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Necessary Condition for Mean Labeling

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Abstract: *Mean labeling of graphs was discussed in [24-28].Different kinds of mean labeling were discussed in [17]. In this paper, we have proved the Necessary Condition for mean labeling.*

Keywords: *mean labeling, mean graph.*

AMS (MSC) ***Subject Classification:*** *05C78*

1. Introduction

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary [20]. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph. Labeled graphs serve as useful models for a broad range of applications [2-4].

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).

Graph labeling was first introduced in the late 1960's. Many studies in graph labeling refer to Rosa's research in 1967 [23].

Labeled graphs serve as useful models for a broad range of applications such as X-ray crystallography, radar, coding theory, astronomy, circuit design and communication network addressing. Particularly interesting applications of graph labeling can be found in [5].

Mean labeling of graphs was discussed in [24-26].

Vaidya [29-32] and et al. have investigated several new families of mean graphs. Nagarajan [31] and et al. have found some new results on mean graphs.

Ponraj, Jayanthi and Ramya extended the notion of mean labeling to super mean labeling in [21].

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Gayathri and Tamilselvi [18-19, 27] extended super mean labeling to k-super mean, (k, d)-super mean, k-super edge mean and (k, d)-super edge mean labeling. Manickam and Marudai [22] introduced the concept of odd mean graph.

Gayathri and Amuthavalli [1, 6-8] extended this concept to k-odd mean and (k, d)-odd mean graphs. Gayathri and Gopi[9-17] extended this concept to k-even mean and (k, d)-Even mean graphs.

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In this paper, we have proved the Necessary Condition for mean labeling.

2. Main Results

Definition 2.1.1:

A graph G with p vertices and q edges is called a **mean labeling** if there is an injective function f from the vertices of G to $\{0, 1, 2, ..., q\}$ such that when each edge uv is labeled with $\frac{(f(u)+f(v))}{2}$ if $f(u) + f(v)$ is even, and $\frac{(f(u)+f(v)+1)}{2}$ if $f(u) + f(v)$ is odd, then the resulting edge labels are distinct.

The induced edge labeling is represented by a function $f^*(E(G)) = \{1, 2, ..., q\}$.

A graph which admits mean labeling is called a **mean graph.**

Theorem 2.1.2:

If G is a mean graph then any vertex x with label $f(x) \in \{0, 1, 2, ..., q\}$ has number of adjacent vertex labels as

1. $\frac{q+1}{2}$ when q is odd, $\frac{q}{2}$ when q is even if $f(x)=0$.
2. $\frac{q+3}{2}$ when q is odd, $\frac{q+2}{2}$ when q is even if $f(x) \equiv 0 (\mathrm{mod}\, 4)$ and $f(x) \neq 0$
3. $\frac{q+1}{2}$ when q is odd, $\frac{q+2}{2}$ when q is even if $f(x) \equiv 1 (\mathrm{mod}\, 4)$.
4. $\frac{q+3}{2}$ when q is odd, $\frac{q+2}{2}$ when q is even if $f(x) \equiv 2 (\mathrm{mod}\, 4)$.
5. $\frac{q+1}{2}$ when q is odd, $\frac{q+2}{2}$ when q is even if $f(x) \equiv 3 (\mathrm{mod}\, 4)$.

Proof:

Case 1: q is odd

Subcase (i): $\boldsymbol{f(x) \equiv 0 (\mathrm{mod}\, 4)}$

subcase (*a*): $\boldsymbol{f(x)=0}$

If $f(x) = 0$, then its adjacent vertex labels are from the set {1, 3, 5, ..., q} ... (*) or from the set {2, 4, 6, ..., q – 1} ∪ {q} ... (**)

Hence, the total number of possible adjacent vertex labels is:

$$\frac{q-1}{2} + 1 = \frac{q+1}{2} \quad \text{[From (*)]}$$

or $$\frac{q-3}{2} + 2 = \frac{q+1}{2} \quad \text{[From (**)]}$$

Therefore, if $\boldsymbol{f(x)=0}$ then the total number of adjacent vertex labels of x is $\frac{q+1}{2}$.

