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Eccentricity properties of the Boolean graphs $BG_2(G)$ and $BG_3(G)$

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Abstract: *Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). $B_{G,\, INC,\, \bar{L}(G)}(G)$ is a graph with vertex set V(G) $\cup$ E(G) and two vertices are adjacent if and only if they correspond to two adjacent vertices of G, a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by $BG_2(G)$, Boolean graph of G-second kind. $B_{\bar{K}_p,\, INC,\, \bar{L}(G)}(G)$ is a graph with vertex set V(G) $\cup$ E(G) and two vertices are adjacent if and only if they correspond to a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by $BG_3(G)$, Boolean graph of G-third kind. In this paper, eccentricity properties of $BG_2(G)$ and $BG_3(G)$ are studied.*

Key words: *Boolean graphs $BG_2(G)$, $BG_3(G)$.*

1. Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [5], Buckley and Harary [4].

The *girth* of a graph G, denoted g(G), is the length of a shortest cycle (if any) in G; the *circumference* c(G) is the length of any longest cycle. The distance d(u, v) between two vertices u and v in G is the minimum length of a path joining them if any; otherwise $d(u, v) = \infty$. A shortest u-v path is called a *u-v geodesic*. A graph G is *geodetic*, if for every pair of vertices (u, v) there exists a unique shortest path connecting them in G.

Let G be a connected graph and u be a vertex of G. The *eccentricity* e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$. The *radius* r(G) is the minimum eccentricity of the vertices, whereas the *diameter* diam(G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. v is a central vertex if e(v) = r(G). The *center* *C*(G) is the set of all central vertices. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a *peripheral vertex* if e(v) = diam(G). The *periphery* *P*(G) is the set of all such vertices. For a vertex v, each vertex at distance e(v) from v is an eccentric node of v.

A graph is *self-centered* if every vertex is in the center. Thus, in a self-centered graph G all nodes have the same eccentricity, so r(G) = diam(G).

An edge uv $\in$ E(G) is a *dominating edge* of G, if all the vertices of G other than u and v are adjacent to either u or v.

Jin Akiyama and Kiyoshi Ando [3] characterized the graphs G, which are self-centered with diameter two such that $\overline{G}$ is also self-centered with diameter 2.

Lemma 1.1 [3] Let both G and $\overline{G}$ be connected and v be a point of G. If $e_G(v) \geq 3$, then $e_{\overline{G}}(v) = e_G(v) = 2$.

Corollary If G is self-centered with diameter d $\geq$ 3, then $\overline{G}$ is self-centered with diameter 2.

An edge uv $\in$ E(G) is a *dominating edge* of G, if all the vertices of G other than u and v are adjacent to either u or v.

Theorem 1.1 [3] The following three statements are equivalent.

(1) Both G and $\overline{G}$ are self-centered with diameter two.

(2) G is self-centered with diameter two having no dominating edge.

(3) Neither G nor $\overline{G}$ contains a dominating edge.

<u>**Motivation:**</u> The Line graphs, Middle graphs, Total graphs and Quasi-total graphs are very much useful in computer networks. In analogous to line graph [6,15], total graph [4], [5], middle graph [1,2] and quasi-total graph [14], thirty-two graphs can be defined using different adjacency relations. Out of these operations, eight were already studied. Among the remaining twenty-four graph operations, two are defined and analyzed in [11] and [12]. All the others have been defined and studied thoroughly and will be submitted elsewhere. This is illustrated below.

G/ $\overline{G}$/K_p/ $\overline{K_p}$	Incident (INC)/ Non-incident (NINC)	L(G)/ $\overline{L(G)}$/K_q/ $\overline{K_q}$

Defining a new graph from a given graph by using the adjacency relation between two vertices or two edges and incident relationship between vertices and edges is known as Boolean operation. It defines new structure from the given graph and adds extra information of the original graph.

In Management and in social networks, the incident and non-incident relations of vertices and edges are used to define various networks. So these are very much applicable in socio-economical problems. In some cases, it is possible to retrieve the original graph from the Boolean graphs in polynomial time. So these graph operations may be used in graph coding or coding of some grouped signal. Also, it is possible to study the structure of original graphs through these Boolean graph operations. This motivates the study for the exploration of various Boolean operations and study of their structural properties.

