of vertices, say $2p-1$. (2) Number of vertices with eccentricity 2 is p and number of vertices with eccentricity 3 is $p-1$. (3) There exists only one vertex $w \in V(H)$ with eccentricity 2 such that, it lies in a shortest path of length two, joining two vertices with eccentricity three.

Proof: Let G be a uni-central tree with radius two. Then $H = BG_2(G)$ satisfies all the three properties by Lemma 2.2. On the other hand, suppose H satisfies (1), (2) and (3), consider $E_3 = \{x \in H : e(x) = 3\}$. $E_2 = \{x \in H : e(x) = 2\}$. $|E_3| = p-1$, $|E_2| = p$. By the properties of $BG_2(G)$, all the vertices of E_3 are only the point vertices and E_2 may contain both point vertices and line vertices.

Claim 1: $H = BG_2(G)$, where G is connected.

Suppose G is disconnected, since $H = BG_2(G)$, all the vertices of H with eccentricity three are only the point vertices. Therefore, w is a line vertex, but in this case, w is adjacent to two point vertices of eccentricity 3, which are also adjacent in $BG_2(G)$, which is a contradiction to (3). Therefore, G must be connected.

Claim 2: G is a tree.

Since G is connected, $|E_2| = p$, $|E_3| = p-1$, E_2 may contain at most one point vertex. All of E_2 cannot be line vertices, since in that case again w is adjacent to two vertices of eccentricity 3, which are adjacent in $BG_2(G)$. Hence, w must be a point vertex. Thus, G is a connected graph with p vertices and p−1 edges, that is, G is a tree.

Claim 3: G is a uni-central tree with radius two.

G is a tree and eccentricity of a point vertex v in $BG_2(G)$ is two implies that the vertex is of eccentricity at most 2 in G. All other vertices in G are of eccentricity at least 3, since eccentricity of these vertices in $BG_2(G)$ is three. Thus, it can be seen that, G is a tree with one vertex of eccentricity two and all other vertices are of eccentricity at least 3, that is, G is a uni-central tree (w as center) with radius two.

Lemma 2.3: Let $H = BG_2(G)$, where $G = K_n$, $n \geq 3$ and $n \neq 7$. Then (1) H consists of n vertices with degree $2(n-1)$ and C(n, 2) vertices with degree $(n-2)(n-3)/2+2$. (2) H is self-centered with diameter two. (3) Every vertex of H lie on a cycle of length 3, 4, ..., 2n. (4) Every vertex with degree $2(n-1)$ lies on exactly $(n-1)$ triangles such that each triangle contains exactly one vertex of degree $((n-2)(n-3)/2)+2$ and two vertices of degree $2(n-1)$. (5) The induced subgraph formed by the vertices of degree $2(n-1)$ is a complete graph of order n, $n \neq 7$.

Proof: Assume $G = K_n$. G has n vertices and C(n, 2) edges with r(G) = diam(G) = 1. Therefore, $BG_2(G)$ has n point vertices and C(n, 2) line vertices each with eccentricity two. Also, degree of a point vertex = $2(p-1) = 2(n-1)$ and degree of a line vertex e = q+3−(deg u+deg v) = $n(n-1)/2+3-[(n-1)+(n-1)] = (n-2)(n-3)/2+2$.

In G, each vertex and edge lie on a cycle of length 3, 4, ..., n. Hence in $BG_2(G)$, each point vertex and line vertex lie on a cycle of length 3, 4, ..., 2n. Thus, $H = BG_2(G)$ satisfies (1), (2) and (3).

Proof of (4): Consider a vertex with degree $2(n-1)$. It is a point vertex. In G, there are $(n-1)$ edges incident with it. Therefore, $(n-1)$ triangles incident with each point vertex containing exactly two point vertices and one line vertex.

Hence (4) follows.

Proof of (5): The vertices with degree $2(n-1)$ are nothing but the point vertices. Hence (5) follows.

Theorem 2.5: $H = BG_2(G)$, where G is a complete graph of order n, $n \neq 7$ if and only if (1) $2h = n(n+1)$, where $h = |V(H)|$. (2) H is self-centered with diameter two, having n vertices with degree $2(n-1)$, $(n-2)$ vertices with degree $(((n-2)(n-3))/2)+2$. (3) The induced subgraph formed by the vertices of degree 2(n−1) is a complete graph of order n, $n \neq 7$.

Proof: If $G = K_n$, then $H = BG_2(G)$ satisfies (1), (2) and (3).

On the other hand, consider the induced subgraph formed by the vertices of degree 2(n−1). It is K_n. Now, $H = BG_2(K_n)$ satisfies properties (1) and (2) also. Hence, $H = BG_2(G)$, where $G = K_n$.

Remark 2.2: H is a regular self-centered graph of diameter 2, if n = 7.

Following theorems give the characterization of $BG_2(G)$ for a given graph G.

Theorem 2.6: $H = BG_2(G)$ if and only if edges of H can be partitioned into q triangles and induced subgraph S with q vertices such that each triangle has exactly one vertex common with S and this has exactly two adjacent vertices in $< V(H)-V(S) >$ and $\overline{S}$ is a line graph of $< V(H)-V(S) >$.

Proof: Suppose $H = BG_2(G)$. Consider, $e \in E(G)$ such that $e = uv$, where $u, v \in V(G)$. In $BG_2(G)$, the vertices e, u, v form a triangle. This is true for each $e \in E(G)$ and the set of all line vertices of H form an induced subgraph S such that $\overline{S} = L(G) = < V(H)-V(S) >$. On the other hand, let $G = < V(H)-V(S) >$. Then $L(G) = \overline{S}$, $S = \overline{L(G)}$. By the given conditions and from the construction of $BG_2(G)$, one can see that $H = BG_2(G)$. (Clearly G has q edges and $|V(H)| = |V(G)| + q = |V(G)| + |V(S)|$.

Theorem 2.7: $H = BG_2(G)$ if and only if V(H) can be partitioned into V_1 and V_2 such that $< V_2 > = \overline{L(< V_1 >)}$ and each vertex in V_2 is adjacent to exactly two adjacent vertices of V_1 such that for $x, y \in V_2$, $N(x) \cap V_1 \neq N(y) \cap V_1$.

Theorem 2.8: $H = BG_2(G)$, where G is a connected graph of radius $r \geq 3$ if and only if (1) every vertex of H lie on some triangle with distinct edges and two of the end points are of even degree and eccentricity 3. (2) H is bi-eccentric with diameter three. (3) The induced subgraph S formed by the peripheral nodes of H is a connected graph of radius r, with $H = BG_2(S)$ $(S = G)$.

Proof: Obvious.

Theorem 2.9: $H = BG_2(G)$, where G is a connected graph of radius one and diameter two, $G \neq K_{1,n}$ then (1) Every vertex of H lie on some triangle with distinct edges and at least two of the end points are of even degree. (2) H is self-centered with radius two. (3) There is at least one vertex v in H such that every triangle with the above property (1) has at least two of its end points in N[v].

(4) There exists points u, v in H such that there exists a path P of even length $2k > 2$ from u to v and a path of length k from u to v such that the later contains the alternate vertices of P.

