International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 4, No. 3, July – September 2013, pp. 53 - 58

Properties of 2-Balanced Graphs

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Abstract: *For any graph G with p vertices, $p \geq 3$, and q edges, define $d_2(G) = 2q/(p-2)$. G is 2-balanced if $d_2(H) \leq d_2(G)$ for every subgraph H of G and G is strictly 2-balanced if $d_2(H) < d_2(G)$ for every proper subgraph H of G. In this paper we study few structural properties of 2-balanced graphs.*

Keywords: *Balanced graphs, strongly balanced graphs, 2-balanced graphs and n-component.*

1. Introduction

In this paper we follow the notations of Harary [1]. All the graphs considered here are simple, finite, non-trivial and undirected. Further $p = p(G) = |V(G)|$ and $q = q(G) = |E(G)|$ respectively denote the number of vertices (or points) and number of edges (or lines). By a (p, q) graph or $G(p, q)$ we mean a graph with p vertices and q edges.

As usual, for a set $S \subseteq V(G)$, $\langle S \rangle$ or $G\langle S \rangle$ denote the subgraph of G induced by S. A cut vertex of a graph G is a vertex of G whose removal increases the number of components of G. A non-separable graph is a connected nor-trivial graph that has no cut vertices.

The connectivity $k = k(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or a trivial graph and the connectivity of disconnected is zero.

A graph G is n-connected if $k(G) \geq n$. A 1-connected graph is a connected graph. A 2-connected graph is a block.

An n-component of a graph is a maximal n-connected subgraph. In particular, the 1-components of G are the non-trivial components of G. The 2-components of G are the blocks of G with at least three vertices. Moreover two distinct n-components of a graph G will have at most $(n - 1)$ vertices in common.

2. Prior Results

For a (p, q) graph G, define the average degree $d(G)$ by $d(G) = 2q/p$. G is called balanced if $d(H) \leq d(G)$ for every subgraph H of G, and G is called strictly balanced if $d(H) < d(G)$ for every proper subgraph H of G.

Received: 21 April, 2013; Revised: 11 August, 2013; Accepted: 22 August, 2013

Similarly, for a (p, q) graph G with $p > 1$, define the strong average degree $d_1(G)$ by $d_1(G) = 2q/(p - 1)$. G is called strongly balanced or 1-balanced if $d_1(H) \leq d_1(G)$ for every subgraph H of G, and G is called strictly strongly balanced if $d_1(H) < d_1(G)$ for every proper subgraph H of G.

The relevance of average degree and strong average degree were first identified by Erdos and Renyi [2] and Rucinski and Vince [3], respectively. Later, their structural properties were studied by Veerapandiyan et al in [4] and [5]. We shall use some of the properties here.

Theorem 2.1: [5] Every 1-balanced graph is connected.

3. 2-Balanced Graphs

For any graph G with p vertices, $p \geq 3$, and q edges, define $d_2(G) = 2q/(p - 2)$. G is defined to be 2-balanced if $d_2(H) \leq d_2(G)$ for all subgraphs H of G and G is strictly 2-balanced if $d_2(H) < d_2(G)$ for every proper subgraph H of G.

Examples: 2-regular, 2-connected, 2-balanced graphs

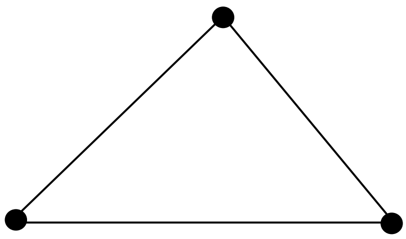

Figure 3.1: C_3

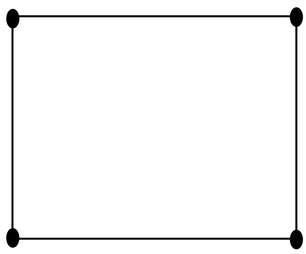

Figure 3.2: C_4

Observations:

1. K_n is 2-balanced for all $n \geq 3$.
2. K_n is strictly 2-balanced only if $n \geq 5$.
3. Trees are not 2-balanced.
4. Unicyclic graphs with pendant vertices are not 2-balanced.
5. If G is 2-balanced, then $d_2(G) \geq 4$
6. If G is a 2-balanced graph and contains a copy of K_3 then $d_2(G) \geq 6$.
7. C_n, cycles on n vertices are not 2-balanced for $n > 4$.

Theorem 3.1: Every 2-balanced graph is strictly 1-balanced.

Proof: Let G be any 2-balanced graph.

Let $H(n, m)$ be a proper connected induced subgraph of G.