In [10] and [11], the Boolean graphs are defined as follows.

Let G be a (p, q) simple, undirected graph with vertex set V(G) and edge set E(G). The Boolean graph $B_{G,\ INC,\ \overline{L(G)}}(G)$ has vertex set $V(G) \cup E(G)$ and two vertices in $B_{G,\ INC,\ \overline{L(G)}}(G)$ are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge incident to it in G or two non- adjacent edges of G. For simplicity, denote this graph by $BG_2(G)$, Boolean graph of G-second kind. The vertices of $BG_2(G)$, which are in V(G) are called point vertices and those in E(G) are called line vertices of $BG_2(G)$.

$V(BG_2(G)) = V(G) \cup E(G)$ and $E(BG_2(G)) = [E(T(G))-E(L(G))] \cup E(\overline{L(G)})$. With an immediate consequence of the definition of $BG_2(G)$, if G is a (p, q) graph, whose vertices are $v_1, v_2, \ldots, v_p$ having degrees d_i, and edges e_{ij}, note that $BG_2(G)$ has p+q vertices and $(q^2+7q-\Sigma d_i^2)/2$ edges with deg $v_i = 2d_i$; deg $e_{ij} = q+3-(d_i+d_j)$. Also, G and $\overline{L(G)}$ are induced subgraphs of $BG_2(G)$.

$B_{\overline{K_p},\ INC,\ \overline{L(G)}}(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by $BG_3(G)$, Boolean graph of G-third kind. The vertices of $BG_3(G)$, which are in V(G) are called point vertices and vertices in E(G) are called line vertices of $BG_3(G)$. $V(BG_3(G)) = V(G) \cup E(G)$ and $E(BG_3(G)) = (E(T(G))-(E(G) \cup E(L(G))) \cup E(\overline{L(G)})$. $BG_3(G)$ has p+q vertices, p-point vertices and q-line vertices. $BG_3(G)$ is a spanning subgraph of $BG_2(G)$. $\overline{K_p}$ and $\overline{L(G)}$ are induced subgraphs of $BG_3(G)$. Let G be a (p, q) graph with vertices $v_1, v_2, \ldots, v_p$. Let d_i, i = 1, 2, ..., p denote the degree of the vertices $v_1, v_2, \ldots, v_p$ in G. Then it follows from the definition that $BG_3(G)$ has $(q(q+5)/2)-(1/2)\Sigma d_i^2$ edges with degree of v_i in $BG_3(G) = \deg_G v_i = d_i$ and degree of a line vertex $e_{ij} = v_iv_j \in E(G)$ in $BG_3(G)$ is $q-d_i-d_j+3$.

2. Eccentricity properties of $BG_2(G)$ and $\overline{BG_2}(G)$

In this section, eccentricity of vertices of $BG_2(G)$ are studied. Radius and diameter of $BG_2(G)$, $\overline{BG_2}(G)$ are also found out. $BG_2(G)$ is disconnected, whenever G has an isolated vertex. Hence, to study the eccentricity of vertices of $BG_2(G)$, assume that G is a graph without isolated vertices.

Proposition 2.1 Eccentricity of every line vertex is two in $BG_2(G)$ if $G \neq K_2$.

Proof: Let e be a line vertex in $BG_2(G)$. In $BG_2(G)$, d(u, e) = 1 if u is a point vertex incident with e and d(v, e) = 2 if v is a point vertex not incident with e. Since, if $e = uu_1 \in E(G)$ and v is adjacent to u or u_1 in G, then e u v or e u_1 v is a shortest path in $BG_2(G)$, where $e = uu_1$ and v is not adjacent to u and u_1. Also, there exists $e_1 \in E(G)$, incident with v such that e and e_1 are not adjacent. In this case, e e_1 v is a shortest path in $BG_2(G)$. Also, $d(e, e_1) = 1$ in $BG_2(G)$, if e_1 is not adjacent to e in G and $d(e, e_1) = 2$ if e_1 is adjacent to e in G. Therefore, distance between a line vertex and other vertices is less than or equal to 2. Also, there exists at least one vertex at distance two since $G \neq K_2$. Hence, eccentricity of a line vertex is two in $BG_2(G)$ if $G \neq K_2$.

