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On the Complement of the Boolean Function Graph $B(\overline{K_p}, NINC, \overline{L(G)})$ of a Graph

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Abstract: *For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph $B(\overline{K_p}, NINC, \overline{L(G)})$ of G is a graph with vertex set $V(G)\cup E(G)$ and two vertices in $B(\overline{K_p}, NINC, \overline{L(G)})$ are adjacent if and only if they correspond to two nonadjacent edges of G or to a vertex and an edge not incident to it in G, where L(G) is the line graph of G. For brevity, this graph is denoted by $B_3(G)$. In this paper, structural properties of the complement $\overline{B_3}(G)$ of $B_3(G)$ including traversability and eccentricity properties are studied. Also covering, independence and chromatic numbers and various domination numbers are determined.*

Keywords: *Boolean Function Graph, Domination Number*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. A graph with p vertices and q edges is denoted by G(p, q). *Eccentricity* of a vertex $u\in V(G)$ is defined as $e_G(u) = \max \{d_G(u, v): v\in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called a *self-centered* graph with radius r, equivalently G is r-self-centered. A vertex u is said to be an eccentric point of v in a graph G, if d(u, v) = e(v). In general, u is called an *eccentric point*, if it is an eccentric point of some vertex. A connected graph G is said to be *geodetic*, if a unique shortest path joins any two of its vertices.

A vertex and an edge are said to *cover* each other, if they are incident. A set of vertices, which covers all the edges of a graph G is called a *point cover* for G. The smallest number of vertices in any point cover for G is called its *point covering number* and is denoted by $\alpha_0(G)$ or α_0. A set of vertices in G is *independent*, if no two of them are

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adjacent. The largest number of vertices in such a set is called the *point independence number* of G and is denoted by $\beta_0(G)$ or β_0.

Sampathkumar and Neeralagi [19] introduced the concept of neighborhood sets in graphs. A subset S of V(G) is a *neighborhood set* (n-set) of G, if $G = \cup_{v\in S}(\langle N[v]\rangle)$, where $\langle N[v]\rangle$ is the subgraph of G induced by N[v]. The *neighborhood number* $n_0(G)$ of G is the minimum cardinality of an n-set of G.

The concept of domination in graphs was introduced by Ore [15]. A set $S \subseteq V$ is said to be a *dominating set* in G, if every vertex in $V - S$ is adjacent to some vertex in S. S is said to be a *minimal dominating set*, if $S - \{u\}$ is not a dominating set, for any $u \in S$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. A dominating set with cardinality $\gamma(G)$ is referred as a γ-set. A dominating set S of a connected graph G is called a *connected dominating set* of G, if the induced subgraph $\langle S\rangle$ is connected. The minimum cardinality of a connected dominating set of G is called the *connected domination number* of G and is denoted by γ_c. A set $S \subseteq V$ is a *restrained dominating set*, if every vertex in $V-S$ is adjacent to a vertex in S and another vertex in $V-S$. The restrained domination number of G, denoted by $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set of G.

A dominating set $S \subseteq V$ is a *cycle dominating set*, if the subgraph $\langle S\rangle$ induced by S has a Hamiltonian cycle; S is also called a dominating cycle. The cardinality of a smallest cycle dominating set in G is called the *cycle domination number* of G and is denoted by γ_0.

Kulli and Janakiram [13, 14] introduced the concept of split and non split domination in graphs. A dominating set D of a connected graph G = (V, E) is a *split dominating set*, if the induced subgraph $\langle V-D\rangle$ is disconnected and is a *non split dominating set*, if $\langle V-D\rangle$ is connected. *The split* (*non split*) *domination number* $\gamma_s(G)$ ($\gamma_{ns}(G)$) is the minimum cardinality of a split (non split) dominating set.

Sampathkumar and Pushpalatha [17] introduced the concept of point set domination number of a graph. For any connected graph G, a set $S \subseteq V$ is called a *point set dominating set* (psd-set), if for every set $T \subseteq V-S$ there exists a vertex $v \in S$ such that the subgraph $\langle T\cup\{v\}\rangle$ induced by $T\cup\{v\}$ is connected. The *point set domination number* $\gamma_{ps}(G)$ of G is defined as the minimum cardinality of a psd-set of G. Note that every psd-set is a dominating set.

Theorem 1.1:[17]

Let G = (V, E) be a graph. A set $S \subseteq V$ is a point-set dominating set of G if and only if for every independent set W in V−S, there exists a vertex u in S such that $W \subseteq N_G(u) \cap (V-S)$.

A set $F \subseteq E$ is an *edge dominating set*, if each edge in E is either in F or is adjacent to an edge in F. The *edge domination number* $\gamma'(G)$ is the smallest cardinality among all minimal edge dominating sets.

The *Boolean function graph* $B(\overline{K_p}, NINC, \overline{L(G)})$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(\overline{K_p}, NINC, \overline{L(G)})$ are adjacent if and only if they correspond to two nonadjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_3(G)$. In other words, $V(B_3(G)) = V(G) \cup V(L(G))$; and $E(B_3(G)) = [E(T(\overline{G})) - (E(\overline{G}) \cup E(L(G)))] \cup E(\overline{L(G)})$, where $\overline{G}$, L(G) and T(G) denote the complement, the line graph and the total graph of G respectively. The vertices of G and L(G) in $B_3(G)$ are referred as point and line vertices respectively and the line vertex in $B_3(G)$ corresponding to an edge e in G is denoted by e'.