Proof: Since $r(G) = 1$, there exists a vertex $v \in V(G)$ with eccentricity one in G. This v satisfies property (3) in the result. If $d(u, v) = 2$ in G, then (2) follows.

Now, let us consider a disconnected graph G. Then the following properties are true for $BG_2(G)$:

(1) $BG_2(G)$ is disconnected if and only if G contains K_1 as a component.

(2) If G has no isolated vertices, every point vertex of $BG_2(G)$ has eccentricity three and line vertex has eccentricity 2.

(3) $BG_2(G) = G$ if and only if G is totally disconnected.

(4) If $G = G_1 \cup G_2$, then $BG_2(G) = BG_2(G_1) \cup BG_2(G_2)$ if and only if any one of G_1, G_2 is rK_1.

(5) If $G = rK_2$, then $BG_2(G)$ has r triangles with end points having eccentricity 3, 3, 2 and degree 2, 2, r+1 respectively and a complete graph K_r. That is, edges of $BG_2(G)$ can be partitioned into r triangles and K_r such that each triangle and K_r has exactly one common point with eccentricity 2.

Theorem 2.10: $H = BG_2(G)$, where $G = rK_2$ if and only if the edges of H can be partitioned into r triangles and K_r such that each triangle and K_r has exactly one common point and other vertices of the triangles are of degree two.

Proof: If $G = rK_2$, $H = BG_2(G)$ satisfies all the properties. On the other hand, suppose H satisfies all the properties, consider the induced subgraph formed by the vertices of $V(H) - V(K_r) = V_1$. Clearly, $< V_1 > = rK_2$ and $BG_2(< V_1 >) = H$.

Hence the theorem is proved.

Theorem 2.11: $H = BG_2(G)$, where G is a disconnected graph having no isolated vertices if and only if (1) H is bi-eccentric with diameter three. (2) Every vertex of H lies on triangles with distinct edges, having two vertices with even degree and eccentricity three, the other vertex with eccentricity two. (3) The induced subgraph S formed by the peripheral vertices is disconnected with p vertices and q edges such that $p+q = |V(H)|$ and $BG_2(S) = H$.

Proof: Follows from the properties of $BG_2(G)$ (refer [5], [7]).

3. Partition of Edges of $BG_2(G)$ and $\overline{BG_2}(G)$

From the definition of $BG_2(G)$, it can be seen that edges of $BG_2(G)$ can be partitioned into edges of G, edges of subdivision graph of G and edges of $\overline{L}(G)$.

Proposition 3.1: (1) If n is odd, then edges of $BG_2(C_n)$ can be partitioned into $((n-1)/2)C_n$ and C_{2n}. (2) If n is even, then edges of $BG_2(C_n)$ can be partitioned into $((n-2)/2)C_n$, C_{2n}, $(n/2)K_2$.

Proof: When G = C_n, edges of $BG_2(G)$ can be partitioned into C_n, C_{2n}, $\overline{C_n}$. But, edges of K_n can be partitioned into $((n-1)/2)C_n$ if n is odd; $((n-2)/2)C_n$ and $(n/2)K_2$ if n is even [by Theorem 1.23]. Hence, edges of $\overline{C_n}$ can be partitioned $((n-3)/2)C_n$ if n is odd; $((n-4)/2)C_n$ and $(n/2)K_2$ if n is even. Hence, the edges of $Q(C_n)$ can be partitioned into $((n-1)/2)C_n$, C_{2n} if n is odd; $((n-2)/2)C_n$, C_{2n}, $(n/2)K_2$ if n is even.

Since, the proof is immediate from the definition, the following propositions are stated without proof.

Proposition 3.2; If G = P_n, edges of $BG_2(P_n)$ can be partitioned into P_n, P_{2n-1}, $\overline{P_{n-1}}$ or $(n-1)$ triangles and $\overline{P_{n-1}}$.

Proposition 3.3: If G = $K_{1,n}$, edges of $BG_2(G)$ can be partitioned into $K_{1,n}$, subdivision graph of $K_{1,n}$ or n triangles each are attached at one vertex.

Proposition 3.4: If G = nK_2, edges of $BG_2(G)$ can be partitioned into n triangles and K_n.

The following theorems give the path or cycle partition of $BG_2(G)$ for some graphs.

Theorem 3.1: (1) If n is odd, then edges of $BG_2(C_n)$ can be partitioned into n paths of length (n+3)/2. (2) If n is even, then edges of $BG_2(C_n)$ can be partitioned into n paths of length (n+2)/2 and $(n/2)K_2'$s.

Proof: Let $v_1, v_2, \ldots, v_n$ be the vertices of C_n and let $v_1v_2 = e_{12}$, $v_2v_3 = e_{23}, \ldots,$

$v_{n-1}v_n = e_{n-1n}$, $v_nv_1 = e_{n1}$ be the edges of C_n.

<u>**Case 1:** n is odd.</u>

$v_1\ v_2\ e_{23}\ e_{45}\ e_{78}\ e_{11\ 12} \ldots e_{lm}\ v_m$ containing $(n-1)/2$ line vertices in this sequence and each subscript is congruent to modulo n.

$v_2\ v_3\ e_{34}\ e_{56}\ e_{89} \ldots e_{(l+1)(m+1)}\ v_{m+1}$

……………………………………………

$v_n v_1 e_{12} e_{34} e_{67} \ldots e_{(l-1)(m-1)} v_{m-1}$ is a partition of n paths of length (n+3)/2.

<u>**Case 2:** n is even.</u>

$v_1 v_2 e_{23} e_{45} e_{78} \ldots e_{lm} v_m$ $v_2 v_3 e_{34} e_{56} e_{89} \ldots e_{(l+1)(m+1)} v_{m+1}$.

… … ……………………………………………

v_n v_1 e_{12} e_{34} e_{67} … $e_{(l-1)(m-1)}$ v_{m-1}, each containing (n−2)/2 line vertices is a partition of n paths of length (n+2)/2 and n/2 K_2's given by e_{12} $e_{((n+2)/2)((n+2)+1/2)}$; … Hence the theorem is proved.

Theorem 3.2: (1) If n is even, then edges of $BG_2(nK_2)$ can be partitioned into n paths of length (n+4)/2 and $(n/2)K_2'$s. (2) If n is odd, then edges of $\overline{BG_2(nK_2)}$ can be partitioned into n paths of length (n+5)/2.

Proof: Number of edges in $BG_2(nK_2) = (q^2+7q-\Sigma d_i^2)/2 = (n^2+7n-2n)/2 = n(n+5)/2$. Let v_1, v_2, …, v_{2n} represent the vertices of $G = nK_2$ and let $e_{12} = v_1v_2$,

$e_{34} = v_3v_4, \ldots, e_{2n-12n} = v_{2n-1}v_{2n}$. Denote e_{12} by e_1, …, e_n by $e_{2n-1\ 2n}$ by e_n.

<u>**Case 1:** n is even.</u>

$BG_2(nK_2)$ can be partitioned into n triangles and K_n. Edges of K_n can be partitioned into n paths of length (n/2)−1 and $(n/2)K_2'$s as below.

$$\underbrace{e_i\, e_{i+1}\, e_{i+3}\, e_{i+6} \ldots e_{i+1+2+3+\ldots+((n/2)-1)}}_{n/2 \text{ in numbers}}$$ of length (n/2)−1 for i = 1, 2, 3, …, n.