Remark 2.1 If $G = K_2$, eccentricity of line vertex in $BG_2(G)$ is one.

Proposition 2.2 Eccentricity of any point vertex in $BG_2(G)$ is 1, 2 or 3.

Proof: Let u be a point vertex of $BG_2(G)$. In $BG_2(G)$, $d(u, e) \leq 2$ by the previous proposition, where e is any line vertex of $BG_2(G)$. Also d(u, v) = 1, if $d_G(u, v)=1$; d(u, v) = 2, if $d_G(u, v) = 2$ and d(u, v) = 3, if $d_G(u, v) \geq 3$, where v is any other point vertex of $BG_2(G)$. Since if $d_G(u, v) \geq 3$, there exist $e, e_1 \in E(G)$ such that e is incident with u, e_1 is incident with u_1 and so u e e_1 v is a shortest path in $BG_2(G)$. Hence the proposition is proved.

Remark 2.2 If $BG_2(G)$ is connected, then diameter of $BG_2(G)$ is at most 3.

Theorem 2.1 (1) Radius of $BG_2(G)$ is one if and only if $G = K_{1,n}$, $n \geq 1$.

(2) $BG_2(G)$ is self-centered with radius two if and only if $G \neq K_{1,n}$ and $diam(G) \leq 2$.

(3) $BG_2(G)$ is bi-eccentric with diameter three if and only if $diam(G) \geq 3$.

Proof of (1): Assume $r(BG_2(G)) = 1$. This implies that there exists a vertex $x \in V(BG_2(G))$ such that e(x) = 1 in $BG_2(G)$. Suppose $G \neq K_2$, x is not a line vertex. Therefore, when $G \neq K_2$, there exists a point vertex, which is adjacent to every other vertices in $BG_2(G)$. That is, every edge in G is incident with x and every other point vertices are adjacent to x in G, that is $G = K_{1,n}$. Converse is obvious.

Proof of (2): Since $G \neq K_{1,n}$, eccentricity of every line vertex in $BG_2(G)$ is two. $diam(G) \leq 2$ implies, eccentricity of every point vertex in $BG_2(G)$ is also two. Hence, $BG_2(G)$ is self-centered with diameter two. On the other hand, $BG_2(G)$ is self-centered with radius two implies that eccentricity of every vertex in $BG_2(G)$ is two. Therefore, $G \neq K_{1,n}$. Eccentricity of every point vertex is two implies that $d_G(u, v) \leq 2$ for all $u, v \in V(G)$. This implies that $diam(G) \leq 2$, where $G \neq K_{1,n}$.

Proof of (3): diam(G) $\geq$ 3 if and only if e(u) $\geq$ 3 for some u $\in$ V(G) in G. This is true if and only if d(u, v) = 3 in $BG_2(G)$, where v is an eccentric point of u in G. That is e(u) = 3 in $BG_2(G)$.Also, eccentricity of every line vertex is two in $BG_2(G)$. Thus, diam(G) $\geq$ 3 if and only if $BG_2(G)$ is bi-eccentric with diameter three.

Remark 2.3 Eccentricity of each point vertex is three and eccentricity of each line vertex is two if and only if radius of G is at least 3.

Next, eccentricity properties of $\overline{BG_2(G)}$ are studied for a graph G.

Proposition 2.3 Eccentricity of a point vertex u in $\overline{BG_2(G)}$ is at most 3.

Proof: Consider a point vertex u in $\overline{BG_2(G)}$.

Case1: If u is an isolated vertex of G. In this case, in $\overline{BG_2(G)}$, u is adjacent to all other vertices. Hence, e(u) in $\overline{BG_2(G)}$ =1.

Case 2: If u is not an isolated vertex of G.

<u>To find distance between u and other point vertices:</u>

Let v $\in$ V(G) such that v is not adjacent to u in G. Then d(u, v) = 1 in $\overline{BG_2(G)}$. Let w $\in$ V(G) such that w is adjacent to u in G. If there exists an edge e $\in$ E(G), such that e is not incident with both u and w, then d(u, w) = 2 in $\overline{BG_2(G)}$. If there exists a point vertex x, which is not adjacent to both u and w in G, then also d(u, w) = 2 in $\overline{BG_2(G)}$. If all the edges are like that they are incident to u or to w in G and G has no vertex, which is not adjacent to both u and w, then d(u, w) = 3 in $\overline{BG_2(G)}$. Therefore, distance between u and any other point vertices is 1, 2 or 3.