***subcase* (b):** $\boldsymbol{f(x) \neq 0}$

Any number $f(x) \in \{4, 8, 12, ...\}$ will have the adjacent vertex labels as $\{0\} \cup \{1, 3, ..., q\} ... (***)$ or from the set

$$\{0\} \cup \{2, 4, 6, ..., q-1\} - \{f(x)\} \cup \{f(x)-1\} \cup \{q\} ... (****)$$

Hence, the total number of possible adjacent vertex labels is:

$$\frac{q-1}{2} + 1 + 1 = \frac{q+3}{2} \quad \text{[From (***)]}$$

or $$\frac{q-3}{2} + 1 + 2 - 1 + 1 = \frac{q+3}{2} \quad \text{[From (****)]}$$

Therefore if $f(x) \neq 0$ then the total number of adjacent vertex labels of x is $\frac{q+3}{2}$.

Subcase (ii): $f(x) \equiv 1 \pmod 4$

Any number $f(x) \in \{1, 5, 9, ...\}$ will have the adjacent vertex labels as $\{0, 2, 4, ..., q-1\}$ ... (*) or from the set $\{1, 3, 5, ..., q\} - \{f(x)\} \cup \{f(x) - 1\}$ ... (**)

Hence, the total number of possible adjacent vertex labels is

$$\frac{q-1}{2} + 1 = \frac{q+1}{2} \text{ [From (*)]}$$

or $$\frac{q-1}{2} + 1 - 1 + 1 = \frac{q+1}{2} \text{ [From (**)]}$$

Therefore, if $f(x) \equiv 1 \pmod 4$ then the total number of adjacent vertex labels of x is $\frac{q+1}{2}$.

Subcase (iii): $f(x) \equiv 2 \pmod 4$

Any number $f(x) \in \{2, 4, 6, ...\}$ will have the adjacent vertex labels as $\{0\} \cup \{1, 3, ..., q\}$ ... (*) or from the set $\{2, 4, 6, ..., q-1\} \cup \{q\} \cup \{0\}$ ... (**)

Hence, the total number of possible adjacent vertex labels is,

$$\frac{q-1}{2} + 2 = \frac{q+3}{2} \text{ [From (*)]}$$

or $$\frac{q-3}{2} + 3 = \frac{q+3}{2} \text{ [From (**)]}$$

Therefore, if $f(x) \equiv 2 \pmod 4$ then the total number of adjacent vertex labels of x is $\frac{q+3}{2}$.

Subcase (iv): $f(x) \equiv 3 \pmod 4$

Any number $f(x) \in \{3, 7, 11, ...\}$ will have the adjacent vertex labels as $\{0, 2, 4, ..., q-1\}$...(*) or from the set $\{1, 3, 5, ..., q\} - \{f(x)\} \cup \{f(x) - 1\}$... (**)

Hence, the total number of possible adjacent vertex labels is

$$\frac{q-1}{2} + 1 = \frac{q+1}{2} \text{ [From (*)]}$$

or $$\frac{q-1}{2} + 1 - 1 + 1 = \frac{q+1}{2} \text{ [From (**)]}$$

Therefore, if $f(x) \equiv 3 \pmod 4$ then the total number of adjacent vertex labels of x is $\frac{q+1}{2}$.

Case (ii): q is even

Subcase (i): $f(x) \equiv 0 \pmod 4$

***subcase* (*a*):** $f(x) \neq 0$

Any number $f(x) \in \{0, 4, 8, ...\}$ will have the adjacent vertex labels as $\{0\} \cup \{1, 3, 5, ..., q - 1\}$ ... (*) or from the set $\{0\} \cup \{2, 4, 6, ..., q\} - \{f(x)\} \cup \{f(x) - 1\}$ ... (**).