Now, add v_{2i-1} v_{2i}, to the beginning. If $e_{i+n(n-2)/8} = e_l = v_{2l-1}v_{2l}$, add v_{2l-1} at the end. Then v_{2i-1} v_{2i} e_i $e_{i+1} \ldots e_{i+n(n-2)/8}$ v_{2l-1} is path of length (n/2)−1+3 = (n+4)/2. Thus, n paths of length (n+4)/2 and $(n/2)K_2'$s can be obtained.

<u>**Case 2:** n is odd.</u>

Consider e_i e_{i+1} e_{i+3} … $e_{i+(1+2+\ldots+(n-1)/2)}$, i = 1, 2, …, n.

That is, e_i e_{i+1} $e_{i+3} \ldots e_{i+(n-1)(n+1))/8}$ of length (n−1)/2. Thus, $\overline{BG_2(nK_2)}$ can be partitioned into n paths of length ((n−1)/2)+2+1 = (n+5)/2 as follows:

v_{2i-1} v_{2i} e_i e_{i+1} e_{i+3} … $e_{i+(n-1)(n+1)/8}$ v_{2l-1}, where i = 1, 2, …, n; and i+(n−1)(n+1)/8 = l. This proves the theorem.

Theorem 3.3: (1) If n = 4k or 4k+1, then edges of $BG_2(K_n)$ can be partitioned into n(n−1)/2 paths of ((n−1)(n−4)/4)+3 and $(n(n-1)/4)K_2'$s.

(2) If n = 4k+2 or 4k+3, then edges of $BG_2(K_n)$ can be partitioned into n(n−1)/2 paths of length of ((n−2)(n−3)/4)+3.

Proof: Number of edges of $BG_2(K_n) = 1/2[(n(n-1)/2)^2+7n(n-1)/2-\Sigma(n-1)^2] = (n(n-1)/2)((n-2)(n-3)/4+3)$. If n is odd, edges of K_n can be partitioned into

$((n-1)/2)C_n$. If n is even, edges of K_n can be partitioned into $((n-2)/2)C_n$ and $(n/2)K_2'$s. Also, each edge in K_n is not adjacent to $((n(n-1)/2-2(n-2)-1) = (n^2-5n+6)/2$ other edges in K_n. Hence, by using Theorem 3.1 and adding other edges, the theorem can be proved.

For example, when n = 3,

$$\left.\begin{matrix} v_1 v_2 e_{23} v_3 \\ v_2 v_3 e_{31} v_1 \\ v_3 v_1 e_{12} v_2 \end{matrix}\right\} \text{is a path partition of } BG_2(G).$$

$$\left.\begin{matrix} v_1 v_2 e_{12} v_1 \\ v_2 v_3 e_{23} v_2 \\ v_3 v_1 e_{13} v_3 \end{matrix}\right\} \text{is a cycle partition of } BG_2(G).$$

n = 4, $v_1 v_2 e_{12} v_1$; $v_2 v_3 e_{23} v_2$; $v_3 v_4 e_{34} v_3$; $v_4 v_1 e_{41} v_4$; $v_1 v_3 e_{13} v_1$; $v_2 v_4 e_{24} v_2$

and $e_{12} e_{34}$; $e_{23} e_{41}$; $e_{13} e_{24}$ is a partition of edges of $BG_2(K_n)$.

n = 5,

$$\left.\begin{matrix} v_1 v_2 e_{23} e_{51} v_1 & v_1 v_3 e_{35} e_{14} v_1 \\ v_2 v_3 e_{34} e_{12} v_2 & v_2 v_4 e_{41} e_{25} v_2 \\ v_3 v_4 e_{45} e_{23} v_3 & v_3 v_5 e_{52} e_{31} v_3 \\ v_4 v_5 e_{51} e_{34} v_4 & v_4 v_1 e_{13} e_{42} v_4 \\ v_5 v_1 e_{12} e_{45} v_5 & v_5 v_2 e_{24} e_{53} v_5 \end{matrix}\right\}$$

and $e_{12} e_{35}$; $e_{23} e_{41}$; $e_{34} e_{52}$; $e_{45} e_{13}$; $e_{51} e_{24}$ is a partition of edges of $BG_2(K_n)$.

Theorem 3.4: (1) If n is even, then edges of $BG_2(K_{n,n})$ can be partitioned into n^2 paths of length $(n^2-2n+6)/2$ and $(n^2)/2$ K_2's.

(2) If n is odd, then edges of $BG_2(K_{n,n})$ can be partitioned into n^2 paths of length $(n^2-2n+7)/2$.

Proof: Number of edges of $BG_2(K_{n,n}) = 1/2[(n^2)^2+7n^2-\Sigma(n)^2] = (n^2/2)(n^2-2n+7)$.

If n is odd, edges of $K_{n,n}$ can be partitioned into $((n-1)/2)C_{2n}$ and nK_2. If n is even, edges of $K_{n,n}$ can be partitioned into $(n/2)C_{2n}$. Also, each edge in $K_{n,n}$ is not adjacent to $n^2-2(n-1) = n^2-2n+2$ other edges. Hence, by using Theorem 3.1 and adding other edges, one can prove the theorem.

Next, edge partitions of $\overline{BG_2(G)}$ can be seen.

Theorem 3.5: The edges of $\overline{BG_2(C_n)}$ can be partitioned into $\overline{C_n}$, C_n, $nK_{1,\, n-2}$.

(1) If n is odd, then edges of $\overline{BG_2(C_n)}$ can be partitioned into $((n-1)/2)C_n$, $nK_{1,n-2}$. (2) If n is even, then edges of $\overline{BG_2(C_n)}$ can be partitioned into $((n-2)/2)C_n$, $(n/2)K_2$ and $nK_{1,n-2}$.

Proof: Proof follows from the definition.

Theorem 3.6: (1) If n is odd, then edges of $\overline{BG_2(C_n)}$ can be partitioned into $((n-1)/2)C_n$, $((n-3)/2)C_{2n}$ and nK_2. (2) If n is even, then edges of $\overline{BG_2(C_n)}$ can be partitioned into $((n-2)/2)C_n$, $((n-2)/2)C_{2n}$ and $(n/2)K_2$.

Proof: When $G = C_n$, the edges of $\overline{BG_2(G)}$ can be partitioned into C_n, $\overline{C_n}$, and the edges joining line vertices and point vertices.

<u>**Case 1:**</u> <u>n is even.</u>

Let $v_1, v_2, ..., v_n$ be the point vertices, $e_{12}, e_{23}, e_{34}, \ldots, e_{n-1n}$ be the line vertices. In $\overline{BG_2(G)}$, each v_j is adjacent to $(n-2)$ line vertices $e_{12}, e_{23}, \ldots, e_{(j-2)(j-1)}, e_{j+1)(j+2)}, \ldots, e_{n-1n}, e_{n1}$. Combine these line vertices in two's as n is even. There are $(n-2)/2$ such collections, which are adjacent to v_j, $j = 1, 2, \ldots, n$.

