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Critical Concept in Strong Convex Dominating Set of Graphs

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Abstract: *In a graph G = (V, E), a set $D \subset V$ is a strong convex set if $d_{<D>}(u,v) = d_G(u,v)$ for any two vertices u, v in D and the induced graph <D> consists of all shortest path connecting every pair of vertices of D. A strong convex set is called as a strong convex dominating (SCD) set if each vertex of V-D is adjacent to at least one vertex in D. The strong convex domination number $\gamma_{sc}(G)$ is the smallest order of a strong convex dominating set of G and the codomination number of G, denoted by $\gamma_{sc}(\overline{G})$, is the strong convex domination number of its complement. In this paper, we found various bounds of these parameters and characterized the graphs, for which bounds are attained.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self-centered, neighbourhood, strong convex set, strong convex dominating set.*

1. Introduction

Graphs discussed in this paper are undirected and simple. Unless otherwise stated the graphs which we consider are connected graphs only. For a graph G, let V(G) and E(G) denote its vertex and edge set respectively and p and q denote the cardinality of those sets respectively. The *degree* of a vertex v in a graph G is denoted by $\deg_G(v)$. The minimum and maximum degree in a graph is denoted by δ and Δ respectively. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max\{d_G(u, v): u \in V(G)\}$. If there is no confusion, we simply use the notion deg (v), $d(u, v)$ and $e(v)$ to denote degree, distance and eccentricity respectively for the concerned graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When these two are equal, the graph is called *self-centered* graph with radius r, equivalently is *r self-centered*. A vertex u is said to be an *eccentric vertex* of v in a graph G, if $d(u, v) = e(v)$. In general, u is called an eccentric vertex, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighbourhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighbourhood* of v. A set S of edges in a graph is

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said to be *independent* if no two of the edges in S are adjacent. An edge e = (u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

The concept of domination in graphs was introduced by Ore. A set $D \subseteq V(G)$ is called *dominating set* of G if every vertex in V(G)-D is adjacent to some vertex in D. D is said to be a *minimal* dominating set if D-{v} is not a dominating set for any $v \in D$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called *connected* (*independent*) dominating set if the induced subgraph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D.

A cycle of D of a graph G is called a *dominating cycle* of G, if every vertex in V–D is adjacent to some vertex in D. A dominating set D of a graph G is called a *clique dominating set* of G if <D> is complete. A set D is called an *efficient dominating set* of G if every vertex in V – D is adjacent to exactly one vertex in D. A set $D \subseteq V$ is called a *global dominating set* if D is a dominating set in G and $\overline{G}$. A set D is called a *restrained dominating set* if every vertex in V(G)-D is adjacent to a vertex in D and another vertex in V(G)-D. A dominating set D with $d_{<D>}(u,v) = d_G(u,v)$ for any two vertices u, v in D is called as a weak convex dominating (WCD) set. By $\gamma_c, \gamma_i, \gamma_t, \gamma_o, \gamma_k, \gamma_e, \gamma_g$, γ_{wc} and γ_r, we mean the minimum cardinality of a connected dominating set, independent dominating set, total dominating set, cycle dominating set, clique dominating set, efficient dominating set, global dominating set, weak convex dominating and restrained dominating set respectively.

In this paper, we introduce a new dominating set called strong convex dominating set of a graph through which we analyse the properties of the graph such as variation in radius and diameter of the graph. We find upper and lower bounds for the new domination number in terms of various already known parameters. Also we obtain several interesting properties like Nordhaus-Gaddum type results relating the graph and its complement.

2. Prior Results

Theorem A:[14] Let G be any graph and D be any dominating set of G. Then $|V\text{-}D| \leq \sum_{u \in V(D)} \deg(u)$ and equality holds in this relation if and only if D has the following properties :

(i) D is independent.

(ii) For every $u \in V\text{-}D$, There exist a unique vertex $v \in D$ such that $N(u) \cap D = \{v\}$.

3. Main Results

Definition 3.1: A dominating set D is said to be a Strong Convex Dominating (SCD) set if the induced graph <D> consists of all shortest path connecting every pair of vertices of D.

The cardinality of a minimum strong convex dominating set of G is called the strong convex dominating number of G and is denoted by γ_{sc}.

Most of the parameters so far defined on domination in graphs is a subclass of this strong convex domination, because the strong convex dominating set is a dominating distance hereditary set of a graph in which it is defined.

Observation 1: Clearly from the definition, $1 \leq \gamma_{sc} \leq p$.

Observation 2: If G is geodetic, then for any spanning sub graph H, $\gamma_{sc}(G) \leq \gamma_{sc}(H)$.