Hence, the total number of possible adjacent vertex labels is:

$$\frac{q-2}{2}+2 = \frac{q+2}{2} \text{ [From (*)]}$$

or $$\frac{q-2}{2}+2-1+1 = \frac{q+2}{2} \text{ [From (**)]}$$

Therefore if $f(x) \neq 0$ then the total number of adjacent vertex labels of x is $\frac{q+2}{2}$.

***subcase* (*b*):** $f(x) = 0$

If $f(x) = 0$, then the adjacent vertex labels are $\{1, 3, 5, ..., q - 1\}$ ... (***) or from the set $\{2, 4, , ..., q\}$ ... (****)

Hence, the total number of possible adjacent vertex labels is,

$$\frac{q-2}{2}+1 = \frac{q}{2} \text{ [From (***)]}$$

or $$\frac{q-2}{2}+1 = \frac{q}{2} \text{ [From (****)]}$$

Therefore, if $f(x) = 0$ then the total number of adjacent vertex labels of x is $\frac{q}{2}$.

Subcase (ii): $f(x) \equiv 1 \pmod 4$

Any number $f(x) \in \{1, 5, 9, ...\}$ will have the adjacent vertex labels as $\{0, 2, 4, ..., q\}$ ... (*) or from the set $\{1, 3, 5, ..., q - 1\} - \{f(x)\} \cup \{f(x) - 1\} \cup \{q\}$ ... (**)

Hence, total number of possible adjacent vertex labels is

$$\frac{q-0}{2}+1 = \frac{q+2}{2} \text{ [From (*)]}$$

or $$\frac{q-2}{2}+2-1+1 = \frac{q+2}{2} \text{ [From (**)]}$$

Therefore, if $f(x) \equiv 1 \pmod 4$ then the total number of adjacent vertex labels x is $\frac{q+2}{2}$.

Subcase (iii): $f(x) \equiv 2 \pmod 4$

Any number $f(x) \in \{2, 6, 10, ...\}$ will have the adjacent vertex labels as $\{0\} \cup \{1, 3, 5, ..., q - 1\}$ ... (*) or from the set $\{0\} \cup \{2, 4, 6, ..., q\} - \{f(x)\} \cup \{f(x) - 1\}$ ... (**)

Hence, the total number adjacent vertex labels is,

$$\frac{q-2}{2}+2 = \frac{q+2}{2} \text{ [From (*)]}$$

or $\frac{q-2}{2}+2-1+1 = \frac{q+2}{2}$ [From (**)]

Therefore, if $f(x) \equiv 2 \pmod 4$ then the total number of adjacent vertex labels of x is $\frac{q+2}{2}$.

Subcase (iv): $f(x) \equiv 3 \pmod 4$

Any number $f(x) \in \{3, 7, 11, ...\}$ will have the adjacent vertex labels as $\{0, 2, 4, ..., q\}$ ... (*) or from the set $\{1, 3, 5, ..., q - 1\} - \{f(x)\} \cup \{f(x) - 1\} \cup \{q\}$ ... (**)

Hence, the total number of possible adjacent vertex labels is,

$\frac{q}{2}+1 = \frac{q+2}{2}$ [From (*)]

or $\frac{q-2}{2}+2 = \frac{q+2}{2}$ [From (**)]

Therefore, if $f(x) \equiv 3 \pmod 4$ then the total number of adjacent vertex labels of x is $\frac{q+2}{2}$.

Corollary 2.1.3:

If $G = (p, q)$ is a mean graph with q is odd then the maximum possible adjacent vertex labels for any vertex x with label $f(x) \in \{0, 1, 2, ..., q\}$ is

1. $\frac{q+1}{2}$ if $f(x) = 0$.
2. $\frac{q+3}{2}$ if $f(x) > 0, f(x) \equiv 0 \pmod 4$
3. $\frac{q+1}{2}$ if $f(x) \equiv 1, 3 \pmod 4$
4. $\frac{q+3}{2}$ if $f(x) \equiv 2 \pmod 4$

Proof:

Proof follows from Theorem 2.1.2.