Consider the point vertex v_1. v_1 is adjacent to $e_{23}, e_{34}, ..., e_{n-1n}$. Now, combine these into (e_{23}, e_{34}); (e_{45}, e_{56}); $(e_{(n-2)(n+1)}, e_{(n-1)n})$. Get the $(n-2)/2$ cycles of length 2n in $\overline{BG_2(G)}$ as follows:

1. $v_1\ e_{(n-1)n}\ v_2\ e_{n1}\ v_3\ e_{12}\ v_4 \ldots v_n\ e_{(n-2)(n-1)}\ v_1$

2. $v_1\ e_{(n-3)(n-2)}\ v_2\ e_{(n-2)(n-1)}\ v_3\ e_{(n-1)n}\ v_4\ e_{n1}\ v_5\ e_{12} \ldots v_n\ e_{(n-4)(n-3)}\ v_1$

3. $v_1\ e_{(n-5)(n-4)}\ v_2\ e_{(n-4)(n-3)}\ v_3\ e_{(n-3)(n-2)}\ v_4\ e_{(n-2)(n-1)} \ldots v_n\ e_{(n-6)(n-5)}\ v_1$

.

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$(n-2)/2$. $v_1\ e_{34}\ v_2\ e_{45}\ v_3\ e_{56}\ v_4 \ldots e_{12} v_n\ e_{23}\ v_1$

From this construction and from the definition of $\overline{BG_2(G)}$, it follows that edges of $\overline{BG_2(G)}$ are union of edges of C_n, $\overline{C_n}$ and these cycles.

<u>**Case 2:**</u> <u>n is odd.</u>

There are n point vertices $v_1, v_2, \ldots, v_n$ and n line vertices $e_{12}, e_{23}, \ldots, e_{n1}$. Combine $(n-1)$ line vertices into two's. v_1 is adjacent to $e_{23}, e_{34}, \ldots, e_{(n-1)n}$. Leaving $e_{(n-1)n}$, combine these into (e_{23}, e_{34}); (e_{45}, e_{56}); $(e_{n-3\ n-2}, e_{n-2\ n-1})$. Similarly, v_2 is adjacent to (e_{34}, e_{45}); (e_{56}, e_{67}); $\ldots$;$(e_{n-2\ n-1}, e_{n-1\ n})$, and $e_{n1}\ldots$ As in case (1) there are $((n-1)-2)/2$ cycles of length 2n and $nK_2{}'s$ $v_1e_{(n-1)n}$; v_2e_{n1}; v_3e_{12}; $\ldots$; $v_ne_{n-2\ n-1}$. Therefore, edges of $\overline{BG_2(G)}$ can be partitioned into C_n, $\overline{C_n}$, $((n-3)/2)C_{2n}$ and nK_2, when n is odd. Hence the proof of the theorem follows.

Theorem 3.7: (1) If n is odd, then edges of $\overline{BG_2}(K_{1,n})$ can be partitioned into $2K_n$, $((n-1)/2)C_{2n}$. (2) If n is even, then edges of $\overline{BG_2}(K_{1,n})$ can be partitioned into $2K_n$, $((n-2)/2)C_{2n}$ and nK_2.

Proof: When $G = K_{1,n}$, edges of $\overline{BG_2}(G)$ can be partitioned into K_n, K_n and $nK_{1,n-2}$. Let v_1, v_2, …, v_n, v be the point vertices, $e_1 = vv_1$, $e_2 = vv_2$,…, $e_n = vv_n \in E(G)$ be the n line vertices, where v is the central node of G.

Case 1: n is odd.

In $BG_1(G)$, each v_j is adjacent to (n−1) line vertices e_1, e_2, …, e_{j-1}, e_{j+1}, …, e_n. Combine these vertices into two by two. Thus, there are (n−1)/2 such collections, which are adjacent to v_j, j = 1, 2, …, n. Consider v_1, combine the edges as (e_2, e_3); (e_4, e_5); …; (e_{n-1}, e_n). The n−1 cycles of length 2n in $\overline{BG_2}(G)$ can be obtained as follows:

(1) $v_1\, e_3\, v_2\, e_4\, v_3\, e_5\, v_4\, e_6 \ldots e_{n-1}\, v_{n-2}\, e_n\, v_{n-1}\, e_1\, v_n\, e_2\, v_1$.

(2) $v_1\, e_5\, v_2\, e_6\, v_3\, e_7\, v_4\, e_8 \ldots e_n\, v_{n-3}\, e_1\, v_{n-2}\, e_2\, v_{n-1}\, e_3\, v_n\, e_4\, v_1$.

.

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((n−1)/2) $v_1\, e_n\, v_2\, e_1\, v_3\, e_2\, v_4\, e_3 \ldots v_{n-2}\, e_{n-3}\, v_{n-1}\, e_{n-2}\, v_n\, e_{n-1}\, v_1$.

From this construction and from the definition of $\overline{BG_2}(G)$, it follows that edges of $\overline{BG_2}(G)$ can be grouped into edges of $2K_n$ and these (n−1)/2 cycles of length 2n.

Case 2: n is even.

By the definition of $\overline{BG_2}(G)$ each v_j is adjacent to (n−1) line vertices, e_1, e_2, e_3, …, e_{j-1}, e_{j+1}, …, e_n. Among this, (n−2) line vertices can be grouped into pairs.

Consider v_1. v_1 is adjacent to (n−1) line vertices, e_2, e_3,…, e_{n-1}, e_n; leaving e_n, get (n−2)/2 pairs (e_2, e_3), (e_4, e_5),…, (e_{n-2}, e_{n-1}). Similarly, v_2 is adjacent to e_1, e_3, …, e_{n-1}, e_n. Leaving e_1, get (n−2)/2 pairs (e_3, e_4), (e_5, e_6), …, (e_{n+1}, e_n), …

As in case1, there are ((n−1)−1)/2 cycles of length n and nK_2's given by v_1e_n, v_2e_1, v_3e_2, …, v_ne_{n-1}. Therefore, edges of $\overline{BG_2}(G)$ can be partitioned into $2K_n$, $((n-2)/2)C_{2n}$ and nK_2 when $G = K_{1,n}$, where n is even.

Theorem 3.8: (1) If n is even, then edges of $\overline{BG_2}(K_n)$ can be partitioned into $L(K_n)$, $((n-2)^2/4)C_{2n}$ and $(n/2)K_{1,n-2}$. (2) If n is odd, then edges of $\overline{BG_2}(K_n)$ can be partitioned into $L(K_n)$, $((n-1)(n-3)/4)C_{2n}$, $(n(n-1)/2)K_2$.

Proof: When $G = K_n$, $\overline{BG_2}(G)$ can be partitioned into $L(K_n)$, $(n(n-1)/2)K_{1,n-2}$.

Proof: $G = K_n$. $\overline{BG_2}(G)$ has n point vertices and n(n−1)/2 line vertices.

<u>**Case 1:**</u> <u>n is even.</u>

When n is even, edges of K_n can be partitioned into $((n-2)/2)C_n$ and $(n/2)K_2$′s. Therefore, as in Theorem 3.6, edges of $\overline{BG_2(K_n)}$ can be partitioned into $L(K_n)$, $((n-2)^2/4)C_{2n}$ and $(n/2)K_{1,n-2}$.