Observation 3: For any tree T, $\gamma_{sc}(T) = \gamma_c(T)= p\text{-}e$, where e is the number of pendant vertices of T.

Observation 4: Every strong convex dominating set contains a minimal WCD set.

Observation 5: Every strong convex dominating set of a connected graph contains a minimal W.C.D set.

Remark 3.1: Clearly, from observations 4 and 5 we have the relation that $\gamma \leq \gamma_t \leq \gamma_c \leq \gamma_{wc} \leq \gamma_{sc}$.

Proposition 3.1: Let D be any strong convex dominating set of G. Then $|V\text{-}D| \leq \sum_{u \in V(D)} \deg(u)$, for all $u \in D$.

Proof: Let D be any strong convex dominating set of G. Then clearly D is a dominating set of G. Hence from theorem A, we have $|V\text{-}D| \leq \sum_{u \in V(D)} \deg(u)$.

Remark 3.2 : Equality holds for any complete graph.

Proposition 3.2 : $|V\text{-}D| = \sum_{u \in V(D)} \deg(u) \Leftrightarrow |D| = 1$.

Proof: As D is connected dominating set, from theorem A, |D| cannot be greater than one.

Corollary 3.1: If $|V\text{-}D| = \sum_{u \in V(D)} \deg(u) \Leftrightarrow$ there exists a vertex of degree p-1.

Theorem 3.1: Let G be a connected graph. If $\gamma_{sc}(G) = p\text{-}q \Leftrightarrow$ G is isomorphic to $K_{1,r}$.

Proof : Let γ_{sc} = p-q. We know that for unicyclic p-q =0. Therefore, G must be a tree.

$\Rightarrow$ p-q =1

$\Rightarrow$ γ_{sc} = 1

$\Rightarrow$ radius (G) = 1

$\Rightarrow$ G = $K_{1,r}$.

4. Some Bounds on γ_{sc}

Proposition 4.1: [p/Δ+1] $\leq$ γ_{sc}.

Proposition 4.2: Let G be a graph of order p. Then k = γ_{sc}(G) = [p/Δ+1] if and only if γ_{sc}(G) = 1.

Corollary 4.1: Let G be a graph of order p such that γ_{sc}(G) = [p/(Δ+1)]. Then Δ+1 divides p.

Theorem 4.1: Let G be any distance hereditary graph. Then for any spanning sub graph H of G, $\gamma_{sc}(G) \leq \gamma_{sc}(H)$.

Remark 4.1: Usually this inequality is not true for any graph G. There are so many graphs, which have the other inequality.

If d and r are the diameter and radius of the graph G then what about the diameter and radius of the induced graph induced by the strong convex dominating set of G? The following two theorems give the answer for this question.

Theorem 4.2: Let G be a graph and D be a strong convex dominating set. Then the radius of the induced graph <D>, induced by D, is at the lowest by r-1, where r is the radius of G.

Remark 4.2: Clearly, we know that the radius of the induced graph cannot be increased more than the diameter of the original graph.

Corollary 4.2: Let d, r be the diameter and radius of the given graph G and let r_c be the radius of the induced graph induced by the strong convex dominating set of G. Then $r-1 \leq r_c \leq d$.

Theorem 4.3: There exists no graph, which has a strong convex dominating set that induces a sub graph of diameter less than are equal to d-3, where d is the diameter of the original graph.

Theorem 4.4: Let d, d_c be the diameters of the given graph G and the induced sub graph of the strong convex dominating set of G respectively. Then $d-2 \leq d_c \leq d$.
Proof: Proof follows from the fact of Theorem 4.3.

Corollary 4.3: Let G be a connected graph with diameter d. Then $d-1 \leq \gamma_{sc}(G)$.
Proof : Proof follows from Theorem 4.4.

Proposition 4.2: Let cut(G) denote the number of cut vertices of a connected graph G. Then $cut(G) \leq \gamma_{sc}(G)$.

Theorem 4.5: Let G be a self-centered graph of diameter 2. If G has two non-adjacent vertices of degree 2 with a common neighbour of degree 2, then $\gamma_{sc} = 2$ or 3.
Proof: Let G be a self-centered graph of diameter 2, which has two non-adjacent vertices of degree 2 with a common neighbour of degree 2, then G = C_4 or C_5. Hence $\gamma_{sc} = 2$ or 3.

Proposition 4.3: If $\gamma_{sc}(\overline{G}) \geq 3$, then $diam(G) \leq 2$.

Proof: If $\gamma_{sc}(\overline{G}) \geq 3$, then $\overline{G}$ has no dominating edge. This implies that G has diameter less than or equal to 2.

Theorem 4.6: If a graph G is connected and $diam(G) \geq 3$, then $\gamma_{sc}(\overline{G}) = 2$.