In this paper, we study structural properties of the complement $\overline{B_3(G)}$ of $B_3(G)$ including traversability and eccentricity properties. Also, covering, independence, chromatic and various domination numbers are determined. The definitions and details not furnished in this paper may found in [2].

2. Main Results

In this section, the complement of $B(\overline{K_p}, NINC, \overline{L(G)})$ is denoted by $\overline{B_3(G)}$. The properties of $\overline{B_3(G)}$ including traversability and eccentricity properties are studied. Also decomposition of $\overline{B_3(G)}$, for some known graphs are given.

Observation 2.1:

2.1.1. If G has p vertices, then the complete graph on p vertices is an induced subgraph of $\overline{B_3(G)}$. Also L(G) is an induced subgraph.

2.1.2. The degree of a point vertex v in $\overline{B_3(G)}$ is $p - 1 + \deg_G(v)$ and the degree of a line vertex e' is $\deg_{L(G)}(e') + 2$ and hence $\Delta(\overline{B_3(G)}) = p - 1 + \Delta(G)$ and $\delta(\overline{B_3(G)}) = \delta'(G)+2$, where $\delta'(G) = \delta(L(G))$.

2.1.3. $\overline{B_3(G)}$ is a connected graph, for any graph G.

2.1.4. $\overline{B_3(G)}$ is biregular if and only if G is a regular graph other than a complete graph and is regular if and only if G is a complete graph.

2.1.5. If G is a complete graph, then $\overline{B_3(G)} \cong T(G)$, where T(G) is the total graph of G.

2.1.6. No vertex of $\overline{B_3(G)}$ is a cut-vertex.

2.1.7. Each vertex of $\overline{B_3}(G)$ lies on a triangle and hence girth of $\overline{B_3}(G)$ is 3. Also each edge of $\overline{B_3}(G)$ lies on a triangle and hence $L(\overline{B_3}(G))$ is Hamiltonian.

2.1.8. If G contains $K_2 \cup K_1$ as a subgraph, then $\overline{B_3}(G)$ contains $K_4 - e$ as an induced subgraph and hence not geodetic. Thus, $\overline{B_3}(G)$ is geodetic if and only if $G \cong nK_1$ or K_2, where $n \geq 2$.

2.1.9. For a (p, q) graph G, $\overline{B_3}(G)$ is Eulerian if and only if one of the following holds.
(i). p is odd and G or each of its components is Eulerian; and
(ii). p is even and each vertex in G is of odd degree.

In the following, hamiltonicity of $\overline{B_3}(G)$ is discussed.

Theorem 2.2:

For any connected graph G, $\overline{B_3}(G)$ is Hamiltonian.

Proof:

The subgraph of $\overline{B_3}(G)$ induced by all the point vertices is complete and L(G) is an induced subgraph of $\overline{B_3}(G)$. Choose any vertex v_1 in V(G). Let $e_{11}, e_{12}, \ldots, e_{1t}$ be the edges in G incident with v_1 ($t \geq 1$), v_t be a vertex incident with e_{1t}, and $e_{t1}, e_{t2}, \ldots, e_{ts}$ be the edges in G incident with v_t ($s \geq 1$), where $e_{tj} \neq e_{1t}$, $j = 1,2,\ldots,s$ and so on. Then $v_1, v_t, \ldots \in V(\overline{B_3}(G))$. Let e_{1i}', e_{tj}' be the line vertices in $\overline{B_3}(G)$ corresponding to the edges e_{1i}, e_{tj} ($i = 1, 2, \ldots, t$; $j = 1, 2, \ldots, s$) respectively. Then form a path $v_1\, e_{11}'\, e_{12}' \ldots e_{1t}'\, v_t\, e_{t1}'\, e_{t2}' \ldots e_{ts}'\, v_s \ldots$ in $\overline{B_3}(G)$ and then place the remaining point vertices (if any) in the above path so as to form a Hamiltonian cycle in $\overline{B_3}(G)$. Hence, $\overline{B_3}(G)$ is Hamiltonian.

Theorem 2.3:

If G is a disconnected graph, then also $\overline{B_3}(G)$ is Hamiltonian.

Proof:

Form a Hamiltonian path in each $\overline{B_3}(G_i)$, where G_i is a component of G, starting and ending with a point vertex. Since any two point vertices in $\overline{B_3}(G)$ are adjacent, these Hamiltonian paths can be linked to form a Hamiltonian cycle in $\overline{B_3}(G)$.

In the following, the eccentricity properties of $\overline{B_3}(G)$ are discussed. A characterization of a graph G for which $\overline{B_3}(G)$ is self-centered with radius 2 is obtained. For simplicity, the distance between two vertices u and v in $\overline{B_3}(G)$ is denoted by $d_3'(u, v)$

Theorem 2.4:

Let G be any graph with $\beta_1(G) \geq 2$. Then $\overline{B_3}(G)$ is self-centered with radius 2 if and only if diameter of L(G) is two.