<u>**Case 2:**</u> <u>n is odd.</u>

When n is odd, edges of K_n can be partitioned into $((n-2)/2)C_n$ Hence, as in Theorem 3.6, edges of $\overline{BG_2(K_n)}$ can be partitioned into $L(K_n)$, $((n-1)(n-3)/4)C_{2n}$, $(n(n-1)/2)K_2$.

Following theorems give the path partition of $\overline{BG_2(G)}$ for some special graphs.

Theorem 3.9: (1) If n is odd, then edges of $\overline{BG_2(C_n)}$ can be partitioned into n paths of length $(3n-5)/2$. (2) If n is even, then edges of $\overline{BG_2(C_n)}$ can be partitioned into n paths of length $(3n-6)/2$ and $(n/2)\ K_2$.

Proof: <u>**Case 1:**</u> <u>n is odd.</u>

$\lceil n/2 \rceil = (n+1)/2$. Edges of $\overline{BG_2(C_n)}$ can be partitioned into

(1) $\underbrace{\ldots v_{10}\ v_6\ v_3\ v_1}_{(n-1)/2 \text{ in numbers.}}\ e_{23}\ v_4\ e_{12}\ v_5\ e_{1n} \ldots\ v_{(n+3)/2}\ e_{((n+1)/2)+3\ ((n+1)/2)+4},\ e_{((n+1)/2)+4\ ((n+1)/2)+5}$

....

(n) $\ldots v_9\ v_5\ v_2\ v_n\ e_{12}\ v_3\ e_{n1}\ v_5\ e_{n-1n} \ldots v_{\lceil n/2 \rceil} e_{(\lceil n/2 \rceil+2)(\lceil n/2 \rceil+3)}\ e_{(\lceil n/2 \rceil+3)(\lceil n/2 \rceil+4)}$.

These are n paths of length $((n-1)/2)-1+(n-2)+1 = ((n-1)/2)+n-2 = (3n-5)/2$.

<u>**Case 2:**</u> <u>n is even.</u>

Edge set of $\overline{BG_2(C_n)}$ can be partitioned as follows:

(1) $\ldots\ e_{34}\ e_{23}\ v_4\ e_{12}\ v_5\ e_{1n}\ v_6\ \ldots\ v_{(n/2)+1}\ e_{((n/2)+4)((n/2)+5)} v_{((n/2)+2)}\ \underbrace{v_{((n/2)+4))}\ v_{(n/2)+7} \ldots}_{(n-2)/2 \text{ in number}}$

....

(n) $\ldots\ e_{23}\ e_{12}\ v_3\ e_{n1}\ v_4\ e_{n-1n}, \ldots\ e_{(n/2)+3\ (n/2)+4}\ v_{(n/2)+1}\ \underbrace{v_{(n/2)+3}\ v_{(n/2)+6}\ v_{(n/2)+10} \ldots}_{(n-2)/2 \text{ in number}}$

Hence, each path is of length $(n-2)+(n-2)/2 = (2n-4+n-2)/2 = (3n-6)/2$ and $(n/2)K_2$′s. Hence the theorem is proved.

Theorem 3.10: When $G = K_{1,n}$, edges of $\overline{BG_2(G)}$ can be partitioned into n path of length $2n-2$.

Proof: <u>**Case 1:**</u> <u>n is even.</u>

$\overline{BG_2(K_{1,n})}$ can be partitioned into edges of K_n, K_n and $nK_{1,n-2}$. Now, each

K_n can be partitioned into (n/2) paths of length (n−1)-------- I and

edges in $nK_{1,n-2}$ can be partitioned into n paths of length (n−1)-------- II.

Now, joining path of length (n−1) from I to a path of length (n−1) in II, one can get n distinct paths of length (n−1)+(n−1) = 2n−2.

Case 2: n is odd.

$nK_{1,n-2}$ can be partitioned into n paths of length n−1--------II and since n is odd, edges of each K_n can be partitioned into n paths of length (n−1)/2 ------III.

So, join n paths of length (n−1)/2 from each K_n to n paths of length n−1 in II. This is always possible. Thus, n paths of length n−1+(n−1)/2+(n−1)/2 = 2n−2 can be obtained. This proves the theorem.

Theorem 3.11: (1) If n is odd, then edges of $\overline{BG_2(nK_2)}$ can be partitioned into n paths of length 4n−4. (2) If n is even, then edges of $\overline{BG_2(nK_2)}$ can be partitioned into n paths of length 4n−5 and nK_2.

Proof: Let $v_1, v_2, \ldots v_{2n} \in V(G)$, $e_{12} = v_1v_2, e_{34} = v_3v_4, \ldots e_{2n-1\ 2n} = v_{2n-1}v_{2n} \in E(G)$.

Case 1: n is odd.

Consider the following partitions:

(1) $v_2\ e_{34}\ v_{2n-1}\ e_{56}\ v_{2n-3} \ldots e_{n\ n+1}\ v_{n+2}\ e_{n+4\ n+5}\ v_{n+1}\ e_{(n+6)(n+7)}\ v_{n-1} \ldots e_{(2n-1)(2n)}\ v_6\ e_{12}\ v_4$.

....

(n) $v_{2n}e_{12}\ v_{2n-3}\ e_{34}\ v_{2n-5}\ e_{56} \ldots e_{n-2\ n-1}v_ne_{n+2\ n+3}v_{n-1}\ e_{n+4\ n+5}\ v_{n-3} \ldots e_{2n-3\ 2n-2}\ v_4e_{2n-1\ 2n}v_2$

Each of this is a path of length 2n−2. Now, join the path $v_2\ v_4\ v_7\ v_{11}\ v_6 \ldots$ to the left. This path has 2n−1 distinct vertices and has length 2n−2. Thus, n paths of length 2n−2+2n−2 = 4n−4 can be obtained.

Case 2: n is even.

Consider

(1) $v_2\ e_{34}\ v_{2n}\ e_{56}\ v_{2n-2}\ e_{78}\ v_{2n-4}, \ldots e_{n+1\ n+2}\ v_{n-1}\ e_{n+3\ n+4}\ v_{n-3} \ldots v_3\ e_{2n-1\ 2n}\ v_1$.

(2) $v_4\ e_{56}\ v_2 \ldots v_{2n-2}\ e_{n+3\ n+4} \ldots v_{n-1} \ldots e_{12}\ v_3$.

....

(n) $v_{2n}\ e_{12}\ v_{2n-2}\ e_{34}\ v_{2n-4}\ e_{56}\ e_{2n-6} \ldots e_{n-1\ n}\ v_{n-3}\ e_{n+1\ n+2}\ v_{n-1} \ldots e_{2n-3\ 2n-2}\ v_{2n-1}$. These are n paths of length 2n−2. Consider $v_2\ v_4\ v_7\ v_{11}\ v_{15} \ldots$. This has length 2n−3. Joining this to (1) to the left, a path of length 2n−2+2n−3 = 4n−5 can be obtained. Similarly, one can get n paths of length 4n−5, and remaining nK_2's. This proves the theorem.

Theorem 3.12: When $G = K_n$, edges of $\overline{BG_2}(G)$ can be partitioned into $(n(n-1))/2$ paths of length $2n-4$.

