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# Boolean graph $BG_3(G)$ of a graph G

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**Abstract:** *Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G).* $B_{\bar{K}_p, INC, \bar{L}(G)}(G)$ *is a graph with vertex set* $V(G) \cup E(G)$ *and two vertices are adjacent if and only if they correspond to a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by* $BG_3(G)$*, Boolean graph of G-third kind. In this paper, some properties of* $BG_3(G)$ *are studied.*

Key words: Boolean graph $BG_3(G)$.

## 1. Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [10], Buckley and Harary [7].

The *girth* of a graph G, denoted g(G), is the length of a shortest cycle (if any) in G; the *circumference* c(G) is the length of any longest cycle. The distance d(u, v) between two vertices u and v in G is the minimum length of a path joining them if any; otherwise $d(u, v) = \infty$. A shortest u-v path is called a *u-v geodesic*. A graph G is *geodetic*, if for every pair of vertices (u, v) there exists a unique shortest path connecting them in G.

Let G be a connected graph and u be a vertex of G. The *eccentricity* e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max \{d(u, v) : u \in V\}$. The *radius* r(G) is the minimum eccentricity of the vertices, whereas the *diameter* diam(G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. v is a central vertex if e(v) = r(G). The *center C(G)* is the set of all central vertices. The central subgraph $< C(G) >$ of a graph G is the subgraph induced by the center. v is a *peripheral vertex* if e(v) = diam(G). The *periphery P(G)* is the set of all such vertices. For a vertex v, each vertex at distance e(v) from v is an eccentric node of v.

A graph is *self-centered* if every vertex is in the center. Thus, in a self-centered graph G all nodes have the same eccentricity, so r(G) = diam(G).

An edge $uv \in E(G)$ is a *dominating edge* of G, if all the vertices of G other than u and v are adjacent to either u or v.

A vertex (point) and an edge are said to *cover* each other, if they are incident. A set of vertices, which cover all the edges of a graph G is called a (*vertex*) *point cover of G*,

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while a set of lines (edges), which covers all the vertices is a *line cover*. The smallest number of points in any point cover for G is called its *point covering number* and is denoted by $\alpha_o(G)$ or $\alpha_o$. Similarly, $\alpha_1(G)$ or $\alpha_1$ is the smallest number of lines in any line cover of G and is called its *line covering number*. Clearly, $\alpha_o(K_p) = p-1$ and $\alpha_1(K_p) = \lfloor (p+1)/2 \rfloor$. A point cover (line cover) is called *minimum*, if it contains $\alpha_o$ (respectively $\alpha_1$) elements.

A set of points in G is *independent*, if no two of them are adjacent. The largest number of points in such a set is called the *point independence number* of G and is denoted by $\beta_o(G)$ or $\beta_o$. Analogously, an independent set of lines (matching) of G has no two of its lines adjacent and the maximum cardinality of such a set is the *line independence number* $\beta_1(G)$ or $\beta_1$, $\beta_o(K_p) = 1$ and $\beta_1(K_p) = \lfloor p/2 \rfloor$. A set of independent edges covering all the vertices of a graph G is called a *1-factor* or a *perfect matching* of G.

A *coloring* of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ is defined to be the minimum n for which G has n coloring.

The minimum number of complete subgraphs of G needed to cover the vertices of G is known as the *clique cover number of G* and is denoted $\theta(G)$.

The maximum number of mutually adjacent vertices, that is the size of the largest complete subgraphs of G is known as the *clique number of G* and is denoted $\omega(G)$.

A graph G is *Berge* [11] if it does not contain odd cycles of length at least five or their respective complement as induced subgraphs. A graph is *perfect* if $\beta_o(H) = \theta(H)$ for every induced subgraph H of G. This implies that $\omega(H) = \chi(H)$ for every induced subgraph H. Clearly, every bipartite graph is perfect.

**Theorem 1.1** (Gallai) [8]:

For any connected graph G, $\alpha_o+\beta_o = p = \alpha_1+\beta_1$.

**Theorem 1.2** [8]:

$\beta_o(L(G)) = \beta_1(G)$, $\alpha_o(L(G)) = q-\beta_1(G)$ and $\alpha_1(L(G)) = \lceil q/2 \rceil$.

**Theorem 1.3** [10]:

$\chi(G) \leq 1+\Delta(G)$.

**Theorem 1.4** [8]:

For any simple graph G, $\chi(G) = \theta(\overline{G})$ and $\beta_o(G) = \omega(\overline{G})$.

**Theorem 1.5** (Hayward) [17]:

If G is Berge and if it contains neither a cycle of length at least 6 nor its complement as an induced subgraph, then G is perfect.

Jin Akiyama and Kiyoshi Ando [3] characterized the graphs G, which are self-centered with diameter two such that $\overline{G}$ is also self-centered with diameter 2.

**Lemma 1.1** [3]:

Let both G and $\overline{G}$ be connected and v be a point of G. If $e_G(v) \geq 3$, then $e_{\overline{G}}(v) = e_G(v) = 2$.