Proof:

Assume G is a graph with $\beta_1(G) \geq 2$ and $\overline{B_3}(G)$ is self- centered with radius 2. If diam(L(G)) ≥ 3, then there exists two vertices e_1', e_2' in L(G) with $d_{L(G)}(e_1', e_2') = m$, where $m \geq 3$. But $d_3'(e_1', e_2') = 3$, which is a contradiction. Therefore, diam(L(G)) = 2. Conversely, assume diam(L(G)) = 2. The distance between any two point vertices in $\overline{B_3}(G)$ is 1. Let v, e' be a point, line vertex in $\overline{B_3}(G)$ respectively and e be the edge in G corresponding to e'. Then,

$$d_3'(v, e') = 1, \quad \text{if } v \in e$$
$$= 2, \quad \text{if } v \notin e$$

Similarly, let e_1', e_2' be two line vertices in $\overline{B_3}(G)$ and e_1, e_2 be the corresponding edges in G. If e_1, e_2 are adjacent edges in G, then $d_3'(e_1', e_2') = 1$. If e_1 and e_2 are not adjacent, since diam(L(G)) = 2, then there exists an edge in G adjacent to both e_1 and e_2. Therefore, $d_3'(e_1', e_2') = 2$. From the above argument, it follows that both point and line vertices in $\overline{B_3}(G)$ have eccentricity 2 and hence $\overline{B_3}(G)$ is self-centered with radius 2.

Remark 2.4.1:

Let β_1G) = 1. If $G \cong C_3$, $\overline{B_3}(G)$ is self-centered with radius 2 and if G is a star, then $\overline{B_3}(G)$ is bi-eccentric with radius 1.

Now, a characterization of a graph G for which $\overline{B_3}(G)$ is bi-eccentric with radius 2 is obtained.

Theorem 2.5:

$\overline{B_3}$((G) is bi-eccentric with radius 2 if and only if diam(L(G)) ≥ 3, where L(G) is the line graph of G..

Proof:

Assume $\overline{B_3}(G)$ is bi-eccentric with radius 2. If diam(L(G)) $\leq$ 2, then either G is self-centered with radius 2 or bi-eccentric with radius 1. Hence, diam(L(G)) ≥ 3. Conversely, assume diam(L(G)) ≥ 3. Then as in Theorem 6.4.4., eccentricity of a point vertex is 2. Since diam(L(G)) ≥ 3, there exists two vertices e_1', e_2' in L(G) with $d_{L(G)}(e_1', e_2') = m$, where $m \geq 3$. Therefore, $d_3'(e_1', e_2') = 3$. Also the distance between any two line vertices in $\overline{B_3}(G)$ is less than or equal to 3. Hence, $\overline{B_3}(G)$ is bi-eccentric with radius 2.

In the following, the graphs G for which $\overline{B_3}(G)$ contains C_n ($n \geq 4$), as an induced subgraph are obtained, where G is any graph which is not totally disconnected.

Proposition 2.6:

$\overline{B_3}(G)$ contains C_n ($n \geq 4$) as an induced subgraph if and only if either G contains C_n or C_{n-1} as a subgraph.

Proof:

Assume $\overline{B_3}(G)$ contains C_n ($n \geq 4$) as an induced subgraph. If all the vertices of C_n in $\overline{B_3}(G)$ are line vertices, then G contains C_n as a subgraph. If not, since any two point vertices in $\overline{B_3}(G)$ are adjacent, any cycle in $\overline{B_3}(G)$ contains exactly two adjacent point vertices and the other vertices are line vertices. Then G contains C_{n-1} as a subgraph. Converse can be proved easily.

In the following, the edge partitions of $\overline{B_3}(G)$ for some known graphs G are given.

Theorem 2.7:

(1) The edge set of $\overline{B_3}(P_n)$, for $n \geq 4$, can be partitioned into $((n-1)/2)C_n$, P_{n-1} and P_{2n-1}, if n is odd; and $((n-2)/2)C_n$, P_{n-1}, P_{2n-1} and $(n/2)K_2$, if n is even.

(2) The edge set of $\overline{B_3}(C_n)$ ($n \geq 3$) can be partitioned into $((n+1)/2)C_n$, C_{2n}, if n is odd; and $((n-1)/2)C_n$, C_{2n} and $(n/2)K_2$, if n is even such that all the vertices of C_n are either point or line vertices.

(3) The edge set of $\overline{B_3}(K_{n,n})$, for $n \geq 3$ can be partitioned into $((n-1)/2)C_{n+1}$, $((n-1)/2)C_n$, $((n+1)/2)K_2$ and nP_3, if n is odd; and $(n/2)C_{n+1}$, $((n-1)/2)C_n$, $(n/2)K_2$ and nP_3, if n is even.

(4) The edge set of $\overline{B_3}(nK_2)$, for $n \geq 2$, can be partitioned into $((2n-1)/2)C_n$ and nP_3, if n is odd and $((2n-3)/2)C_n$, nP_3 and nK_2, if n is even.