Proof: __Case 1:__ __n is odd.__

Since n is odd, edges of K_n can be partitioned into $(n-1)/2$ cycles of length n. Corresponding to a cycle $v_1\, v_2 \ldots v_n\, v_1$ of length n, one can partition the edges as follows:

(1) $v_1\, e_{23}\, v_4\, e_{12}\, v_5\, e_{1n}\, v_6\, e_{n-1\,n} \ldots v_{(n+3)/2}\, e_{(n+1)/2+3\ \ (n+1)/2+4}\, e_{(n+1)/2+3\ \ (n+1)/2+2}\, e_{(n+1)/2+3\ \ (n+1)/2+1}\, e_{(n+1)/2+3\ \ (n+1)/2} \ldots e_{(n+1)/2+3\ \ (n+1)/2+5}$.

(2) $v_2\, e_{34}\, v_5\, e_{23}\, v_6\, e_{12} \ldots v_{\lceil n/2 \rceil+2}\, e_{\lceil n/2 \rceil+4\ \lceil n/2 \rceil+5} \ldots e_{\lceil n/2 \rceil+4\ \lceil n/2 \rceil+3}\, e_{\lceil n/2 \rceil+4\ \lceil n/2 \rceil+2} \ldots e_{\lceil n/2 \rceil+4\ \lceil n/2 \rceil+6}$.

($\lceil n/2 \rceil$) $v_{\lceil n/2 \rceil}\, e_{\lceil n/2 \rceil+1\ \lceil n/2 \rceil+2} \ldots v_1\, e_{34}\, e_{32}\, e_{31}\, e_{3n}\, e_{3\,n-1}\, e_{3\,n-2}\, e_{3\,n-3}\, e_{3\,n-4} \ldots e_{35}$.

....

(n) $v_n\, e_{12}\, v_3\, e_{1n}\, v_4\, e_{n-1\,n} \ldots v_{\lceil n/2 \rceil}\, e_{\lceil n/2 \rceil+2\ \lceil n/2 \rceil+3}\, e_{\lceil n/2 \rceil+2\ \lceil n/2 \rceil+1}\, e_{\lceil n/2 \rceil+2\ \lceil n/2 \rceil} \ldots e_{\lceil n/2 \rceil+2\ \lceil n/2 \rceil+4}$.

In this, each path is of length $(n-2)+(n-2) = 2n-4$. Thus, corresponding to $(n-1)/2$ cycles of K_n there are $n(n-1)/2$ paths of length $2n-4$, when n is odd.

__**Case 2:**__ __n is even.__

Now, consider K_n as $K_{n-1}+K_1$ where K_{n-1} is the complete graph with vertices $v_1, v_2, \ldots v_{n-1}$ and K_1 is v_n. Edges of $\overline{BG_2}(K_n)$ are edges of $\overline{BG_2}(K_{n-1})$, edges of ($\overline{BG_2}(K_{1,n-1})$ − edges of K_{n-1})) and the edges joining the adjacent line vertices (since each edge in K_{n-1} is adjacent to two edges which are joining end vertices of e to v_n, that is K_1)---------I.

Now, by case 1, edges of $\overline{BG_2}(K_{n-1})$ can be partitioned into $(n-1)(n-2)/2$ paths of length $2(n-1)-4 = 2n-6$. One can extend these paths to paths of length $2n-4$ by adding two more line vertices by property I. Thus, one can get $(n-1)(n-2)/2$ paths of length $2n-4$. Also, edges of $\overline{BG_2}(K_{1,n-1})$ can be partitioned into $(n-1)$ paths of length $2(n-1)-2 = 2n-4$ by Theorem 3.10. Hence, $\overline{BG_2}(K_n)$ can be partitioned into $(n-1)(n-2)/2+(n-1)$ paths of length $2n-4$, that is $(n(n-1))/2$ paths of length $2n-4$.

Theorem 3.13: Edges of $\overline{BG_2}(K_{n,n})$ can be partitioned into n paths of length $(n-1)(3n+1)$.

Proof: Similar to the previous theorem.

4. Characterizations and edge partitions of $BG_3(G)$ and $\overline{BG_3}(G)$

A graph H is a Boolean graph of third kind if it is isomorphic to a graph $BG_3(G)$ for some graph G. For example, $K_{1,2}$ is a Boolean graph of third kind. In this section, characterizations for Boolean graph of third kind and edge partitions of $BG_3(G)$, $\overline{BG_3}(G)$ for some special graphs G are presented.

Theorem 4.1: H = $BG_3(G)$ if and only if V(H) can be partitioned into V_1 and V_2 such that (1) V_1 is a maximal independent set. (2) Each element in V_2 is adjacent to exactly two elements of V_1 such that for $v_{21}, v_{22} \in V_2$, $N(v_{21}) \cap V_1 \neq N(v_{22}) \cap V_1$. (3) $< \overline{V_2} >$ is a line graph.

Proof: If G be a (p, q) graph such that H = $BG_3(G)$, then H contains p+q vertices, p-point vertices and q-line vertices. $V_1 = V(G)$, $V_2 = E(G)$ satisfy all the properties (1), (2) and (3).

On the other hand, assume V(H) satisfies the properties (1), (2) and (3). By (3), $< \overline{V_2} >$ is a line graph. Find out G such that $< \overline{V_2} > = L(G)$ and $BG_3(G) = H$. This proves the theorem.

Remark 4.1: Two vertices v_{21}, v_{22} in V_2 are adjacent if and only if $N(v_{21}) \cap V_1$ and $N(v_{22}) \cap V_1$ are disjoint.

Theorem 4.2: H = $BG_3(G)$ if and only if V(H) can be partitioned into V_1 and V_2 such that (i) V_1 is the maximum independent set (ii) Each element in V_2 is adjacent to exactly two elements of V_1 such that for $v_{21}, v_{22} \in V_2$, $N(v_{21}) \cap V_1 \neq N(v_{22}) \cap V_1$). (iii) For any two adjacent vertices v_{21}, v_{22} in V_2, $N(v_{21}) \cap V_1$, $N(V_{22}) \cap V_1$ are disjoint.

Proof: Let H = $BG_3(G)$. Take $V_1 = V(G)$, $V_2 = E(G)$. V_1, V_2 follow all the properties. On the other hand, suppose H satisfy all the properties, take $V(G) = V_1$. By (ii) each element in V_2 is adjacent to exactly two vertices in V_2. Corresponding to one element in V_2, there exist two elements in V_1.

For $v_2 \in V_2$, there exists $v_1, v_1' \in V_1$ such that v_1, v_1' are adjacent to V_2. Let E(G) = $\{(v_1, v_1') : v_2 \in V_2$ is adjacent to v_1 and $v_1'\}$. Then E(G) contains exactly $|V_2|$ edges, and $BG_3(G) = H$. Hence the theorem is proved.

Theorem 4.3: If H = $BG_3(C_n)$, $n \geq 4$, H is bi-regular with 2n vertices, n vertices with degree 2 and eccentricity 3 and n vertices with degree n−1 and eccentricity 2. Also, edges of $BG_3(C_n)$ can be partitioned into

(i) C_{2n} and $((n-3)/2)C_n$, if n is odd.