<u>Distance between u and line vertices</u>:

Let e $\in$ E(G) be a line vertex, which is incident with u in G. In $\overline{BG_2(G)}$, e and u are not adjacent. (a) If there exists an edge e_1, adjacent to e but not incident to u, then u e_1 e is a shortest path in $\overline{BG_2(G)}$. Therefore, d(u, e) = 2 in $\overline{BG_2(G)}$.

(b) If there does not exist an edge adjacent to e and not incident with u, then let e = uv, e_1 = uv_1. Suppose there exists another vertex v_2 such that it is not adjacent to u, then e v_2 u is a path in $\overline{BG_2(G)}$. Therefore, d(u, e) = 2 in $\overline{BG_2(G)}$. If there does not exist such a vertex, then G = K_3 or $K_{1,n}$. If G = $K_{1,n}$, u is isolated in $\overline{BG_2(G)}$. If G = K_3, d(u, e) = 2 and d(u, v) = 3 in $\overline{BG_2(G)}$. Now, let e be an edge in G, which is not incident with u in G. Then d(u, e) = 1 in $\overline{BG_2(G)}$. Therefore, distance between u and any other line vertices is 1, 2 or 3. Hence, eccentricity of any point vertex is at most 3 in $\overline{BG_2(G)}$.

Proposition 2.4 Eccentricity of a line vertex in $\overline{BG_2(G)}$ is 2 or 3.

Proof: Let $e \in V(\overline{BG_2}(G))$ be a line vertex of $\overline{BG_2}(G)$. Then $e \in E(G)$. Therefore, $e = uv$, where $u, v \in V(G)$. Distance between e and u or e and v in $\overline{BG_2}(G)$ is 2 or 3 and distance between e and any other point vertex is one in $\overline{BG_2}(G)$.

Now, let $e_1 \in V(\overline{BG_2}(G))$ be any other line vertex.

(a) If e and e_1 are adjacent in G, $d(e, e_1) = 1$.

(b) If e and e_1 are not adjacent in G: If there exists a point vertex, not incident with both e and e_1, then $d(e, e_1) = 2$ in $\overline{BG_2}(G)$. If G contains only two non- adjacent edges e and e_1 and no other point vertices not incident with both e and e_1 then $d(e, e_1) = 3$ in $\overline{BG_2}(G)$ ($e = uv$, $e_1 = wx$; e w u e_1 is a path in $\overline{BG_2}(G)$). If G contains another edge e_2, adjacent to e and e_1, then $d(e, e_1) = 2$ in $\overline{BG_2}(G)$. Thus, distance between two line vertices is 1, 2 or 3. Hence, eccentricity of a line vertex is 2 or 3.

Theorem 2.2 Radius of $\overline{BG_2}(G) = 1$ if and only if G has an isolated vertex.

Proof: Suppose G has an isolated vertex $u \in V(G)$. By the definition of $\overline{BG_2}(G)$, u is adjacent to every other vertices of $\overline{BG_2}(G)$. Hence, $r(\overline{BG_2}(G)) = 1$.

On the other hand, let $r(\overline{BG_2}(G)) = 1$. Therefore, there exists a vertex in $\overline{BG_2}(G)$ with eccentricity one. This vertex can not be a line vertex, since eccentricity of a line vertex is two or three in $\overline{BG_2}(G)$. Therefore, $e(u) = 1$ in $\overline{BG_2}(G)$, $u \in V(G)$. This implies e is adjacent to every other point vertices in $\overline{BG_2}(G)$. This implies that e is an isolated vertex in G. Hence the theorem is proved.

Theorem 2.3 $\overline{BG_2}(G)$ is bi-eccentric with diameter three if and only if G satisfies any one of the following: (1) $G = 2K_2$. (2) G ($\neq K_2$) has a dominating edge $e = uv$ such that every other edge is adjacent to this edge.

Proof: Assume that $\overline{BG_2}(G)$ is bi-eccentric with diameter three. Then there exists a vertex u in $\overline{BG_2}(G)$ with eccentricity three.