**Corollary 1.1:**

If G is self-centered with diameter $d \geq 3$, then $\overline{G}$ is self-centered with diameter 2.

An edge $uv \in E(G)$ is a *dominating edge*) of G, if all the vertices of G other than u and v are adjacent to either u or v.

**Theorem 1.6** [3]:

The following three statements are equivalent.

(1) Both G and $\overline{G}$ are self-centered with diameter two.

(2) G is self-centered with diameter two having no dominating edge.

(3) Neither G nor $\overline{G}$ contains a dominating edge.

**Motivation:** The Line graphs, Middle graphs, Total graphs and Quasi-total graphs are very much useful in computer networks. In analogous to line graph [6, 13], total graph [4], [5], middle graph [1, 2] and quasi-total graph [12], thirty-two graphs can be defined using different adjacency relations. Out of these operations, eight were already studied. Among the remaining twenty-four graph operations, two are defined and analyzed here. All the others have been defined and studied thoroughly and will be submitted elsewhere. This is illustrated below.

| $G$/ $\overline{G}$/$K_p$/ $\overline{K_p}$ | — Incident (INC)/ Non-incident (NINC) — | $L(G)$/ $\overline{L(G)}$/$K_q$/ $\overline{K_q}$ |
|---|---|---|

Defining a new graph from a given graph by using the adjacency relation between two vertices or two edges and incident relationship between vertices and edges is known as Boolean operation. It defines new structure from the given graph and adds extra information of the original graph.

In Management and in social networks, the incident and non-incident relations of vertices and edges are used to define various networks. So these are very much applicable in socio-economical problems. In some cases, it is possible to retrieve the original graph from the Boolean graphs in polynomial time. So these graph operations may be used in graph coding or coding of some grouped signal. Also, it is possible to study the structure of original graphs through these Boolean graph operations. This motivates the study for the exploration of various Boolean operations and study of their structural properties.

Let G be a (p, q) graph with vertex set V(G) and edge set E(G). The Boolean graph $B_{\overline{K_p}, INC, \overline{L(G)}}(G)$ is a graph with vertex set V(G) ∪ E(G) and two vertices are adjacent if and only if they correspond to two non-adjacent edges of G or to a vertex and an edge incident to it in G. For simplicity, denote this graph by $BG_3(G)$, Boolean graph of G-third kind. The vertices of $BG_3(G)$, which are in V(G) are called point vertices and vertices in E(G) are called line vertices of $BG_3(G)$. $V(BG_3(G)) = V(G) \cup E(G)$ and $E(BG_3(G)) = (E(T(G)) - (E(G) \cup E(L(G))) \cup E(\overline{L(G)})$. $BG_3(G)$ has p+q vertices, p-point vertices and q-line vertices. $BG_3(G)$ is a spanning subgraph of $BG_2(G)$. $\overline{K_p}$ and $\overline{L(G)}$ are induced subgraphs of $BG_3(G)$.

## 2. Properties of $BG_3(G)$ and $\overline{BG_3(G)}$

Let G be a (p, q) graph with vertices $v_1, v_2, ..., v_p$. Let $d_i$, i = 1, 2, ..., p denote the degree of the vertices $v_1, v_2, ..., v_p$ in G. Then it follows from the definition that $BG_3(G)$ has $(q(q+5)/2) - (1/2)\Sigma d_i^2$ edges with degree of $v_i$ in $BG_3(G) = deg_G v_i = d_i$ and degree of a line vertex $e_{ij} = v_i v_j \in E(G)$ in $BG_3(G)$ is $q - d_i - d_j + 3$.
Since it is easy to follow, the following proposition is stated without proof.

**Proposition 2.1:**
(1) $BG_3(G)$ has an isolated vertex if and only if G has an isolated vertex.
(2) $BG_3(G)$ has a pendant vertex if and only if G has a pendant vertex.
(3) $BG_3(G)$ is regular if and only if G is regular of degree $m = (q+3)/3$. If G is regular of degree $m \neq (q+3)/3$, then $BG_3(G)$ is bi-regular.
(4) $BG_3(G)$ is disconnected if and only if G has isolated vertices.

**Remark 2.1:**

$BG_3(K_4) \cong P$, the Petersen graph.

**Theorem 2.1:**

Girth of $BG_3(G)$ is three if and only if G has at least three mutually non-adjacent edges.

**Proof:**

In $BG_3(G)$, no two point vertices are adjacent and line vertices which represent non-adjacent edges in G are adjacent in $BG_3(G)$. Therefore, girth of $BG_3(G)$ is three implies, there exists $e_1, e_2, e_3 \in E(G)$ such that $e_1, e_2, e_3$ form a cycle in $BG_3(G)$. Hence by the definition of $BG_3(G)$, the edges $e_1, e_2, e_3$ are mutually non-adjacent to each other in G.

Converse is obvious.