(ii) C_{2n}, $((n-4)/2)C_n$ and $(n/2)K_2$ if n is even.

Proof: If H = $BG_3(C_n)$, then H contains n point vertices and n line vertices. Each point vertex has degree two and eccentricity three and each line vertex has degree n−1 and eccentricity two by Theorem 1.3

Now, let $v_1, v_2, ..., v_n \in V(C_n)$ and $v_1v_2 = e_{12}, ..., v_nv_1 = e_{n1}, \in E(G)$. In $BG_3(C_n)$, e_{12} is adjacent to v_1, v_2 and e_{34}, e_{45}, e_{n-1n}. Similar result is true for other e_{jj+1}'s. Also, v_1 is adjacent to e_{12} and e_{n1}; v_2 is adjacent to e_{12} and e_{23} and so on. Hence, $BG_3(C_n)$ contains one C_{2n} and edges of $\overline{C_n}$. Edges of $\overline{C_n}$ can be partitioned into $((n-2)/2-1)C_n$ and $(n/2)K_2$ if n is odd; $(((n-1)/2)-1)C_n$ if n is even. Hence the theorem is proved.

Theorem 4.4: If $H = BG_3(K_{1,n})$, then H contains a vertex with degree n and eccentricity 2, n vertices with degree 2 and eccentricity 3, n vertices with degree 1 and eccentricity 4. Also, edges of H can be partitioned into $K_{1,n}$, nK_2 or $nK_{1,2}$.

Proof: Let $G = K_{1,n}$. In $BG_3(G)$, the central vertex of G is of degree n and eccentricity 2, all other point vertices are of degree one and eccentricity 4. All line vertices are of degree 2 and eccentricity 3. Clearly, edges of $BG_3(K_{1,n})$ can be partitioned into $nK_{1,2}$, each of which is attached at one of their end vertices or $K_{1,n}$, nK_2. When n is even, edges of $BG_3(K_{1,n})$ can be partitioned into n/2 paths of length 4, each of which is attached at their central vertex.

Theorem 4.5: If $H = BG_3(K_n)$, then H has n vertices with degree $n-1$; $n(n-1)/2$ vertices with degree $((n-1)(n-4)/2)+3$. Suppose V_1 represents vertices with degree $n-1$ and V_2 represents vertices with degree $((n-1)(n-4)/2)+3$, any two non-adjacent vertices in V_2 are mutually adjacent to $(1/2)(n-4)(n-3)$ vertices in V_2; any two adjacent vertices in V_2 are mutually adjacent to $(1/2)(n-4)(n-5)$ vertices in V_2. Also, edges of $BG_3(K_n)$ can be partitioned into $((n-1)/2)C_{2n}$, $((n-2)(n-3)/4)C_{n(n-1)/2}$ or $((n-1)/2)C_{2n}$, $((n-2)(n-3)/2)C_{n(n-1)/4}$.

Proof: $H = BG_3(K_n)$. H has n point vertices each with degree $n-1$, and has $n(n-1)$ line vertices each with degree $((n-1)(n-4)/2)+3$. Take two adjacent edges e_{ij}, e_{jk} in G ($e_{ij} = v_iv_j$, $e_{jk} = v_jv_k \in E(G)$). By definition, these two line vertices are not adjacent in H. Each e_{ij} is exactly not adjacent to $2n-4$ line vertices in $BG_3(G)$. Therefore, e_{ij} and e_{jk} are mutually not adjacent to $3n-6$ line vertices, that is, e_{ij} and e_{jk} are mutually adjacent to $(n(n-1)/2)-(3n-6) = (1/2)(n-4)(n-3)$ line vertices. Now, take two non-adjacent edges e_{ij}, e_{kl} in G. By definition, these two line vertices are adjacent in $BG_3(K_n)$. e_{ij} is not adjacent to exactly $2n-4$ line vertices in $BG_3(G)$.Therefore, e_{ij}, e_{kl} are mutually not adjacent to $(2n-4+2n-4-2) = 4n-10$ edges. Therefore, e_{ij}, e_{kl} are mutually adjacent to $(n(n-1)/2)-(4n-10) = (1/2)(n-4)(n-5)$ line vertices in $BG_3(K_n)$. (Leaving i, j, k, l there are $n-4$ point vertices. The edges made by these points, that is $C(n-4, 2)$ edges are adjacent to e_{ij} and e_{kl} in $BG_3(K_n)$, that is $((n-4)(n-5)/2)$ line vertices).

Theorem 4.6: H = $BG_3(nK_2)$ if and only if H = K_n^{2+}. Edges of $BG_3(nK_2)$ can be partitioned into K_n and $nK_{1,2}$.

Proof: Assume H = $BG_3(nK_2)$. G = nK_2 has n independent edges. Hence, in $R_3(nK_2)$, n line vertices are adjacent to each other and each line vertex is adjacent to its incident point vertices and degree of each point vertex is one. This proves H = K_n^{2+}. On the other hand, suppose H = K_n^{2+}. Consider V_1 = set of vertices of H with degree one then $|V_1| = 2n$. Let V(G) = V_1 and E(G) = {(u, v) : u, v $\in V_1$ and u, v are adjacent to the same vertex in H}. Then G contains 2n vertices, with n edges; G = nK_2, and H = $BG_3(G)$. Hence the theorem is proved.

Theorem 4.7: H = $BG_3(K_{1,n})$ if and only if H = F, where F is the subdivision of $K_{1,n}$.

Proof: Suppose H = F. Consider the maximal independent set with maximum cardinality. Take this as V_1 and $V_2 = V(H)-V_1$. Let V_1 = V(G) and E(G) = {(u, v):u $\in V_1$, v $\in V_1$ such that u and v are adjacent to w in V_2}. Clearly, G = $K_{1,n}$. Then F $\cong BG_3(G)$. Hence H $\cong BG_3(K_{1,n})$.

Theorem 4.8: If n is even, edges of $BG_3(K_{1,n})$ can be partitioned into n/2 paths of length four. If n is odd, edges of $BG_3(K_{1,n})$ can be partitioned into n−1 paths of length four and a path of length two.

Proof: $BG_3(K_{1,n})$ = F, where F is the subdivision graph of $K_{1,n}$. Hence the proof follows.

Theorem 4.9: If n is odd, $BG_3(nK_2)$ can be partitioned into n paths of length (n+3)/2; If n is even, $BG_3(nK_2)$ can be partitioned into n paths of length (n+2)/2 and $(n/2)K_2$.

Proof: Follows from Theorem 3.2.

Theorem 4.10: If n is odd, $BG_3(C_n)$ can be partitioned into n paths of length (n+1)/2. If n is even, $BG_3(C_n)$ can be partitioned into n paths of length n/2 and $(n/2)K_2$.

Proof: Follows from Theorem 3.1.

In the following partitions of edges of $\overline{BG_3(G)}$ can be studied.

Theorem 4.11: If n is odd, then edges of $\overline{BG_3(K_{1,n})}$ can be partitioned into edges of K_n, edges of K_{n+1} and $(n-1)C_{2n}$. If n is even, then edges of $\overline{BG_3(K_{1,n})}$ can be partitioned into edges of K_n, edges of K_{n+1}, $((n-2)/2)C_{2n}$ and nK_2.