Corollary 2.1.4:

If $G = (p, q)$ is a mean graph with q is odd then $\Delta(G) \leq \frac{q+3}{2}$.

Proof:

From Corollary 2.1.2, the maximum degree $\frac{q+3}{2}$ corresponds to a vertex x with label $f(x) \equiv 0, 2 \pmod 4$.

Corollary 2.1.5:

If $G = (p, q)$ is a mean graph with q is even then the maximum possible adjacent vertex labels for any vertex x with label $f(x) \in \{0, 1, 2, ..., q\}$ is

1. $\frac{q}{2}$ if $f(x) = 0$
2. $\frac{q+2}{2}$ if $f(x) > 0, f(x) \equiv 0, 1, 2, 3 \pmod 4$

Proof:

Proof follows from Theorem 2.1.2.

Corollary 2.1.6:

If $G = (p, q)$ is a mean graph with q is even then $\Delta(G) \le \frac{q+2}{2}$.

Proof:

From Corollary 2.1.4., the maximum degree $\frac{q+2}{2}$ corresponds to a vertex x with label $f(x) \equiv 0, 1, 2, 3 \pmod 4$.

Corollary 2.1.7:

If $G = (p, q)$ is a mean graph then

$$\Delta(G) \le \begin{cases} \frac{q+3}{2} & q \text{ is odd} \\ \frac{q+2}{2} & q \text{ is even} \end{cases}$$

Proof:

Proof follows from Corollary 2.1.4 and Corollary 2.1.6.

Corollary 2.1.8:

If $G = (p, q)$ is a mean tree then

$$\Delta(G) \le \begin{cases} \frac{p+2}{2} & p \text{ is even} \\ \frac{p+1}{2} & p \text{ is odd} \end{cases}$$

Proof:

By replacing q by $p - 1$ in Corollary 2.1.7, the result follows.

References:

[1] K. Amuthavalli, Graph labeling and it's Application's some generalizations of odd mean labeling, Ph.D. Thesis, Mother Theresa Women's University, Kodaikanal, July (2010).

[2] G.S. Bloom, S.W. Golomb, Applications of numbered undirected graphs, Proc. IEEE, 65 (1977), 562 – 570.

[3] G.S. Bloom, S.W. Golomb, Numbered complete graphs unusual rulers and assorted applications, Theory and Applications of Graphs – Lecture notes in Math., Springer – Verlag, New York, 642 (1978), 53- 65.

[4] G.S. Bloom, D.F. Hsu, On graceful digraphs and a problem in network addressing, Congressus Numerantium, 35 (1982)91 – 103.

[5] J.A. Gallian, A dynamic survey of graph labeling, Electronic Journal of Combinatorics, 18 (2012) # DS6.

[6] B. Gayathri and K. Amuthavalli, *k*-odd mean labeling of crown graphs, *International Journal of Mathematics and Computer Science*, 2(3) (2007) 253-259.

[7] B. Gayathri and K. Amuthavalli, (*k*, *d*)-odd mean labeling of some graphs, *Bulletin of Pure and Applied Sciences*, 26E(2) (2007) 263-267.

[8] B. Gayathri and K. Amuthavalli, *k*-odd mean labeling of $\langle K_{1,n}, K_{1,m} \rangle$, *Acta Ciencia Indica*, 34(2) (2008) 827-834.

[9] B. Gayathri, R. Gopi, *k*-even mean labeling of $D_{m,n}$ @ C_n, International Journal of Engineering Science, Advanced Computing and Bio-Technology, Vol. 1, No. 3, July – September 2010, pp.137 – 145.

[10] B. Gayathri, R. Gopi, *k*-even mean labeling of $D_{m,n}$, Acta Ciencia Indica, Vol.XXXVII, No. 2, 291 – 300, 2011.