**Theorem 2.2:**

Girth of $BG_3(G) = 4$ if and only if G has $K_{1,2} \cup K_2$ as subgraph and G has no three mutually non-adjacent edges.

**Proof:** Assume girth of $BG_3(G) = 4$. Hence by the previous theorem, G has no three mutually non-adjacent edges. $BG_3(G)$ has an induced $C_4$ as subgraph.

**Case 1:** $v_1\ e_1\ v_2\ e_2\ v_1$ represents a $C_4$ in $BG_3(G)$.

This is not possible since both $e_1, e_2$ are incident with same vertices $v_1$ and $v_2$.

**Case 2:** $v_1\ e_1\ e_2\ e_3\ v_1$ represents a $C_4$ in $BG_3(G)$.

In G, $v_1$ is incident with $e_1$ and $e_3$ and $e_2$ is not adjacent with $e_1$ and $e_2$. Therefore, G has $K_{1,2} \cup K_2$ as subgraph.

Other cases are not possible.

**Corollary 2.2:**

(1) If G is a connected graph with more than four vertices and $G \neq K_{1,n}$, girth of $BG_3(G)$ is 3 or 4.

(2) If G is disconnected and has no isolated vertices, girth of $BG_3(G)$ is 3 or 4.

**Remark 2.2:**

If $G = K_{1,n}, 2K_2$, then $BG_3(G)$ is acyclic.

**Theorem 2.3:**

Girth of $BG_3(G) = 5$ if and only if $p = 4$ and G has $P_4$ as subgraph.

**Proof:**

Assume girth of $BG_3(G) = 5$. By the previous two Theorems, p is at most 4. When $p \leq 3$, $BG_3(G)$ has no cycle of length five. Therefore, $p = 4$. Also $G \neq K_{1,n}$. Hence, G must be any of $P_4$, $C_4$, $K_4 - e$, $K_4$.

Conversely, if $G = P_4, C_4, K_4 - e$ or $K_4$, that is $p = 4$ and G has $P_4$ as subgraph, girth of $BG_3(G)$ is five.

**Theorem 2.4:**

Girth of $BG_3(G) = 6$ if and only if $G = K_3$.

**Proof:**

Girth of $BG_3(G) = 6$ implies $p < 4$. The only possibility is $G = K_3$.

**Theorem 2.5:**

$BG_3(G)$ has a dominating edge if and only $G = K_2$ or $2K_2$.

**Proof:**

**Case 1:** uv, where u, v $\in$ V(G) is a dominating edge in $BG_3(G)$.
This is not possible since, no two point vertices are adjacent in $BG_3(G)$.

**Case 2:** $e_1e_2$ is a dominating edge in $BG_3(G)$, where $e_1$, $e_2$ $\in$ E(G).
$e_1e_2$ is an edge in $BG_3(G)$ implies that $e_1$, $e_2$ are non-adjacent edges in G. $e_1e_2$ is a dominating edge in $BG_3(G)$ implies that all other edges in G are not adjacent to $e_1$ or $e_2$ or both. Therefore, there exists no edge adjacent to both $e_1$ and $e_2$ in G. Also, all point vertices are incident with $e_1$ or $e_2$ in G.

Hence G must be $2K_2$.

**Case 3:** ue is a dominating edge in $BG_3(G)$, where u $\in$ V(G), e $\in$ E(G).
By the definition of $BG_3(G)$, e is incident with u in G. Also, all other point vertices are dominated by e in $BG_3(G)$. This implies that $G = K_2$.

Hence the theorem is proved.

Next, properties of $\overline{BG_3}(G)$ can be seen. Following results are easy to follow from the definition.

(1) Degree of a point vertex v in $\overline{BG_3}(G)$ is $p+q-1-\deg_G v$.

(2) Degree of a line vertex $e_{ij}$ in $\overline{BG_3}(G)$ is $(p-4)+d_i+d_j$.

(3) Number of edges in $\overline{BG_3}(G) = p(p-1)/2+q(p-3)+(1/2)\Sigma di^2$.

(4) The induced subgraph formed by the point vertices is a complete graph and the induced subgraph formed by the line vertices is L(G) in $\overline{BG_3}(G)$.

(5) Girth of $\overline{BG_3}(G)$ is always three for $p \geq 3$.

(6) $BG_3(G)$ is disconnected if $G = K_2$, otherwise $\overline{BG_3}(G)$ is connected.

(7) $\overline{BG_3}(G)$ has a pendant vertex if and only if $G = K_2 \cup K_1$ or $K_2$.

**Theorem 2.6:**

$\overline{BG_3}(G)$ has a dominating edge for all G, $G \neq K_n$, $n > 3$.

**Proof:**

Assume $\overline{BG_3}(G)$ has a dominating edge.

**Case 1:** $u_1u_2$ is a dominating edge in $\overline{BG_3}(G)$, where $u_1$, $u_2$ $\in$ V(G).

$u_1u_2$ is a dominating edge in $\overline{BG_3}(G)$ implies that every line vertex is adjacent to either $u_1$ or $u_2$ in $\overline{BG_3}(G)$. This gives, $u_1$ and $u_2$ are not adjacent in G. Hence, diam(G) $\geq 2$.