Proof: Let H = $\overline{BG_3(K_{1,n})}$. In H, all the point vertices form a complete graph K_{n+1}. In $K_{1,n}$, all the edges are adjacent to each other and hence in H, all the line vertices form K_n. Also, each line vertex is adjacent to exactly (n+1−2) point vertices. Let v be the central vertex of $K_{1,n}$ and $v_1, v_2, ..., v_n$ be the other vertices, and let $e_i = vv_i$, i = 1, 2, …, n.

<u>**Case1:** n is odd.</u> Each e_i is adjacent to $n-1$ (even) vertices $v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n$. So, there are $(n-1)/2$ cycles of length 2n.

1. $v_1\ e_n\ v_2\ e_1\ v_3\ e_2 \dots v_n\ e_{n-1}\ v_1$.
2. $v_1\ e_{n-2}\ v_2\ e_{n-1}\ v_3\ e_n\ v_4\ e_2 \dots v_n\ e_{n-3}\ v_1$.

....

$(n-1)/2$. $v_1\ e_3\ v_2\ e_4\ v_3\ e_5 \dots v_n\ e_2\ v_1$.

<u>**Case 2:** n is even.</u> When n is even each e_i is adjacent to $n-1$ (odd) vertices $v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n$. So, as in case 1, there are $((n-1)-1)/2 = (n-2)/2$ cycles of length 2n and nK_2's.

Thus, edges of $\overline{BG_3}(K_{1,n})$ can be partitioned into edges of

$$\begin{cases} K_n, K_{n+1}, ((n-1)/2)C_{2n} \text{ if n is odd.} \\ K_n, K_{n+1}, (n-2)/2C_{2n} \text{ and } nK_2 \text{ if n is even.} \end{cases}$$

Theorem 4.12: Edges of $\overline{BG_3}(K_{1,n})$ can be partitioned into edges of

$((n-1)/2)C_n$, $((n-1)/2)C_{2n}$, $((n-1)/2)C_{n+1}$, $((n+1)/2)K_2$ if n is odd,

$((n-2)/2)C_n$, $((n-2)/2)C_{2n}$, $(n/2)C_{n+1}$, nK_2 if n is even.

Proof: Follows from the previous Theorem and partitions of K_n and K_{n+1}.

Theorem 4.13: (1) If n is odd, edges of $\overline{BG_3}(K_{1,n})$ can be partitioned into edges of n paths of length $2n-1$, each containing exactly one edge from $K_{1,n}$.

(2) If n is even, edges of $\overline{BG_3}(K_{1,n})$ can be partitioned into edges of n paths of length $2n-2$, (each containing exactly one edge from $K_{1,n}$) and nK_2's.

Proof: Follows from Theorem 3.10.

Theorem 4.14: Edges of $\overline{BG_3}(C_n)$ can be partitioned into $((n+1)/2)C_n$, $((n-3)/2)C_{2n}$, nK_2 if n is odd; $(n/2)C_n$, $(n/2)K_2$, $((n-2)/2)C_{2n}$ if n is even.

Proof: Similar to Theorem 3.6.

Theorem 4.15: If n is odd, then edges of $\overline{BG_3}(C_n)$ can be partitioned into n paths of length $(3/2)(n-1)$. If n is even, then edges of $\overline{BG_3}(C_n)$ can be partitioned into n paths of length $(3n-4)/2$ and $(n/2)K_2$.

Proof: Follows from Theorem 3.9.

Theorem 4.16: (1) If n is odd, then $\overline{BG_3}(nK_2)$ can be partitioned into n paths of length $4n-3$. (2) If n is even, $\overline{BG_3}(nK_2)$ can be partitioned into n paths of length $4n-4$ and nK_2's.

Proof: Number of edges in $\overline{BG_3}(nK_2) = 2n(2n-1)/2+n(2n-3)+½\sum 1 = n(4n-3)$.

Proof follows from the partition of $\overline{BG_2}(nK_2)$ (Theorem 3.11).

Theorem 4.17: If n is odd, then edges of $\overline{BG_3}(P_n)$ can be partitioned into edges of K_n, P_{n-1}, $((n-3)/2)P_{2n-1}$, $(n-1)K_2$. If n is even, then edges of $\overline{BG_3}(P_n)$ can be partitioned into K_n, P_{n-1}, $((n-2)/2)P_{2n-1}$.

Proof: Edges of $\overline{BG_3}(P_n)$ can be partitioned into K_n, P_{n-1} and $(n-1)K_{1,n-2}$. If n is odd, then $(n-1)K_{1,n-2}$ can be partitioned into $((n-3)/2)P_{2n-1}$, $(n-1)K_2$. If n is even, then $(n-1)K_{1,n-2}$ can be partitioned into $((n-2)/2)P_{2n}-1$. This proves the theorem.

Theorem 4.18: Edges of $\overline{BG_3}(K_n)$ can be partitioned into $((n-2)/2)C_n$, $(n/2)K_2$, $(n(n-1)(n-2)/4)C_{n-1}$, $((n-2)^2/4)C_{2n}$, $(n/2)K_{1,n-2}$ if n is even; $((n-1)/2)C_n$, $(n(n-1)(n-3)/4)C_{n-1}$, $(n(n-1)^2/4)K_2$, $(n(n-1)^2/4)K_2$, $((n-1)(n-3)/4)C_{2n}$ if n is odd.

Proof: Edges of $\overline{BG_3}(K_n)$ can be partitioned into K_n, $L(K_n)$ and $(n(n-1)/2)K_{1,n-2}$, that is K_n, $(n(n-1)/2)K_{n-1}$, $(n(n-1)/2)K_{1,n-2}$. Now, using the partitions of K_n and $(n(n-1)/2)K_{1,n-2}$ (similar to Theorem 3.6) the theorem can be proved.

Theorem 4.19: If n is odd, then edges of $\overline{BG_3}(K_{n,n})$ can be partitioned into K_{2n}, n^2K_n, $((n-1)^2/2)C_{4n}$, $nK_{1,2n-2}$. If n is even, then edges of $\overline{BG_3}(K_{n,n})$ can be partitioned into K_{2n}, n^2K_n, $(n(n-1)/2)C_{4n}$.

Proof: Edges of $\overline{BG_3}(K_{n,n})$ can be partitioned into edges of K_{2n}, edges of n^2K_n and $n^2K_{1,2n-2}$. Edges of $n^2K_{1,2n-2}$ can be partitioned into $(n/2)(n-1)C_{4n}$ if n is even; $((n-1)(2n-2)/4)C_{4n}$, $nK_{1,2n-2}$ if n is odd (as in Theorem 3.6). Hence, when n is odd, edges of $\overline{BG_3}(K_{n,n})$ can be partitioned into K_{2n}, n^2K_n, $((n-1)^2/4)C_{4n}$, $nK_{1,2n-2}$; when n is even, edges of $\overline{BG_3}(K_{n,n})$ can be partitioned into K_{2n}, n^2K_n, $(n(n-1)/2)C_{4n}$.

Conclusion: Other properties such as domination parameters of $BG_2(G)$ and graph equations connecting $BG_2(G)$, total graphs and Line graphs are studied and submitted elsewhere.

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