Since G is 2-balanced, $d_2(H) \leq d_2(G)$.

$$\Rightarrow \frac{2m}{n-2} \leq \frac{2q}{p-2} \Rightarrow m(p-2) \leq q(n-2)$$

$\Rightarrow mp - 2m \leq nq - 2q \quad \Rightarrow mp - nq \leq 2m - 2q$

Now $\frac{d_1(H)}{2} - \frac{d_1(G)}{2} = \frac{m}{n-1} - \frac{q}{p-1} = \frac{m(p-1)-q(n-1)}{(n-1)(p-1)}$

$= \frac{mp-m-nq+q}{(n-1)(p-1)} \leq \frac{2m-2q-m+q}{(n-1)(p-1)} = \frac{m-q}{(n-1)(p-1)} < 0$, since H is proper.

∴ $d_1(H) < d_1(G)$. Thus G is strictly 1-balanced.

Corollary 3.2: Every 2-balanced graph is connected.

Proof: Since every 2-balanced graph is 1-balanced, by Theorem 2.1 every 2-balanced graph is connected.

Theorem 3.3: Every 2-balanced graph is 2-connected.

Proof: Let $G(p, q)$ be a 2-balanced graph.

By Corollary 3.2, G is 1-connected.

It is enough if we prove that G has no cut vertex.

If possible, let u be a cut vertex of G.

Then G-u is disconnected and so G-u has at least two components.

Let the vertex set of one of the component is V_1 and V_2 be the union of the vertex sets of the remaining components.

Let $H_1 = H_1(n_1, m_1)$ and $H_2 = H_2(n_2, m_2)$ are respectively the subgraphs induced by $V_1 \cup \{u\}$ and $V_2 \cup \{u\}$.

Then $p = n_1 + n_2 - 1$, $q = m_1 + m_2$ and $d_2(G) = \frac{m_1 + m_2}{n_1 + n_2 - 3}$ ...(1)

Suppose both $d_2(H_1) \leq d_2(G)$ and $d_2(H_2) \leq d_2(G)$ are true. ...(2)

Then $\frac{m_1}{n_1-2} \leq \frac{q}{p-2}$ and $\frac{m_2}{n_2-2} \leq \frac{q}{p-2}$

which implies $\frac{m_1+m_2}{n_1+n_2-4} \leq \frac{q}{p-2}$ ...(3)

(1) and (3) implies $\frac{q}{p-2} = \frac{m_1+m_2}{n_1+n_2-3} < \frac{m_1+m_2}{n_1+n_2-4} = \frac{q}{p-2}$ which is a contradiction.

Thus (2) cannot be true. i.e. either $d_2(H_1) > d_2(G)$ or $d_2(H_2) > d_2(G)$ implying G is not 2-balanced. Hence G cannot have a cut vertex and so G is 2-connected.

Theorem 3.4: If $p > 4$, $G(p, q)$ is connected $(p - 2)$-regular, then G is 2-balanced.

Proof: $d_2(G) = \frac{2q}{p-2} = \frac{p(p-2)}{p-2} = p$

Let $H(n, m)$ be a connected proper subgraph of G. Then $n < p$.

If $n = p - 1$, $d_2(H) = \frac{p(p-2) - 2(p-2)}{p-3} = \frac{p^2-3p-p+4}{p-3}$

$= \frac{p(p-3)}{p-3} - \frac{p-4}{p-3} = p - \frac{p-4}{p-3} < p$

∴ $d_2(H) < d_2(G) = p$.

If $n = p - 2, d_2(H) = \frac{p(p-2)-4(p-2)+2}{p-4} = p - 2\,\frac{p-5}{p-4} < p.$

$\therefore d_2(H) < d_2(G)$.

If $n \le p-3,\ d_2(K_{p-3}) < p-1$

$\Rightarrow d_2(H) < p - 1 < p = d_2(G).$

$\therefore$ *G* is strictly 2-balanced.

Theorem 3.5: Let *G*(p, q) be a 2-connected 2-balanced graph and S = {u, v} be 2-vertex cut of *G*. Then $\langle u, v\rangle = \overline{K_2}$.

Proof: Suppose $\langle u,v\rangle = K_2$. Let $G_1(p_1, q_1)$ be one of the 3-component of $G - S$ and $G_1(p_2, q_2)$ be the union of the remaining 3-components of $G - S$.

Then $p = p_1 + p_2 - 2$ and $q = q_1 + q_2 - 2$

G 2-balanced $\Rightarrow d_2(G_1) \le d_2(G)$ and $d_2(G_2) \le d_2(G)$

$\Rightarrow \frac{q_1}{p_1-2} \le \frac{q}{p-2}$ and $\frac{q_2}{p_2-2} \le \frac{q}{p-2} \Rightarrow \frac{q_1+q_2}{p_1+p_2-2-2} \le \frac{q}{p-2}$

$\Rightarrow \frac{q+1}{p-2} \le \frac{q}{p-2} \Rightarrow q + 1 \le q$, which is a contradiction.

Therefore $\langle u,v\rangle = \overline{K_2}$

Theorem 3.6: Let *G*(*p*, *q*) a 2-balanced graph. Then for any 3-component G_1 of *G*, $d_2(G_1) = d_2(G)$.