**Case 2:** $e_1e_2$ is a dominating edge in $\overline{BG_3}(G)$, where $e_1$, $e_2$ are line vertices.

Since $e_1e_2$ is a dominating edge, $e_1$ and $e_2$ are adjacent edges in G. Hence, $e_1$ and $e_2$ are incident with a common vertex u in G. Therefore, this u is not dominated by $e_1$ or $e_2$ in $\overline{BG_3}(G)$. Hence, this case is not possible.

**Case 3:** ue is a dominating edge, where $u \in V(G)$ and $e \in E(G)$.

Since ue is a dominating edge in $\overline{BG_3}(G)$, $e \in E(G)$, u is not incident with e in G. If u is isolated, then diam(G) > 2. If u is not isolated, there exists an edge $e_1$ incident with u, this must be dominated by e in $\overline{BG_3}(G)$. Therefore, e must be adjacent to $e_1$ and not incident with u. If $deg_G$ u is more than 2, then ue cannot be a dominating edge. Therefore, ue is a dominating edge only when $deg_G\, u \leq 2$.

Hence, by case1, $\overline{BG_3}(G)$ has a dominating edge when diam(G) $\geq 2$. By case 3, when diam(G) = 1, $\overline{BG_3}(G)$ has a dominating edge only when $deg_G\, u \leq 2$. That is when $G = K_3$. If $G = K_n$, n > 3, $\overline{BG_3}(G)$ has no dominating edge. So, $\overline{BG_3}(G)$ has a dominating edge for $G \neq K_n$, n > 3. (when n = 2, $BG_3(G)$ is disconnected).

**Theorem 2.7:**

Let G be a (p, q) graph (non-trivial) such that $\overline{BG_3}(G)$ is connected. Then $\overline{BG_3}(G)$ is geodetic if and only if $G = K_2 \cup K_1$.

**Proof:**

$\overline{BG_2}(G)$ is a subgraph of $\overline{BG_3}(G)$ and $\overline{BG_2}(G)$ is geodetic if and only if $G = K_3$ or $K_2 \cup K_1$. But, $\overline{BG_3}(K_3)$ is not geodetic. Hence the theorem is proved.

**Theorem 2.8:**

$BG_3(G)$ has $C_5$ as induced subgraph if and only if G has $P_4$ as subgraph.

**Proof:**

Suppose $BG_3(C_5)$ has $C_5$ as induced subgraph.

**Case 1:** All the vertices of $C_5$ in $BG_3(G)$ are line vertices.

Let $e_1\, e_2\, e_3\, e_4\, e_5\, e_1$ form an induced $C_5$ in $BG_3(G)$. Hence in G, $e_1$ is adjacent with $e_3$, $e_4$; $e_2$ is adjacent with $e_4$, $e_5$; $e_3$ is adjacent with $e_1$, $e_5$; $e_4$ is adjacent with $e_1$, $e_2$; $e_5$ is adjacent with $e_2$, $e_3$. Therefore, G contains $C_5$ as subgraph.

**Case 2:** $C_5$ in $BG_3(G)$ contains four line vertices.

Let $v_1\, e_1\, e_2\, e_3\, e_4\, v_1$ form an induced $C_5$ in $BG_3(G)$. In this case, G has $P_5$ as subgraph.

**Case 3:** $C_5$ in $BG_3(G)$ has three line vertices.

Let $v_1\, e_1\, v_2\, e_2\, e_3\, v_1$, form an induced $C_5$ in $BG_3(G)$. $e_1$ is incident with $v_1$, $v_2$ in G; $e_2$ is incident with $v_2$; $e_1$, $e_2$ are adjacent; $e_3$ is not adjacent to $e_2$ and adjacent to $e_1$. Therefore, G contains $P_4$ as subgraph. Other cases are not possible.

Combining all these, it is seen that $BG_3(G)$ has $C_5$ as induced subgraph if and only if G has $P_4$ as subgraph.

**Theorem 2.9:**

$BG_3(G)$ has no induced $C_5$ if and only if the components of G are $nK_1$, $nK_2$, $K_3$ or $K_{1,n}$, $n \geq 2$.

**Theorem 2.10:**

$BG_3(G)$ has $C_6$ as induced subgraph if and only if G has $C_3$ or $\theta$-graph as subgraph.

**Remark 2.3:**

$BG_3(G)$ cannot have $C_n$, $n > 7$ as induced subgraph.

**Theorem 2.11:**

$BG_3(G)$ has $P_4$ as induced subgraph whenever $G \neq K_2, K_2 \cup nK_1$.

**Proof:**

Assume $G \neq K_2 \cup nK_1$. Since G is non-trivial, G has atleast two edges.