Covering, independence, chromatic and neighborhood numbers in $\overline{B_3}(G)$.

Theorem 2.8:

$\beta_0(\overline{B_3}(G)) = \beta_1(G)$ or $\beta_1(G) + 1$.

Proof:

Any two point vertices in $\overline{B_3}(G)$ are adjacent and L(G) is an induced subgraph of $\overline{B_3}(G)$. Therefore, any independent set in $\overline{B_3}(G)$ contains either all line vertices or one point vertex and line vertices. If there exists a perfect matching in G, then $\beta_0(\overline{B_3}(G)) \geq \beta_1(G)$ and there is no independent set in $\overline{B_3}(G)$ having more than $\beta_1(G)$ vertices. Hence, $\beta_0(\overline{B_3}(G)) = \beta_1(G)$. Let there exist no perfect matching in G and let $D = \{e_1, e_2, \ldots, e_n\} \subseteq E(G)$ with $|D| = \beta_1(G)$ be the set of independent edges in G where $n < [p/2]$. Let v be a vertex in G not incident with any of the edges in D. If D' is the set of line vertices in $\overline{B_3}(G)$ corresponding to the edges in D, then $D' \cup \{v\}$ is a set of independent vertices in $\overline{B_3}(G)$. Hence, $\beta_0(\overline{B_3}(G)) \geq \beta_1(G) + 1$. Since there exists no independent set in $\overline{B_3}(G)$ having more than $\beta_1(G) + 1$ vertices, then $\beta_0(\overline{B_3}(G)) = \beta_1(G) + 1$.

Remark 2.8.1:

Using the relation, $\alpha_0(\overline{B_3}(G)) + \beta_0(\overline{B_3}(G)) = p + q$, it follows that

$$\alpha_0(\overline{B_3}(G)) = q + \alpha_1(G) \text{ or } q + \alpha_1(G) - 1.$$

Proposition 2.9:

If G is any (p, q) graph, then $\alpha_1(\overline{B_3}(G)) = \{(p + q)/2\}$ and $\beta_1(\overline{B_3}(G)) = [(p + q)/2]$.

Proof:

Since $\alpha_1(L(G)) = \{q/2\}$ and $\overline{B_3}(G) \cong L(G + K_1)$, by Theorem 6.3.3., it follows that $\alpha_1(\overline{B_3}(G)) = \alpha_1(L(G + K_1)) = \{(p + q)/2\}$. Also, $\alpha_1(\overline{B_3}(G)) + \beta_1(\overline{B_3}(G)) = p + q$ implies that $\beta_1(\overline{B_3}(G)) = [(p + q)/2]$.

Proposition 2.10:

$\chi(\overline{B_3}(G)) = p$ or $p + 1$.

Proof:

$\chi(\overline{B_3}(G)) = \chi(L(G + K_1)) = \chi'(G + K_1) = \Delta(G + K_1)$ or $\Delta(G + K_1) + 1 = p$ or $p + 1$.

Note 2.10.1:

$$\chi(\overline{B_3}(K_n)) = \chi'(K_{n+1}) = \begin{cases} n, & \text{if } n \text{ is even; and} \\ n + 1, & \text{if } n \text{ is odd.} \end{cases}$$

In the following, the neighborhood number n_0 of $\overline{B_3}(G)$ is obtained.

Observation 2.11:

2.11.1. $n_0(\overline{B_3}(G)) = 1$ if and only if $G \cong nK_1$, $n \geq 2$ or $K_{1,n} \cup mK_1$, where $n \geq 1$ and $m \geq 0$.

2.11.2. $n_0(\overline{B_3}(G)) \leq \alpha_0(G) + n_0(L(G))$.

In the following, a necessary and sufficient condition that an n-set of L(G) to be an n-set of $\overline{B_3}(G)$ is obtained.

Theorem 2.12:

Let G be any graph having no isolated vertices. Then $n_0(\overline{B_3}(G) \leq n_0(L(G))$ if and only if there exists an neighborhood set (n-set) S of L(G) with $|S| = n_0(G)$ such that each vertex in G is incident with at least one of the edges in G corresponding to the vertices in S.

Proof:

Assume $n_0(\overline{B_3}(G)) \leq n_0(L(G))$. Let S be an n-set of L(G) with $|S| = n_0(G)$. Then S is also an n-set of $\overline{B_3}(G)$. Let S' be the set of edges in G corresponding to the vertices in S and $v \in V(G)$ be such that v is not incident with any of the edges in S'. Let $e \in E(G)$ be such that e is incident with v and e' be the corresponding line vertex in $\overline{B_3}(G)$. Then the edge $(v, e') \in E(\overline{B_3}(G))$ does not belong to $\cup_{w \in s} E(<N[w]>)$, which is a contradiction. Converse follows easily.

This bound is attained, if $G \cong C_n$, $n \geq 4$.

Corollary 2.12.1:

$n_0(\overline{B_3}(G)) \leq n_0(L(G)) + k$, where k is the number of vertices in G not incident with the edges in G corresponding to the vertices in the n-set S of L(G), where $|S| = n_0(L(G))$. For example, $n_0(\overline{B_3}(P_5)) = n_0(L(P_5)) + 1$.

