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Further Results on Eccentric domination in Graphs

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Abstract: *A subset D of the vertex set V(G) of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every v ∈ V–D, there exists at least one eccentric point of v in D. The minimum of the cardinalities of the eccentric dominating sets of G is called the eccentric domination number* $\gamma_{ed}(G)$ *of G. In this paper, some new bounds for eccentric domination number of a graph is studied.*

Key words*: Eccentric dominating set, Eccentric domination number.*

1. Introduction

Let G be a finite, simple, undirected graph on n vertices with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [4], Buckley and Harary [1].

Definition 1.1 Let G be a connected graph and u be a vertex of G. The **eccentricity** e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max {d(u, v) : u ∈ V}. The **radius** r(G) is the minimum eccentricity of the vertices, whereas the **diameter** diam(G) is the maximum eccentricity. For any connected graph G, r(G) ≤ diam(G) ≤ 2r(G). v is a central vertex if e(v) = r(G). The **center C(G)** is the set of all central vertices. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a peripheral vertex if e(v) = d(G). The **periphery P(G)** is the set of all peripheral vertices.

For a vertex v, each vertex at a distance e(v) from v is an **eccentric vertex**. **Eccentric set of a vertex v** is defined as E(v) = {u ∈ V(G) / d(u, v) = e(v)}.

Definition 1.2 [4, 9] A set S ⊆ V is said to be a **dominating set** in G, if every vertex in V−S is adjacent to some vertex in S. A dominating set D is an **independent dominating set**, if no two vertices in D are adjacent that is D is an independent set. A dominating set

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D is a **connected dominating set**, if < D > is a connected subgraph of G. A set D $\subseteq$ V(G) is a **global dominating set**, if D is a dominating set in G and $\overline{G}$.

The **domination number** γ of G is defined to be the minimum cardinality of a dominating set in G. Similarly, we can define the connected domination number γ_c, independent domination number γ_i and **global domination number** γ_g.

A dominating set with cardinality γ(G) is known as a **γ-dominating set or γ-set .**

Definition 1.3 [6]**:** A set D $\subseteq$ V(G) is an **eccentric dominating set** if D is a dominating set of G and for every v $\in$ V – D, there exists at least one eccentric point of v in D.

An eccentric dominating set D is a **minimal eccentric dominating set** if no proper subset D'' $\subseteq$ D is an eccentric dominating set.

Definition 1.4 [7]**:** The **eccentric domination number** γ_{ed}**(G)** of a graph G equals the minimum cardinality of an eccentric dominating set. An eccentric dominating set with cardinality γ_{ed}(G) is known as a **γ_{ed}- set**.

Definition 1.5 [7]**:**

Let S $\subseteq$ V(G). Then S is known as an **eccentric point set of G** if for every v $\in$ V–S, S has at least one vertex u such that u $\in$ E(v). An eccentric point set S of G is a **minimal eccentric point set** if no proper subset S' of S is an eccentric point set of G. S is known as a **minimum eccentric point set** if S is an eccentric point set with minimum cardinality. The minimum cardinality of an eccentric point set of G denoted as **e(G)** is known as **eccentric number of G**.

Theorem 1.1 [9]**:** If G is a connected graph and n$\geq$ 3, then γ_c(G) = n$-\varepsilon_T$(G), where ε_T(G) is the maximum number of pendent edges in any spanning tree of G.

Theorem 1.2 [8]**:** If H is any self-centered unique eccentric point graph then every vertex of H is an eccentric vertex.

Theorem1.3[7]**:** If r(G) $>$ 1, then γ_{ed} set of G is also a dominating set of $\overline{G}$.

2. Bounds of $\gamma_{ed}(G)$ in terms of number of vertices of G

Following theorem gives the bound of $\gamma_{ed}(G)$ in terms of number of vertices of the graph G.

Theorem 2.1: If G is a connected graph with n vertices then $\gamma_{ed}(G) \leq \lfloor 2n/3 \rfloor$.

Proof: If D is a minimum eccentric dominating set, then for v $\in$ V−D there exists u $\in$ D and w $\in$ D such that u is adjacent to v in G and w is eccentric to v in G. Hence D contains atmost 2n/3 vertices. This proves the theorem.