**Case 1:** $e_1, e_2 \in E(G)$ are adjacent edges of G.

In G, $e_1, e_2$ are incident at $v \in V(G)$. Hence, in $BG_3(G)$, $v_1\ e_1\ v\ e_2$ is an induced $P_4$.

**Case 2:** $e_1, e_2 \in E(G)$ are non-adjacent edges of G.

Let $e_1 = u_1v_1 \in E(G)$, $e_2 = u_2v_2 \in E(G)$. In $BG_3(G)$, $\{u_1\ e_1\ e_2\ u_2\}$ forms an induced $P_4$. Hence the theorem is proved.

**Theorem 2.12:**

$\overline{BG_3(G)}$ has $C_4$ as induced subgraph, whenever $G \neq nK_2$ or $nK_2 \cup mK_1$.

**Proof:**

Let G is be a non-trivial graph and $G \neq nK_2$. Then $\Delta(G) \geq 2$. Hence, there exists atleast two adjacent edges $e_1, e_2 \in E(G)$. Let $e_1 = vv_1$, $e_2 = vv_2$. In $\overline{BG_3(G)}$, $\{v_1\ e_1\ e_2\ v_2\ v_1\}$ forms an induced $C_4$. This proves the theorem.

**Lemma 2.2:**

$\overline{BG_3(G)}$ contains $C_5$ as induced subgraph if and only if G contains $P_4$ as subgraph.

**Proof:**

Assume $\overline{BG_3(G)}$ has $C_5$ as induced subgraph. $C_5$ cannot have more than two point vertices.

**Case 1:** $C_5$ in $\overline{BG_3}(G)$ has only one point vertex.

Let $v_1$ $e_1$ $e_2$ $e_3$ $e_4$ $v_1$ be a $C_5$ in $\overline{BG_3}(G)$. In G, $v_1$ is not incident with $e_1$, $e_4$ and $e_2$, $e_3$ are incident with $v_1$. Also, $e_1$ is adjacent to $e_2$ and $e_2$ is adjacent to $e_3$; $e_3$ is adjacent to $e_4$. Therefore, G has induced $P_5$ as subgraph.

**Case 2:** $C_5$ in $\overline{BG_3}(G)$ has only two point vertices.

Let $v_1$ $v_2$ $e_1$ $e_2$ $e_3$ $v_1$ be a $C_5$ in $\overline{BG_3}(G)$. In G, $e_2$ is incident with $v_1$, $v_2$; $e_1$ is incident with $v_1$; $e_3$ incident with v1; e1, e3 cannot be adjacent. Hence, $P_4$ is a subgraph of G. Other cases are not possible

**Case 3:** $C_5$ in $\overline{BG_3}(G)$ contains no point vertex.

$e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_1$ form a $C_5$ in $\overline{BG_3}(G)$. In this case, G contains a $C_5$ as subgraph. Combining all these cases, $\overline{BG_3}(G)$ has $C_5$ as induced subgraph if and only G has $P_4$ as subgraph.

**Remark 2.4:**

If G is connected and has more than three vertices and $G \neq K_{1,n}$, then G has $P_4$ as subgraph.

**Theorem 2.13:**

$\overline{BG_3}(G)$ has no induced $C_5$ if and only if $G = K_{1,n}$, $K_3$, $K_{1,2}$, $K_2$.

**Theorem 2.14:**

$\overline{BG_3}(G)$ has $C_n$, $n \geq 6$ as induced subgraph if and only if G has $C_n$, $n \geq 6$ as subgraph.

**Theorem 2.15:**

$BG_3(G)$ is perfect if and only if the components of G are $nK_1$, $nK_2$, $K_3$ or $K_{1,n}$, $n \geq 2$.

**Theorem 2.16:**

$BG_3(G)$ has $C_5$ as induced subgraph if and only if G has $P_4$ as subgraph.

**Proof:**

Suppose $BG_3(C_5)$ has $C_5$ as induced subgraph.

**Case 1:** All the vertices of $C_5$ in $BG_3(G)$ are line vertices.

Let $e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_1$ form an induced $C_5$ in $BG_3(G)$. Hence in G, $e_1$ is adjacent with $e_3$, $e_4$; $e_2$ is adjacent with $e_4$, $e_5$; $e_3$ is adjacent with $e_1$, $e_5$; $e_4$ is adjacent with $e_1$, $e_2$; $e_5$ is adjacent with $e_2$, $e_3$. Therefore, G contains $C_5$ as subgraph.

**Case 2:** $C_5$ in $BG_3(G)$ contains four line vertices.

Let $v_1$ $e_1$ $e_2$ $e_3$ $e_4$ $v_1$ form an induced $C_5$ in $BG_3(G)$. In this case, G has $P_5$ as subgraph.

**Case 3:** $C_5$ in $BG_3(G)$ has three line vertices.

Let $v_1$ $e_1$ $v_2$ $e_2$ $e_3$ $v_1$, form an induced $C_5$ in $BG_3(G)$. $e_1$ is incident with $v_1$, $v_2$ in G; $e_2$ is incident with $v_2$; $e_1$, $e_2$ are adjacent; $e_3$ is not adjacent to $e_2$ and adjacent to $e_1$. Therefore, G contains $P_4$ as subgraph. Other cases are not possible.