Example 2.13:

(i). $n_0(\overline{B_3}(P_n)) = \{n/2\}$, if $n \geq 4$.

(ii). $n_0(\overline{B_3}(C_n)) = \{n/2\}$, if $n \geq 4$.

(iii). $n_0(\overline{B_3}(K_n)) = n - 1$, if $n \geq 4$.

(iv). $n_0(\overline{B_3}(nK_2)) = n$, if $n \geq 2$.

Domination Numbers and other parameters for the complement of $B(\overline{K_p}, NINC, \overline{L(G)})$

In the following, domination number of $\overline{B_3}(G)$ is determined

Proposition 2.13:

$\gamma(\overline{B_3}(G)) = 1$ if and only if $G \cong K_{1,n}$ or mK_1, where $n \geq 1$ and $m \geq 2$.

Proof:

$r(G) = 1$ if and only if $G \cong K_{1,n}$ or mK_1, where $n \geq 1$ and $m \geq 2$.

Theorem 2.14:

Let G be a graph other than a star. Then $\gamma(\overline{B_3}(G)) = 2$ if and only if one of the following holds.

(i). There exists a point cover for G containing two vertices;

(ii). There exists a vertex $v \in V(G)$ and an edge $e \in E(G)$ such that e is not incident with v and the edges of G are either adjacent to e or incident with v or both;

(iii). $r(L(G)) = 1$; and

(iv). There exists a line cover for G containing two edges.

Proof:

Let D be a minimal dominating set of $\overline{B_3}(G)$ containing two elements.

(i) Let $D = \{v_1, v_2\} \subseteq V(\overline{B_3}(G))$, $v_1, v_2 \in V(G)$. D dominates all line vertices of $\overline{B_3}(G)$, which implies that each edge in G is incident with v_1, v_2 or both. That is, D is a point cover for G.

(ii) Let $D = \{v, e\}$, where $v \in V(G)$ and $e \in E(G)$.

(a) Let $e \in E(G)$ be incident with $v \in V(G)$. Then, edges of G must be adjacent to e.

(b) Let e be not incident with $v \in V(G)$. Since, D dominates all line vertices of $\overline{B_3}(G)$, all the edges of G are either adjacent to e or incident with v.

(iii) Let $D = \{e_1, e_2\}$, where $e_1, e_2 \in E(G)$. D is dominating set of $\overline{B_3}(G)$ implies that D is a line cover for G.

Proposition 2.15:

If G is any graph having no isolated vertices, then

$$\gamma(\overline{B_3}(G)) \leq \min\{\alpha_0(G), \alpha_1(G), \beta_1(G) + 1\}.$$

Proof:

(i) Let D be a point cover for G such that $|D| = \alpha_0(G)$. Since any two point vertices in $\overline{B_3}(G)$ are adjacent, D dominates all point vertices in $\overline{B_3}(G)$. By the construction of $\overline{B_3}(G)$, D dominates all the line vertices in $\overline{B_3}(G)$. Hence, D is a dominating set of $\overline{B_3}(G)$ and therefore, $\gamma(\overline{B_3}(G)) \leq \alpha_0(G)$.

(ii) Let D be a line cover for G. The line vertices corresponding to the edges in D dominates all point vertices of $\overline{B_3}(G)$. Since L(G) is an induced subgraph of $\overline{B_3}(G)$, D dominates all the line vertices of $\overline{B_3}(G)$. Hence, $\gamma(\overline{B_3}(G)) \leq \alpha_1(G)$.

(iii) Since L(G) is an induced subgraph of $\overline{B_3}(G)$ and any two point vertices are adjacent, $\gamma(\overline{B_3}(G)) \leq \gamma(L(G)) + 1 = \gamma'(G) + 1 \leq \beta_1(G) + 1$.

From (i), (ii) and (iii), $\gamma(\overline{B_3}(G)) \leq \min\{\alpha_0(G), \alpha_1(G), \beta_1(G) + 1\}$.

Proposition 2.16:

Let G be any graph having no isolated vertices. Then $\gamma(\overline{B_3}(G)) \leq \gamma(G)$ if and only if $\gamma(G) = \alpha_0(G)$.

Proof:

Assume $\gamma(G) = \alpha_0(G)$. Let d be a minimal dominating set of G such that $|D| = \gamma(G)$. then d is a point cover for G. Therefore, D is a dominating set of $\overline{B_3}(G)$ and hence $\gamma(\overline{B_3}(G)) \leq \gamma(G)$.

Conversely, let $D \subseteq V(G)$ be a minimal dominating set of G such that D is a dominating set of $\overline{B_3}(G)$. then D is appoint cover for G. That is, $\alpha_0(G) \leq \gamma(G)$. But, $\gamma(G) \leq \alpha_0(G)$. Hence, $\gamma(G) = \alpha_0(G)$.