[11] B. Gayathri and R. Gopi, *k*-even mean labeling of $C_n \cup P_m$, Elixir International Journal of Applied Sciences, No. 36, 2011, P. No. 3102-3105.

[12] B. Gayathri and R. Gopi, *k*-even mean labeling of $T_{n,m,t}$, International Journal of Engineering Science, Advanced Computing and Bio-Technology, Vol. 2, No. 2, April-June 2011, P. No. 77-87.

[13] B. Gayathri, R. Gopi, International Conference on Mathematics and Computer Science, *k*-even mean labeling of C_m □ $\overline{K_n}$, Loyola College, Chennai, January 7-8, 2011, Proc. Page No:519-524.

[14] B. Gayathri, R. Gopi, (*k*, *d*)-even mean labeling of P_m □ nK_1, International Journal of Mathematics and soft computing, Vol. 1, No. 1, 17 – 23, August 2011.

[15] B. Gayathri, R. Gopi, *k*-even mean labeling of some graphs, Heber International Conference on Applications of Mathematics and Statistics, Bishop Heber College (Autonomous), January 5-7, 2012 Proc. Page No:420-425.

[16] B. Gayathri, R. Gopi, International Conference on Mathematics in Engineering and Business Management, *k*-even mean labeling of some trees, Stella Maris College (Autonomous), Chennai, March 9-10, 2012, Proc. Page No:99-102.

[17] R. Gopi, A Study on Different kinds of Mean Labeling, Ph.D. Thesis, Bharathidasan University, Trichy, February (2013).

[18] B. Gayathri, M. Tamilselvi and M. Duraisamy, (*k*, *d*)-super mean labeling of some graphs, *International Journal of Mathematics and Computer Science*, 2(3) (2007) 245-252.

[19] B. Gayathri, M. Tamilselvi, *k*-Super mean labeling of some trees and cycle related graphs, *Bulletin of Pure and Applied Sciences*, Volume 26E(2) (2007) 303-311.

[20] F. Harary Graph Theory, Addision – Wesley, Reading Massachusetts, 1972.

[21] R. Ponraj, Jayanthi and D. Ramya, On super mean graphs of order ≤ 5, *Bulletin of Pure and Applied Sciences*, 25E (2006) 143-148.

[22] K. Manickam and M. Marudai, Odd mean labeling of graphs, *Bulletin of Pure and Applied Sciences*, 25E(1) (2006) 149-153.

[23] Rosa, On certain valuations of the vertices of a graph Theory of Graphs (Internet Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Duhod, Paris (1967) 349 – 355.

[24] S. Somasundaram, R. Ponraj, Mean labeling of graphs, National Academy Science Letter, 26 (7 – 8) (2003), 10 – 13.

[25] S. Somasundaram and R. Ponraj, Mean labeling of graphs, *Natl. Acad. Sci. Let.*, 26 (2003), 10-13.

[26] S. Somasundaram and R. Ponraj, Non-existence of mean labeling for a wheel, *Bull. Pure and Appl. Sciences* (*Mathematics & Statistics*), 22E (2003) 103-111.

[27] M. Tamilselvi, A study in Graph Theory-Generalization of super mean labeling, Ph.D. Thesis, Vinayaka Mission University, Salem, August (2011).

[28] R. Vasuki and A. Nagarajan, Further results on mean graphs, *Scientia Magna*, 6(2) (2010) 26-39.

[29] S.K. Vaidya and Lekha Bijukumar, Mean labeling for some new families of graphs, *Journal of Pure and Applied Sciences*, 1(18) (2010) 115-116.

[30] S.K. Vaidya and Lekha Bijukumar, Some new families of mean graphs, *Journal of Mathematics Research*, 2(3) (2010) 169-176.

[31] S.K. Vaidya and Lekha Bijukumar, New mean graphs, *International J. Math. Combin.*, 3 (2011) 107-113.

[32] S.K. Vaidya and Kanani, Some new mean graphs, *International Journal of Information Science and Computer Mathematics*, 1(1) (2010) 73-80.