Combining all these, it is seen that $BG_3(G)$ has $C_5$ as induced subgraph if and only if G has $P_4$ as subgraph.

**Theorem 2.17:**

$BG_3(G)$ has no induced $C_5$ if and only if the components of G are $nK_1$, $nK_2$, $K_3$ or $K_{1,n}$, $n \geq 2$.

**Theorem 2.18:**

$BG_3(G)$ has $C_6$ as induced subgraph if and only if G has $C_3$ or $\theta$-graph as subgraph.

**Remark 2.5:**

$BG_3(G)$ cannot have $C_n$, $n > 7$ as induced subgraph.

**Theorem 2.19:**

$BG_3(G)$ has $P_4$ as induced subgraph whenever $G \neq K_2$, $K_2 \cup nK_1$.

**Proof:**

Assume $G \neq K_2 \cup nK_1$. Since G is non-trivial, G has at-least two edges.

**Case 1:** $e_1$, $e_2 \in E(G)$ are adjacent edges of G.

In G, $e_1$, $e_2$ are incident at $v \in V(G)$. Hence, in $BG_3(G)$, $v_1$ $e_1$ v $e_2$ is an induced $P_4$.

**Case 2:** $e_1$, $e_2 \in E(G)$ are non-adjacent edges of G.

Let $e_1 = u_1v_1 \in E(G)$, $e_2 = u_2v_2 \in E(G)$. In $BG_3(G)$, $\{u_1\ e_1\ e_2\ u_2\}$ forms an induced $P_4$. Hence the theorem is proved.

**Theorem 2.20:**

$\overline{BG_3(G)}$ has $C_4$ as induced subgraph, whenever $G \neq nK_2$ or $nK_2 \cup mK_1$.

**Proof:**

Let G is be a non-trivial graph and $G \neq nK_2$. Then $\Delta(G) \geq 2$. Hence, there exists atleast two adjacent edges $e_1$, $e_2 \in E(G)$. Let $e_1 = vv_1$, $e_2 = vv_2$. In $\overline{BG3(G)}$, $\{v_1\ e_1\ e_2\ v_2\ v_1\}$ forms an induced $C_4$. This proves the theorem.

**Lemma 2.2:**

$\overline{BG_3(G)}$ contains $C_5$ as induced subgraph if and only if G contains $P_4$ as subgraph.

**Proof:**

Assume $\overline{BG_3(G)}$ has $C_5$ as induced subgraph. $C_5$ cannot have more than two point vertices.

**Case 1:** $C_5$ in $\overline{BG_3(G)}$ has only one point vertex.

Let $v_1$ $e_1$ $e_2$ $e_3$ $e_4$ $v_1$ be a $C_5$ in $\overline{BG_3(G)}$. In G, $v_1$ is not incident with $e_1$, $e_4$ and $e_2$, $e_3$ are incident with $v_1$. Also, $e_1$ is adjacent to $e_2$ and $e_2$ is adjacent to $e_3$; $e_3$ is adjacent to $e_4$. Therefore, G has induced $P_5$ as subgraph.

**Case 2:** $C_5$ in $\overline{BG_3(G)}$ has only two point vertices.

Let $v_1$ $v_2$ $e_1$ $e_2$ $e_3$ $v_1$ be a $C_5$ in $\overline{BG_3(G)}$. In G, $e_2$ is incident with $v_1$, $v_2$; $e_1$ is incident with $v_1$; $e_3$ incident with v1; e1, e3 cannot be adjacent. Hence, $P_4$ is a subgraph of G. Other cases are not possible

**Case 3:** $C_5$ in $\overline{BG_3(G)}$ contains no point vertex.

$e_1$ $e_2$ $e_3$ $e_4$ $e_5$ $e_1$ form a $C_5$ in $\overline{BG_3(G)}$. In this case, G contains a $C_5$ as subgraph. Combining all these cases, $\overline{BG_3(G)}$ has $C_5$ as induced subgraph if and only G has $P_4$ as subgraph.

**Remark 2.6:**

If G is connected and has more than three vertices and $G \neq K_{1,n}$, then G has $P_4$ as subgraph.

The following theorems can be easily proved as in the previous theorem.

**Theorem 2.21:**

$\overline{BG_3(G)}$ has no induced $C_5$ if and only if $G = K_{1,n}$, $K_3$, $K_{1,2}$, $K_2$.

**Theorem 2.22:**

$\overline{BG_3(G)}$ has $C_n$, $n \geq 6$ as induced subgraph if and only if G has $C_n$, $n \geq 6$ as subgraph.

**Theorem 2.23:**

$BG_3(G)$ is perfect if and only if the components of G are $nK_1$, $nK_2$, $K_3$ or $K_{1,n}$, $n \geq 2$.