Case 1: If x is a point vertex u in $\overline{BG_2}(G)$.

From the proof of Proposition 2.3, one see that eccentricity of u in $\overline{BG_2}(G)$ is three only when there exists an edge $e = uw$, $w \in V(G)$ and all other edges of G are such that they are incident with either u or w or $G = K_3$. This implies that G has a dominating edge, such that all other edges are adjacent to this edge.

Case 2: If x is a line vertex e in $\overline{BG_2}(G)$.

From Proposition 2.4, one see that eccentricity of a line vertex is three only when G contains only two non-adjacent edges, that is $G = 2K_2$. This proves the theorem.

Theorem 2.4 $\overline{BG_2(G)}$ is self-centered with diameter two if and only if (1) G has no isolated vertices. (2) $G \neq nK_2$, n = 1, 2. (3) G has no dominating edge, adjacent to all other edges.

Proof: Proof follows from Theorems 2.3 and 2.4.

3. Eccentricity Properties of $BG_3(G)$ and $\overline{BG_3(G)}$

Let G be a graph without isolated vertices. In this section, eccentricity properties of $BG_3(G)$ and $\overline{BG_3(G)}$ are studied. Characterization of graphs for which $BG_3(G)$ is self-centered of diameter two and three are studied.

Proposition 3.1 Radius of $BG_3(G) = 1$ if and only if $G = K_2$.

Proof: Assume that $r(BG_3(G)) = 1$. Eccentricity of a point vertex in $BG_3(G)$ cannot be one, since any two point vertices are not adjacent in $BG_3(G)$. Therefore, $r(BG_3(G)) = 1$ implies that there exists a line vertex with eccentricity one. Let $e \in E(G)$ such that $e(e) = 1$ in $BG_3(G)$. $e(e) = 1$ implies that e is adjacent to every point vertices and line vertices. This implies that e is incident to all point vertices. Hence, $G = K_2$.

On the other hand, $G = K_2$ implies, $r(BG_3(G)) = 1$.

Theorem 3.1 Eccentricity of every point vertices of $BG_3(G)$ is two if and only if $G = K_2$.

Proof: Let $v \in V(G)$ be a point vertex of G and let $e(v) = 2$ in $BG_3(G)$. Hence, distance of any other vertex from v is one or two. Thus, all point vertices are adjacent to v in G and all line vertices are incident to v in G. Suppose there are more than one point vertex adjacent to v in G, then the distance between that two vertices (adjacent to v in G) in $BG_3(G)$ is greater than two. Thus, the eccentricity of that point vertex cannot be two. Hence, G must be K_2.

On the other hand, if $G = K_2$, eccentricity of line vertex is one and eccentricity of the point vertices is two. Hence the theorem is proved.

Theorem 3.2 $BG_3(G)$ is self-centered with diameter three if and only if $G = K_3$.

Proof: If $G = K_3$, $BG_3(G) = C_6$, which is self-centered with diameter three. Now assume, $BG_3(G)$ is self-centered with diameter three. Hence, eccentricity of each vertex is three. Consider a point vertex v. Distance from v to $v^1 \in V(G)$ is 2 in $BG_3(G)$ if v^1 is adjacent to v in G. $d(v, v^1) = 3$ or 4, if v and v^1 are not adjacent in G. Also, $d(v, e) = 1$, if e is incident with v in G; $d(v, e) = 2$, if e is not adjacent to an edge incident with v in G; and $d(v, e) = 3$, if e is adjacent to an edge incident with v in G. Also, $d(e, e^1) = 1$ or 2. Hence, eccentricity of each vertex is three implies, for every edge e there exists a point v such that e and an edge e^1 incident with v are adjacent and there exist no e^{11} incident with v, which is not adjacent to e. (Otherwise, d(e, v) = 2, (e e^1 v is a path). Hence, G must be K_3. This proves the theorem.

Theorem 3.3 In $BG_3(G)$, eccentricity of point vertex is three and eccentricity of line vertex is two if and only if G satisfies the following conditions.

(1) $r(G) > 1$.

(2) For u, v $\in$ V(G) either uv $\in$ E(G) or there exists non-adjacent edges e_u and e_v such that e_u is incident with u and e_v is incident with v.