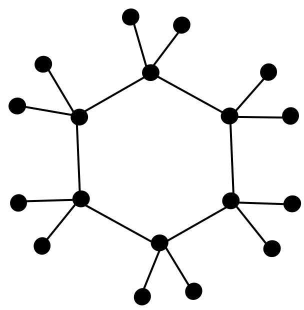

Figure 2.1

Remark 2.1: This upper bound is sharp, since for G = $C_6 \circ 2K_1$ in Figure 2.1, $\gamma_{ed}(G)$ = 12 = 2n/3. Here H = C_6 is an unique eccentric point graph, but G is not an unique eccentric point Graph. Also, $\gamma_{ed}(G) = \gamma(G)+e(G)$.

Theorem 2.2: If H is any self-centered unique eccentric point graph with m vertices and G = $H \circ 2K_1$ then $\gamma_{ed}(G)$ = 2n/3 = 2m.

Proof: If H is any self-centered unique eccentric point graph, then every vertex of H is an eccentric vertex. Hence m is even and G has 3m vertices. Let $v_1, v_2, v_3, v_4, \ldots, v_m$ represent the vertices of H, and $\{v_i', v_i''\}$ for i = 1, 2, 3 ..., m be the vertices of m copies of $2K_1$. Then in G, v_i', v_i'' are adjacent to v_i and if v_j is the eccentric vertex of v_i in H, then v_i', v_i'' are eccentric vertices of v_j in G and v_j', v_j'' are the eccentric vertices of v_i. It is clear that $\{v_1, v_2, v_3, v_4, \ldots, v_m\} \cup \{v_1', v_2', v_3', v_4', \ldots, v_m'\}$ and $\{v_1', v_2', v_3', v_4', \ldots, v_m'\} \cup \{v_1'', v_2'', v_3'', v_4'', \ldots, v_m''\}$ are minimum eccentric dominating sets of G. Hence, $\gamma_{ed}(G)$ = 2n/3 = 2m.

Remark 2.2: If H is self-centered but not a unique eccentric point graph then γ_{ed} of G = $H \circ 2K_1$ need not be 2n/3. For example, when G = $C_5 \circ 2K_1$, $\gamma_{ed}(G) = 8 < 2n/3 = 10$.

Bounds of $\gamma_{ed}(G)$ interms of number of vertices and eccentric number of G.

Let us assume that G be a graph on n vertices, with eccentric number e(G), Let G' be a graph on n+e(G) vertices obtained from G as follows:

Let $S = \{u_1, u_2, ..., u_{e(G)}\}$ be the minimum eccentric point set of G. Attach a new vertex x_1 to u_1 by an edge and x_2 to u_2 by an edge, ..., $x_{e(G)}$ to $u_{e(G)}$ by an edge. Denote the new graph obtained as G'. Then $|V(G')| = n+e(G) = |V(G)|+e(G)$ and $|E(G')| = |E(G)|+e(G)$. Since in G', degree of each x_i's is one, there is a minimum dominating set D of G' containing the set S. This dominating set D is clearly an eccentric dominating set of G. Therefore, $\gamma_{ed}(G) \leq \gamma(G') \leq (n+ e(G))/2$. Hence the following theorem.

Theorem: 2.3: For any connected graph G on n vertices, $\gamma_{ed}(G) \leq (n+ e(G))/2$.

Theorem 2.4: If G is a caterpillar such that each non-pendent vertices is of degree three then $\gamma_{ed}(G) = n/2+1$.

Proof: Since degree of each non-pendent vertex is three, G is of the following form.

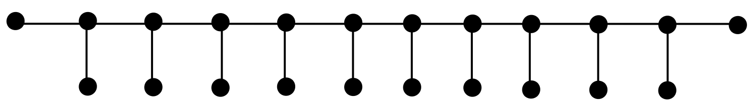

Figure 2.2

It is clear that, $\gamma_{ed}(G) = n/2+1$.

Theorem 2.5: If H is a connected graph with m vertices and eccentric number e(H), then eccentric domination number of $G = H \circ kK_1$, $k > 1$ is not greater than m+e(H).