Proposition 2.17:

Let G be any graph having no isolated vertices with $\gamma(G) \neq \alpha_0(G)$ and D be a minimal dominating set of G with $|D| = \gamma(G)$. Then $\gamma(\overline{B_3}(G)) \leq \gamma(G) + 1$, if either

(i) <V(G)–D> is a star or

(ii) There exists an edge e in G such that e is adjacent to all the edges in <V(G)–D>.

Proof:

Let D be a minimal dominating set of G such that $|D| = \gamma(G)$. If (i) is true, then $D \cup \{u\}$ is a dominating set of $\overline{B_3}(G)$, where u is the center vertex of the star. If (ii) holds, then $D \cup \{e\}$is a dominating set of $\overline{B_3}(G)$.

Theorem 2.18:

$\gamma(\overline{B_3}(G)) \leq \gamma'(G)$ or $\gamma'(G) + 1$, where $\gamma'(G)$ is the edge domination number of G.

Proof:

Let D be a minimal dominating set of L(G) with $|D| = \gamma(L(G)) = \gamma'(G)$. If the set of edges in G corresponding to the vertices in D is a perfect matching or a line cover for G, then that set is a dominating set of $\overline{B_3}(G)$. Let $D' \subseteq D$ be such that $|D'| = \gamma'(G)$-1.

(i). Let $V(L(G)) - N[D'] = M$. If the edges in G corresponding to the vertices in M are the edges of a star in G, then D' together with the center vertex of the star is a dominating set of $\overline{B_3}(G)$;

(ii). Let N be the set of edges in G corresponding to the vertices in D'. If D'' is the set of vertices in G incident with the edges in N and if $\langle V(G) - D'' \rangle \cong K_{1,n}$, $n \geq 1$, then $D' \cup \{$center vertex of $K_{1,n}\}$ is a dominating set of $\overline{B_3}(G)$.

If (i) or (ii) holds, then $\gamma(\overline{B_3}(G)) \leq \gamma'(G)$.

Otherwise, $D \cup \{v\}$ is a dominating set of $\overline{B_3}(G)$, where $v \in V(G)$. Thus, $\gamma(\overline{B_3}(G)) \leq \gamma'(G)$ or $\gamma'(G) + 1$.

Note 2.18.1:

(i). If $G \cong 2P_3$, then $\gamma(\overline{B_3}(G)) = \alpha_0(G) = \gamma(G) = 2$.

(ii). If $G \cong K_n$, then $\gamma(\overline{B_3}(G)) = \gamma'(G) + 1$, if n is odd; and

$= \gamma'(G)$, if n is even.

(ii). If $\text{diam}(L(G)) = 2$, then $\gamma(\overline{B_3}(G)) \leq \delta(L(G)) + 1$.

(iv). If $\text{diam}(\overline{L(G)}) \geq 3$, then $\gamma(\overline{B_3}(G)) \leq 3$, where $\overline{L(G)}$ has no isolated vertices.

Example 2.19:

(i). $\gamma(\overline{B_3}(P_n)) = [n/2]$, if $n \geq 3$.

(ii). $\gamma(\overline{B_3}(C_n)) = [n/2]$, if $n \geq 3$.

(iii). $\gamma(\overline{B_3}(K_n)) = \{n/2\}$, if $n \geq 4$.

(iv). $\gamma(\overline{B_3}(nK_2)) = n$, if $n \geq 2$.

Independent, connected, cycle, restrained and point set domination numbers in $\overline{B_3}(G)$.

The following propositions are stated without proof.

Proposition 2.20:

If G is any graph having no isolated vertices and has a perfect matching, then $\gamma_i(\overline{B_3}(G)) \leq p/2$, where γ_i is the independent domination number.

Proposition 2.21:

$\gamma_i(\overline{B_3}(G)) \leq \min\{\beta_1(G) + 1, \gamma_i(L(G)) + 1\}$.

Proposition 2.22:

$\gamma_i(\overline{B_3}(G)) = 2$ if and only if either there exists a vertex $v \in V(G)$ and an edge $e \in E(G)$ such that e is not incident with v and all the edges of G are either adjacent to e or incident with v or there exists a line cover for G having two vertices.

Note 2.22.1:

(i). If $G \cong C_6$, then $\gamma_i(\overline{B_3}(G)) = \beta_1(G) = 3$.

(ii). Any independent dominating set of $\overline{B_3}(G)$ contains at most one point vertex.

In the following, connected domination number γ_c, cycle domination number γ_0 and the restrained domination number γ_r of $\overline{B_3}(G)$ are determined. The following propositions are stated without proof.

Proposition 2.23:

Let G be a graph other than a star. Then $\gamma_c(\overline{B_3}(G)) = 2$ if and only if either there exists a point cover for G containing two vertices or radius of L(G) is 1, where γ_c is the connected domination number.

Proposition 2.24:

$\gamma_c(\overline{B_3}(G)) \leq \gamma_c(L(G)) + 1$.

Proposition 2.25:

$\gamma_0(\overline{B_3}(G)) \leq \gamma_0(L(G)) + 1$, where γ_0 is the cycle domination number.

Proposition 2.26:

Let G be any graph having no isolated vertices. Then $\gamma_0(\overline{B_3}(G)) = 3$ if and only if one of the following holds.