## Covering numbers of $BG_3(G)$ and $\overline{BG_3}(G)$

**Proposition 2.1:**

$\chi(BG_3(G)) = \chi(\overline{L}(G))+1$ or $\chi(\overline{L}(G))$

**Proof:**

In $BG_3(G)$, each line vertex is adjacent to exactly two point vertices and no other vertex is adjacent to exactly these two point vertices. Also, the induced subgraph formed by the set of all point vertices is totally disconnected.

Therefore, $\chi(BG_3(G)) = \chi(\overline{L}(G))+1$ or $\chi(\overline{L}(G))$.

**Proposition 2.2:**

$$\omega(BG_3(G)) = \omega(\overline{L}(G)) = \begin{cases} \beta_1(G) & \text{if } \beta_1(G) \geq 2; \\ 2 & \text{otherwise.} \end{cases}$$

**Proof:**

The induced subgraph formed by the set of all point vertices is totally disconnected and if any two line vertices are adjacent in $BG_3(G)$, there is no point vertex which is adjacent to these line vertices. Therefore, $\omega(BG_3(G)) = \omega(\overline{L}(G))$ if $\beta_1(G) \geq 2$, 2 otherwise.

**Proposition 2.3:**

$\theta(BG_3(G)) \geq p$.

**Proof:**

(1) If $p \geq q$. Consider an edge joining a point vertex and a line vertex. In this way, there are $qK_2'$s and $(p-q)$ remaining $K_1'$s. Therefore, $\theta(BG_3(G)) = p$.

(2) If $q > p$. As in the previous case, there are $pK_2'$s. Consider the remaining $q-p$ line vertices.

Let $k = \min_{G_1} \{\theta(G_1): G_1 \subseteq \overline{L}(G)\}$ containing $q-p$ vertices. Hence, $\theta(BG_3(G)) = p+k$.

Thus, $\theta(BG_3(G)) \geq p$.

**Proposition 2.4:**

$\beta_o(BG_3(G)) = p$.

**Proof:**

Obvious, since $V(G)$ is independent in $BG_3(G)$, which is a maximum independent set and each line vertex is adjacent to its incident point vertices.

**Remark 2.7:**

$\alpha_o(BG_3(G)) = q$ (since $\alpha_o(BG_3(G))+\beta_o(BG_3(G)) = p+q$).

**Proposition 2.5:**

(1) $\alpha_1(BG_3(G)) \geq p$ and

(2) $\beta_1(BG_3(G)) \leq q$.

**Proof of (1):**

**Case 1:** $p \geq q$: To cover all the vertices of $BG_3(G)$, take q edges joining a point vertex to a line vertex (distinct). To cover the remaining $p-q$ point vertices, consider the $p-q$ edges incident with the $p-q$ point vertices. Totally these $q+(p-q) = p$ edges covers the vertices of $BG_3(G)$. Therefore, $\alpha_1(BG_3(G)) = p$.

**Case 2:** $q > p$: Take p edges joining a point vertex to a line vertex (distinct). To cover the remaining $q-p$ line vertices, consider a line cover for $G_1$, where $G_1$ is the induced subgraph formed by these $q-p$ vertices. Thus, $\alpha_1(BG_3(G)) \geq p$.

**Proof of (2)** is similar.

**Proposition 2.6:**

$\chi(\overline{BG_3(G)}) \geq p$.

**Proof:**

**Case1:** If $p \geq q$. Take a line vertex e and a point vertex u, incident with e. {u, e} form an independent set. Similarly form q independent sets each containing a line vertex and a point vertex incident with it in G. The remaining $p-q$ point vertices form a $K_{p-q}$. Therefore, $\chi(\overline{BG_3(G)}) \leq q+(p-q) = p$.

**Case 2:** If $p < q$. As in the previous case, form p independent sets. There are $q-p$ remaining line vertices. Hence, $\chi(\overline{BG_3(G)}) \geq p$.

**Proposition 2.7:**

(1) $\omega(\overline{BG_3(G)}) = p$.

(2) $\theta(\overline{BG_3(G)}) = \theta(L(G))$ or $1+\theta(L(G))$

**Proof of (1):**

$\omega(G) = \beta_o(\overline{G})$, by Theorem 1.4. Therefore, $\omega(\overline{BG_3(G)}) = \beta_o(BG_3(G)) = p$ (proved). Hence, $\omega(\overline{BG_3(G)}) = p$.

**Proof of (2):**

Since $K_p$ is an induced subgraph, the minimum number of complete subgraphs needed to cover the vertices of $\overline{BG_3(G)}$ is at most $1+\theta(L(G))$.

If the point vertices are divided into $V_1, V_2, ..., V_k$ such that they form complete subgraphs with $E_1, E_2, ..., E_m$, $m = \theta(L(G))$. Then $\theta(\overline{BG_3}(G)) = \theta(L(G))$, otherwise $\theta(L(G))+1$. Therefore, $\theta(\overline{BG_3}(G)) = \theta(L(G))$ or $\theta(L(G))+1$.