Proof: Since $k > 1$, vertices of H is a minimum dominating set and $\gamma(G) = m$. Let S be a minimum eccentric set of H. Then $|S| = e(H)$. Let $S = \{v_1, v_2, v_3, \ldots, v_{e(H)}\}$. Consider $S' = \{v_1', v_2', v_3', \ldots, v_{e(H)}'\}$, where v_i' is a vertex of kK_1 which is adjacent to v_i in G. Then $V(H) \cup S'$ is a minimum eccentric dominating set of G. Hence $\gamma_{ed}(G) \leq m+e(H)$.

Graphs with $\gamma_{ed}(G) = \lfloor n/2 \rfloor +1$:

1. Let $\mathbf{G}_1$ be the collection of graphs C_5, C_5+e, P_5 and $K_1+K_1+K_1+2K_1$.

2. Let $\mathbf{G}_2$ be collection of caterpillars such that each non-pendent vertices is of degree three.

3. For any connected self-centered unique eccentric point graph H, let $\mathbf{G}_3$ denote the collection of connected graphs G, each of which can be formed from $H \circ K_1$ by adding a new vertex x and edges joining x to one or more of vertices of H such that eccentricities of vertices of $H \circ K_1$ are not affected.

4. For any connected graph H, consider $G = H \circ K_1$. Let u_1, v_1 be vertices of G which are at distance = diameter G to each other and let u, v be their support vertices respectively. Join a new vertex x to u and a new vertex y to v by edges. Let $\mathbf{G}_4$ denote the collection of such new graphs.

5. Let $X \in \mathbf{G}_1$ and $Y \in \mathbf{G}_2$. Let G(X, Y) be a connected graph which may be formed from X and Y by joining each vertex of $U \subseteq V(X)$ to one or more non-pendent vertices of Y such that no set with fewer than $\gamma(X)$ vertices of X dominate $V(X)-U$. Let $\mathbf{G}_5$ be collection of such graphs.

Theorem 2.6: For the graphs in G_1, G_2, G_3, G_4 and G_5 $\gamma_{ed}(G) = \lfloor n/2 \rfloor + 1$.

Proof: It can be easily verified that $\gamma_{ed}(G) = \lfloor n/2 \rfloor + 1$.

Theorem 2.7: If $r(G) > 2$, then γ_{ed}-set of G is also an eccentric dominating set of $\overline{G}$

Proof: Let D be a γ_{ed}-set of G. Then any $v \in V-D$ has an adjacent vertex $u \in D$ and an eccentric vertex $w \in D$. Since $r(G) \geq 3$, $\overline{G}$ is self-centered of diameter two and in $\overline{G}$ for $v \in V-D$, $w \in D$ is adjacent to v and $u \in D$ is eccentric to v. Hence D is an eccentric dominating set of $\overline{G}$ also.

Theorem 2.8: For any positive integer m there exists a graph G with $\gamma(G) = 1$ and $\gamma_{ed}(G) = m$.

Proof: Consider $H = K_{2m} - 1$ factor. $G = H + K_1$. Then $\gamma(G) = 1$ and $\gamma_{ed}(G) = m$.

Theorem 2.9: For any positive integer m there exists a graph G with $\gamma(G) = 2$ and $\gamma_{ed}(G) = m$.

Proof: Consider $G = K_{2m} - 1$ factor. Then $\gamma(G) = 2$ and $\gamma_{ed}(G) = m$.

Here we list some problems related to $\gamma_{ed}(G)$

Problem 1: If G is a connected graph with $\delta(G) \geq 2$, then $\gamma_{ed}(G) \leq \lfloor n/2 \rfloor + 1$.

Problem 2: If G is a connected graph with $\delta(G) \geq 2$, then $\gamma_{ed}(G) = \lfloor n/2 \rfloor + 1$ if and only if G is C_5 or C_5+e.

Problem 3: If $\gamma_{ed}(G) = 2n/3 = 2m$, then $G = H \circ 2K_1$, where H is any self-centered unique eccentric point graph with m vertices.

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