Proof: If *G* does not contain 2-vertex cut, then there is nothing to prove.

Therefore assume that there is a 2-vertex cut {u, v}.

Let $G = G_1(p_1, q_1) \cup G_2(p_2, q_2)$ where $G_1(p_1, q_1)$ is any 3-component and $G_2(p_2, q_2)$ be the union of remaining 3-components.

By Theorem 3.5, $\langle u, v\rangle = \overline{K_2}$. Therefore, $p = p_1 + p_2 - 2$ and $q = q_1 + q_2$.

Since *G* is 2-balanced, $d_2(G_1) \le d_2(G)$. Suppose $d_2(G_1) < d_2(G)$.

Then $\frac{q_1}{p_1-2} < \frac{q}{p-2}$. Further *G* is 2-balanced $\Rightarrow \frac{q_2}{p_2-2} \le \frac{q}{p-2}$

$\Rightarrow \frac{q_1+q_2}{p_1+p_2-2-2} < \frac{q}{p-2} \Rightarrow \frac{q}{p-2} < \frac{q}{p-2}$,which is absurd.

$\therefore d_2(G_1) \nless d_2(G) \Rightarrow d_2(G_1) = d_2(G)$.

The results that are discussed in Theorems 3.5 and 3.6 are illustrated in the following example.

Consider the following graph (Figure 3.3) with a unique 2-vertex cut, say S. There are three 3-components having same $d_2(\ )$ value 7. Further, $\langle S\rangle$ is empty.

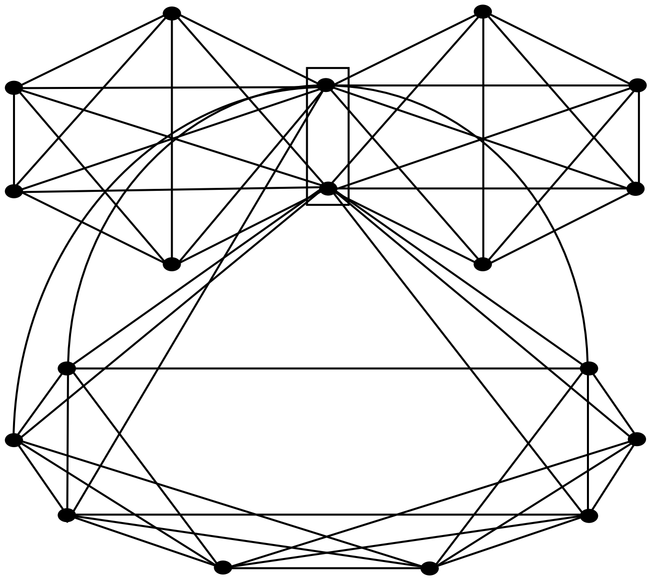

Figure 3.3: A 2-balanced graph with three 3-components.

Corollary 3.7: A graph *G* is 2-balanced if and only if it is 2-connected and for every 3-component *H* of *G*, *H* is 2-balanced and $d_2(H) = d_2(G)$.

Planarity versus 2-balanced graphs

Theorem 3.8: Let *G*(p, q), $p \geq 3$ be a maximal planar graph. Then *G* is 2-balanced. If $p > 3$, then *G* is not strictly 2-balanced.

Proof: Let *G*(p, q), $p \geq 3$ be a maximal planar graph. Then each face of *G* is a triangle and $q = 3p - 6 = 3(p - 2)$ which implies $d_2(G) = 6$.

Let $H(p_1, q_1)$ be a connected induced subgraph of *G* with $p_1 < p$.

Since *H* is also planar, $q_1 \leq 3p_1 - 6$ and so $d_2(H) \leq 6$. Thus $d_2(H) \leq d_2(G)$ implying that *G* is 2-balanced.

In particular, when $p > 3$, since K_3 is a subgraph of *G* and $d_2(K_3) = 6$, $d_2(K_3) = d_2(G)$ and so *G* is not strictly balanced.

Theorem 3.9: Let *G*(p, q), $p \geq 4$, be a planar graph in which each face is an n-cycle. Then for n = 3 and 4 *G* is 2-balanced but not strictly whereas for $n \geq 5$ *G* is not at all 2-balanced.

Proof: For n = 3, *G* become a maximal planar graph and the result is nothing but Theorem 3.8.

For n = 4, the proof is similar to the proof of Theorem 3.8.

For n ≥ 5, the proof is as follows:

Since each face is an n-cycle, $q = \frac{n(p-2)}{n-2}$

$\Longrightarrow d_2(G) = 2 + \frac{4}{n-2} < 3.5$ (because n ≥ 5)

Now the path on three vertices P_3 is a subgraph of G and $d_2(P_3) = 4$, which is greater than $d_2(G)$. Therefore G is not 2-balanced.

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