(3) Each e $\in$ E(G) is not adjacent with at least one edge in G, incident with other vertices.

Proof: Assume in $BG_3(G)$, eccentricity of every point vertex is 3 and eccentricity of every line vertex is two. Let u be a point vertex of G.

e(u) = 3 for all u $\in$ V(G) and e(e) = 2 in $BG_3(G)$, where e $\in$ E(G). Hence, eccentric point of u must be another point vertex. Therefore, there exists v $\in$ V(G) such that d(u, v) = 3 in $BG_3(G)$. For u $\in$ V(G), in $BG_3(G)$, d(u, e) = 1, if u and e are incident in G; d(u, e) = 2, if e is not adjacent with an edge incident with u in G; d(u, v) = 2, if u and v are not adjacent in G; d(u, v) = 3, if u and v are not adjacent and there exists edges e and e_1 such that e is incident with u, e_1 is incident with v and e and e_1 are not adjacent in G; d(e, e_1) = 1, if e and e_1 are non-adjacent in G; d(e, e_1) = 2, if there exits u $\in$ V(G), such that e and e_1 are incident with u (or there exists an edge not adjacent to both e and e_1).

Therefore, G must satisfy: (1) $r(G) > 1$. (2) for u, v $\in$ V(G) either uv $\in$ E(G) or there exists non-adjacent edges e_u and e_v such that e_u is incident with u and e_v is incident with v.

(3) Each e $\in$ E(G) is not adjacent with at least one edge incident with other vertices (otherwise, d(u, e) = 3. Hence, eccentricity of e is also 3). This proves the theorem.

Theorem 3.4 Radius of $BG_3(G)$ is 2 and diameter of $BG_3(G)$ is 4 if and only if r(G) =1 and G has at least two pendant vertices.

Proof: Distance from a line vertex to other vertices is always less than or equal to 3. Therefore, diam($BG_3(G)$) = 4 implies that there exist u, v $\in$ E(G) such that d(u, v) = 4 in $BG_3(G)$. Hence, u and v are not adjacent in G and there exists no pair e_1, e_2 such that e_1 is incident with u, e_2 is incident with v such that e_1 and e_2 are not adjacent in G. Therefore, all the edges incident with u are adjacent to edges incident with v. This is possible only when $d_G(u, v) = 2$. This implies that diam(G) = 2.

Claim: r(G) = 1.

Since diam(G) = 2, r(G) = 1 or 2. Suppose r(G) = 2. Then, there exists vertices u and v at distance two such that there exists edges e_u and e_v incident with u and v respectively such that e_u and e_v are not adjacent in G, which is a contradiction. Therefore, r(G) must be one.

Claim: G has at least two pendant vertices.

Suppose not, then in $BG_3(G)$, eccentricity of every point vertex is at most 3. Hence, diam($BG_3(G)$) is not four. Thus the claim is proved. This proves the theorem.

Theorem 3.5 Radius of $BG_3(G)$ is 2, diameter of $BG_3(G)$) is 4 and eccentricity of every line vertex is three if and only if $G = K_{1,n}$.

Proof: As in the previous theorem, it can be proved that r(G) = 1. Now, suppose $G \neq K_{1,n}$. Let u be a vertex of G with eccentricity one. Since diam($BG_3(G)$) = 4, there exists at least two pendant vertices in G. Let $e \in E(G)$ be such that e is not incident with u. Then eccentricity of e in $BG_3(G)$ is two, which is a contradiction to the hypothesis. Hence, G must be $K_{1,n}$. Converse is obvious.

Remark 3.1 Eccentricity of all point vertices is two and eccentricity of all line vertices is 3 in $BG_3(G)$ is not possible.

Theorem 3.6 $BG_3(G)$ is self-centered with diameter two if and only if $G = K_n$, $n > 3$.

Proof: Eccentricity of every point vertex is two in $BG_3(G)$, implies point vertices are adjacent in G. That is, $G = K_n$. Eccentricity of each line vertex is also two in $BG_3(G)$. Hence, d(e, u) = 2 in $BG_3(G)$, if e is not incident with u. Hence, for every u, there exists an edge incident with u but not adjacent to e in G. Hence, $n \geq 4$. This proves the theorem.