(i). There exists a point cover D for G with $|D| = 3$;

(ii). There exists a connected point cover for G containing at least two vertices; and

(iii). $\gamma_c(L(G)) \leq 2$.

Proposition 2.27:

If G is a connected graph, then $\gamma_r(\overline{B_3}(G)) \leq \alpha_0(G)$, where γ_r is the restrained domination number.

Proposition 2.28:

If G is a graph having no isolated vertices, then $\gamma_r(\overline{B_3}(G)) \leq \min\{\alpha_1(G), \gamma'(G) + 1, \gamma_r(L(G)) + 1\}$.

Proposition 2.29:

If G is a connected graph, then any dominating set of $\overline{B_3}(G)$ containing point vertices only is a restrained dominating set of $\overline{B_3}(G)$.

Next, point set domination number γ_{ps} of $\overline{B_3}(G)$ is determined.

Proposition 2.30:

$\gamma_{ps}(\overline{B_3}(G)) = 1$ if and only if $G \cong K_{1,n}$, where $n \geq 2$.

Theorem 2.31:

$\gamma_{ps}(\overline{B_3}(G)) = \alpha_0(G)$ if and only if $G \cong K_{1,n}$ or C_3, where $n \geq 2$.

Proof:

Assume $\gamma_{ps}(\overline{B_3}(G)) = \alpha_0(G)$. The there exists a point cover D for G which is also a point set dominating set (psd-set) of $\overline{B_3}(G)$. If $\beta_1(G) \geq 2$, then there exists at least two independent edges, say e_1 and e_2 in G. Let e_1' and e_2' be the corresponding line vertices in $\overline{B_3}(G)$. Then $\{e_1', e_2'\} \subseteq \langle V(\overline{B_3}(G)) - D\rangle$ is an independent set in $\langle V(\overline{B_3}(G)) - D\rangle$ and there exists no vertex in D adjacent to both e_1' and e_2', which is a contradiction. Hence, $\beta_1(G) = 1$ and $G \cong K_{1,n}$ or C_3, where $n \geq 2$. Converse follows easily.

Theorem 2.32:

$\gamma_{ps}(\overline{B_3}(G)) \leq q + 1 - \Delta(G)$.

Proof:

For any graph G, $\gamma_{ps}(G) \leq p - \Delta(G)$. Hence, $\gamma_{ps}(\overline{B_3}(G)) \leq p + q - \Delta(\overline{B_3}(G)) = p + q - (p - 1 + \Delta(G)) = q + 1 - \Delta(G)$.

This bound is attained, if $G \cong K_{1,n}$, for $n \geq 2$ and C_3.

In the following theorem, for simplicity, the neighborhood of a vertex v in $\overline{B}_3(G)$ is denoted by $\overline{N}_3(v)$.

Theorem 2.33:

Let G be any graph having no isolated vertices and D be a line cover for G with $\beta_1(\langle E(G)-D\rangle) = 1$. Then the set D' of line vertices in $\overline{B}_3(G)$ corresponding to the edges in D is a point set dominating set of $\overline{B}_3(G)$, if for every pair of vertices v, e' in $V(\overline{B}_3(G))-D$, where the edge e in G corresponding to the line vertex e' is not incident with $v\in V(G)$, $|\overline{N}_3(v)\cap \overline{N}_3(e')\cap D'| = 1$ or 2 in $\overline{B}_3(G)$.

Proof:

Let D be a line cover for G with $\beta_1(\langle E(G)-D\rangle) = 1$ and D' be the set of line vertices in $\overline{B}_3(G)$ corresponding to the edges in D. Let $W \subseteq V(\overline{B}_3(G))-D$ be independent. Then W contains exactly two vertices, namely one point vertex and one line vertex, since $\beta_1(\langle E(G)-D\rangle) = 1$ and any two point vertices in $\overline{B}_3(G)$ are adjacent. By the hypothesis, there exists a vertex in D' adjacent to all the two vertices in W. Hence, D' is a psd-set of $\overline{B}_3(G)$.

Theorem 2.34:

Let G be any (p, q) graph having no isolated vertices and D be a line cover for G with $\beta_1(\langle E(G)-D\rangle) = 2$. Then the set D' of all line vertices in $\overline{B}_3(G)$ corresponding to the edges in D is a psd-set of $\overline{B}_3(G)$ if $G \cong C_4$.

Proof:

Assume D' is a psd-set of $\overline{B}_3(G)$. Since $\beta_1(\langle E(G)-D\rangle) = 2$, any independent set in $V(\overline{B}_3(G))-D$ contains at most two line vertices.

(i). Let $W = \{e_1', e_2'\} \subseteq V(\overline{B}_3(G))-D'$ be independent, where the edges e_1 and e_2 in G corresponding to the vertices e_1' and e_2 are independent in G. Then there exists a vertex in D' adjacent to both $e_1'$and e_2'.