**Example 2.1:**

Consider $G = P_7$. Let $v_1, v_2, ..., v_7$ denote the vertices of G and $e_{ij} = v_iv_j$. In $BG_3(G)$, consider $(v_1, v_2, v_3, e_{56}, e_{67})$, $(v_4, v_5, v_6, v_7, e_{12}, e_{23})$, $(e_{34}, e_{45})$. The induced subgraph formed by these vertices are complete and cover the vertices of $\overline{BG_3}(G)$. Therefore, $\theta(\overline{BG_3}(G)) = 3 = \theta(L(G))$.

**Proposition 2.8:**

(1) $\alpha_o(\overline{BG_3}(G)) = p+\alpha_o(L(G)) = p+q-\beta_o(L(G))$

$$= \begin{cases} p+q-\beta_1(G) & \text{if } G \neq K_{1,n}, K_3 \\ p+q-2 & \text{if } G = K_{1,n} \text{ or } K_3 \end{cases}$$

(2) $\beta_o(\overline{BG_3}(G)) = \beta_o(L(G)) = \begin{cases} \beta_1(G) & \text{if } \beta_1(G) \geq 2, \text{ that is } G \neq K_{1,n}, K_3 \\ 2, & \text{if } G = K_{1,n} \text{ or } K_3. \end{cases}$

**Proof of (1):**

The p point vertices covers all the edges in $K_p$ and edges incident with a point vertex and line vertex. $\alpha_o(L(G))$ line vertices covers all the edges in L(G) if $G \neq K_{1,n}, K_3$. Hence, $\alpha_o(\overline{BG_3}(G)) = p+\alpha_o(L(G)) = p+q-\beta_1(G)$ if $G \neq K_{1,n}, K_3$.

If $G = K_3$, $\alpha_o(\overline{BG_3}(G)) = 4 = p+q-2$ and If $G = K_{1,n}$, then $\alpha_o(\overline{BG_3}(G)) = n+n-1 = 2n-1 = p+q-2 = p+(p-1)-2 = 2p-3$.

**Proof of (2):** $\beta_o(\overline{BG_3}(G)) = p+q-\alpha_o(\overline{BG_3}(G)) = \beta_1(G)$ or 2.

**Remark 2.8:**

Set of all point vertices form a complete subgraph of $\overline{BG_3}(G)$. If $e = uv \in E(G)$, then {u, e}, {v, e} are independent in $\overline{BG_3}(G)$. Therefore, the maximum number of mutually non-adjacent vertices of $\overline{BG_3}(G)$ is the maximum number of mutually non-adjacent edges of G if $\beta_1(G) \geq 2$.

**Proposition 2.9:**

(1) $\alpha_1(\overline{BG_3}(G)) \leq \begin{cases} \min\{q+\lfloor(p-q+1)/2\rfloor), \lfloor(p+1)/2\rfloor+\alpha_1(L(G))\}, \text{ if } p \geq q. \\ \min\{p+k, \lfloor(p+1)/2\rfloor+\alpha_1(L(G))\}, \text{ if } p < q. \end{cases}$

(2) $\beta_1(\overline{BG_3}(G)) \geq \begin{cases} \max\{q+(p-q)/2, (p/2)+\beta_1(L(G))\}, \text{ if } p \geq q. \\ \max\{p+k_1, (p/2)+\beta_1(L(G))\}, \text{ if } p < q. \end{cases}$

**Proof of (1):**

**Case 1:** $p \geq q$. Consider the set $D_1$ of q edges each joining a line vertex to a point vertex (distinct). Consider the set of p−q point vertices. Consider a line cover $D_2$ for $K_{p-q}$. $D = D_1 \cup D_2$ is a line cover for $\overline{BG_3(G)}$ and $|D| = q+\lfloor (p-q+1)/2 \rfloor$. Hence, $\alpha_1(\overline{BG_3(G)}) \leq q+\lfloor (p-q+1)/2 \rfloor$. Consider the set of point vertices. Since they form a complete subgraph, $\lfloor (p+1)/2 \rfloor$ edges cover all the point vertices and $\alpha_1(L(G))$ edges cover all the line vertices.

Hence, $\alpha_1(\overline{BG_3(G)}) < \lfloor (p+1)/2 \rfloor + \alpha_1(L(G))$.

Hence, result (1) follows if $p \geq q$.

**Case 2:** $p < q$. Consider the set $D_1$ of p edges each joining a line vertex to a point vertex (distinct). Consider the remaining q−p line vertices.

Let $k = \min_{G_1} \alpha_1(G_1)$, where $G_1$ is the subgraph induced by these q−p line vertices.

Therefore, $\alpha_1(\overline{BG_3(G)}) \leq \min \{(p+k), \lfloor (p+1)/2 \rfloor + \alpha_1(L(G)\}$.

**Proof of (2):** Follows from (1).

## Conclusion:
Other properties such as traversability, connectivity, edge partition and domination parameters are studied and submitted elsewhere.

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