Theorem 3.7 $BG_3(G)$ is bi-eccentric with diameter three if and only if G satisfies any one of the following: (1) r(G) = 1, diam(G) = 2, G has at most one pendant vertex. (2) diam(G) $\geq$ 2.

Proof: Follows from the previous Theorems 3.4., 3.5 and 3.6.

Theorem 3.8 Radius of $\overline{BG_3}(G)$ is one if and only if G has an isolated vertex.

Proof: Assume r($\overline{BG_3}(G)$) = 1. Therefore, there exists a vertex with eccentricity one in $\overline{BG_3}(G)$. This cannot be a line vertex, since by definition, it cannot be adjacent to its incident vertices. Hence, the vertex must be a point vertex and is adjacent to every line vertex also in $\overline{BG_3}(G)$. This implies that it is not incident with any edge in G. That is, it is an isolated vertex.

On the other hand, if v is an isolated vertex in G, then in $\overline{BG_3}(G)$, e(v) = 1. Hence the theorem is proved.

Lemma 3.1 If G has no isolated vertex and $G \neq K_2$, then eccentricity of every point vertex is two in $\overline{BG_3}(G)$.

Proof: Let v be a point vertex of $\overline{BG_3}(G)$. (i) By definition, $d(v, v^1) = 1$ for $v^1 \in V(G)$ in $\overline{BG_3}(G)$. (ii) If e is not incident with v in G, then d(e, v) = 1 in $\overline{BG_3}(G)$. (iii) If e is incident with v in G and if $p \geq 3$, then e is adjacent to some other point vertex in $\overline{BG_3}(G)$ and hence d(e, v) = 2. Hence, eccentricity of a point vertex is two in $\overline{BG_3}(G)$, if $G \neq K_2$.

Remark 3.2 If $G = K_2$, then $\overline{BG_3}(G)$ is disconnected.

Lemma 3.2 Eccentricity of every line vertex in $\overline{BG_3}(G)$ is two if $G \neq K_2, 2K_2$.

Proof: If $G = K_2$, $\overline{BG_3}(G)$ is disconnected. Consider a line vertex e. In $\overline{BG_3}(G)$, d(e, v) = 1 or 2 for $v \in V(G)$, $e \in E(G)$. Also, in $\overline{BG_3}(G)$, $d(e, e^1) = 1$ if e and e^1 are adjacent in G. $d(e, e^1) = 2$ if e and e^1 are not adjacent in G and there is another edge in G adjacent to both e and e^1 or there exists another vertex not incident with both e and e^1 and d(e, v) = 2, if e is not incident with v in G. Hence, if $p > 5$, eccentricity of line vertex is always two. When $p \leq 4$, distance from a line vertex to other point vertices are at most 2 and distance from a line vertex e to other vertex e^1 is three only when e and e_1 are the non-adjacent edges and have no common non-incident vertex. This is true only when $G = 2K_2$. This proves the lemma.

Theorem 3.9 If G is a non-trivial graph having no isolated vertices and $G \neq K_2, 2K_2$, then $\overline{BG_3}(G)$ is self-centered with diameter two.

Proof: Follows from the previous Lemmas 3.1 and 3.2.

Corollary 3.9 If G is a non-trivial connected graph having more than two vertices, then $\overline{BG_3}(G)$ is self-centered with diameter two.

Theorem 3.10 $\overline{BG_3}(G)$ is bi-eccentric with diameter three if and only if $G = 2K_2$.

Proof: Assume that diam($\overline{BG_3}(G)$) = 3. Therefore, there exists a vertex in V($\overline{BG_3}(G)$) whose eccentricity is three. This cannot be a point vertex, since eccentricity of a point vertex is either 1 or 2. Therefore, there exists a line vertex $e_1 \in V(\overline{BG_3}(G))$ such that $e(e_1) = 3$ in $\overline{BG_3}(G)$. This implies, there exists e_2 such that $d(e_1, e_2)$ in $\overline{BG_3}(G) = 3$. Hence, $e(e_2) = 3$. Hence, the edges e_1, e_2 are not adjacent in G and there exist no other vertex (not incident with e_1 and e_2) or edge in G.

Hence, $G = 2K_2$. Converse is obvious.

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