(ii). Let $W = \{v, e_1', e_2'\} \subseteq V(\overline{B}_3(G))-D'$ be independent. Then e_1 and e_2 are independent edges in G and $v\in V(G)$ is not incident with both e_1 and e_2. But there exists no vertex in D adjacent to the vertices v, e_1' and e_2', which is a contradiction. Hence, $p < 5$ and $W = \{v, e'\}$, where the edge in G corresponding to the line vertex e' is not incident with v. Then there exists a vertex in D' adjacent to both v and e'. Since D is a line cover for G, $\beta_1(\langle E(G)-D\rangle) = 2$ and

$p = 4$, it follows that $G \cong C_4$.

Remark 2.34.1:

(i). Let D be a line cover for G with $\beta_1(\langle E(G)-D\rangle) \geq 3$. Then the set of line vertices in $\overline{B_3}(G)$ corresponding to the edges in D is not a psd-set of $\overline{B_3}(G)$.

(ii). If $\beta_1(G) \geq 2$, then any dominating set of $\overline{B_3}(G)$ containing point vertices only is not a psd-set of $\overline{B_3}(G)$.

Split and Nonsplit domination numbers in $\overline{B_3}(G)$.

Theorem 2.35:

$\gamma_s(\overline{B_3}(G)) \leq q - \alpha_1(G) + 2$, if there exists a line cover D for G with $|D| = \alpha_1(G)$ containing at least two independent edges, where γ_s is the split domination number.

Proof:

Let D be a line cover for G with $|D| = \alpha_1(G)$ containing at least two independent edges, say e_1 and e_2. Let $e_1 = (u_1, v_1) \in E(G)$, where $u_1, v_1 \in V(G)$. Then $u_1, v_1 \in V(\overline{B_3}(G))$. If D' is the set of line vertices corresponding to the edges in $E(G)-D$, then

$D'' = D' \cup \{u_1, v_1\}$ is a split dominating set of $\overline{B_3}(G)$, since the line vertex e_1' in $\overline{B_3}(G)$ corresponding to the edge e_1 is isolated in $\langle V(\overline{B_3}(G)) - D''\rangle$.

Hence, $\gamma_s(\overline{B_3}(G)) \leq |D''| = q - \alpha_1(G) + 2$.

This bound is attained, if $G \cong C_n$, for $n \geq 4$.

In the following, an upper bound of $\gamma_s(\overline{B_3}(G))$ is given in terms of the minimum degree of G.

Theorem 2.36:

$\gamma_s(\overline{B_3}(G)) \leq p + \delta(G) - 1$.

Proof:

Let v be a vertex of minimum degree in G. If D' is the set of line vertices in $\overline{B_3}(G)$ corresponding to the edges in G incident with v, then $D = D' \cup (V(G)-v)$ is a split dominating set of $\overline{B_3}(G)$ and $\gamma_s(\overline{B_3}(G)) \leq p + \delta(G) - 1$.

This bound is attained, if $G \cong C_3$.

Theorem 2.37:

If $diam(G) \geq 3$, then $\gamma_s(\overline{B_3}(G)) \leq p + q - k - 1$,

where $k = \max\{ deg_G(u) + deg_G(v) : d_G(u, v) \geq 3\}$, where $u, v \in V(G)$.

Proof:

Let u and v be any two vertices in G with $d_G(u, v) \geq 3$ and $deg_G(u) + deg_G(v)$ is maximum and $k = deg_G(u) + deg_G(v)$. If D' is the set of line vertices corresponding to the edges in G not incident with u and v, then $D = D' \cup (V(G) - u)$ is a split dominating set of $\overline{B_3}(G)$, since $<V(\overline{B_3}(G)) - D> \cong K_n \cup K_m$, where m, n ≥ 2 and $|D| = p - 1 + q - (deg_G(u) + deg_G(v)) = p + q - (k + 1)$. Thus, $\gamma_s(\overline{B_3}(G)) \leq p + q - k - 1$.

The following propositions are stated without Proof:

Proposition 2.38:

If G is a connected graph and if $\kappa'(G)$ is the connectivity of L(G), then $\gamma_s(\overline{B_3}(G)) \leq p - 1 + \kappa'(G)$.

Proposition 2.39:

$\gamma_s(\overline{B_3}(G)) \leq p + q - \Delta(G) - 1$.

Proposition 2.40:

If G is a disconnected graph, then $\gamma_s(\overline{B_3}(G)) \leq p - 1$.

Proposition 2.41:

$\gamma_{ns}(\overline{B_3}(G)) \leq \min\{\alpha_0(G), \alpha_1(G), \beta_1(G) + 1\}$, for any graph G having no isolated vertices where γ_{ns} is the non split domination number.

Example 2.42:

(i). $\gamma_s(\overline{B_3}(P_n)) = n - 1$, if $n \geq 4$.

(ii). $\gamma_s(\overline{B_3}(K_{1,n})) = n + 1$, if $n \geq 2$.

(iii). $\gamma_s(\overline{B_3}(K_n)) = 2n - 2$, if $n \geq 3$.

Conclusion:

We have established structural properties of the complement $\overline{B_3}(G)$ of $B_3(G)$ including traversability and eccentricity properties Also covering, independence and chromatic and neighborhood numbers are found. Moreover, domination, Independent, connected, cycle and restrained, point set, split and nonsplit domination numbers of $\overline{B_3}(G)$ are determined..

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