Theorem 4.7: If a graph G has $\delta \geq 2$ and girth $g(G) \geq 6$, then $\gamma_{sc}(G) = p$.

Proof : Let D be a γ_{sc} set of G. We know that $\gamma_{sc}(G) \leq p$. If $\gamma_{sc}(G) \neq p$, then $\gamma_{sc}(G) < p$. This implies that $|V-D| \geq 1$. Let $u \in V-D$. Two vertices of D cannot dominate u, as D is strong convex set. Therefore, only one vertex of D dominates u. Since $\delta(G) \geq 2$, there must exists another vertex $v \in V-D$ such that u and v are adjacent. And also u and v are not be dominated by the same vertex of D (if possible, then C_3 arises). Therefore, u and v are dominated by two different vertices say some u' and v' of D respectively. This implies that $d(u', v') \leq 3$ (since length of the path u'uvv' = 3). But d(u', v') cannot be equal to 3, otherwise u, v also must be included in D. Therefore $d(u', v') \leq 2$ only. This implies that there must be a C_4 or C_5 exist in G, which is a contradiction to $g(G) \geq 6$. Hence $\gamma_{sc}(G) = p$.

Remark 4.3: Converse of the above is not true.

Theorem 4.8: For any connected graph G with diameter greater than or equal to 3, $\gamma_{sc}(G) + \gamma_{sc}(\overline{G}) \leq p + 2$, where $\gamma_{sc}(G)$ and $\gamma_{sc}(\overline{G})$ is the cardinality of minimal strong convex set of G and $\overline{G}$ respectively.

Proof : If G is of diameter greater than or equal to 3, then there must exist at least two vertices, which has distance greater than or equal to 3. Then that two vertices form a strong convex dominating set for $\overline{G}$. Hence $\gamma_{sc}(G) + \gamma_{sc}(\overline{G}) \leq p+2$.

Remark 4.4: The cycles of order greater than or equal to 7 attain this bound p+2.

Definition 4.1: A graph is said to be k-strong convex dominating special graph, if it has exactly k-disjoint strong convex dominating sets.

If the whole vertex set of G is the only SCD set of G, then G is said to be a 0-strong convex graph.

Remark 4.5: There are infinite numbers of 0-strong convex dominating graphs, which are not 2-connected.

Proposition 4.4: A graph G is 0-strong convex dominating graph, then $\overline{G}$ is not 0-strong convex dominating graph.

Proof: As the graph has diameter greater than or equal to 3, $\overline{G}$ has a dominating edge and hence $\gamma_{sc}(\overline{G})=2$. Hence the proposition.

Corollary 4.4: There is no graph G such that both G and $\overline{G}$ are 0-strong convex dominating graphs.

Proposition 4.4: If G is a (p, q) - graph, then $q \geq \frac{1}{2}(p + \gamma_{sc})$.

Proof: Clearly from the previous proposition, every pendant vertex of G belongs to V – D. This implies that $\deg(u) \geq 2$ for all $u \in D$.

Therefore,
$$\begin{aligned} 2q &= \sum_{u\in V} \deg(u) \\ &= \sum_{u\in D} \deg(u) + \sum_{u\in V-D} \deg(u) \\ &\geq 2\,|D| + |V - D| \\ &= 2\gamma_{sc} + p - \gamma_{sc} \\ &= \gamma_{sc} + p \end{aligned}$$

Hence $q \geq \frac{1}{2}(p + \gamma_{sc})$.

5. Strong convex domination critical Graphs

Definition 5.1: A Graph is G said to be Strong Convex Domination (SCD) Critical, if for every edge $e \notin E(G)$, $\gamma_{sc}(G+e) < \gamma_{sc}(G)$.

Definition 5.2: A SCD critical graph G is called as k – strong convex domination critical if $\gamma_{sc}(G) = k$.

Proposition 5.1: A Graph G is 1 – SCD critical $\Leftrightarrow G = K_p$.

Proposition 5.2:

If G is 2 - S.C.D critical if and only if G is a 2-W.C.D critical.

Theorem 5.1 :G is 2 –SCD critical if and only if

(i) G is connected

(ii) G has $\gamma_{sc} = 2$ and

(iii) For any two non-adjacent vertices one of them is of degree $|V(G)|$- 2.

Proof: Let G be a 2- SCD critical graph on p vertices. To prove, for any two non-adjacent vertices x and y, one of them is of degree p-2, let us take x, y be any two non-adjacent vertices. Now the inclusion of the edge xy reduces the domination number to one. That is either x or y must dominate all the vertices of G+xy. This implies that either deg(x) or deg(y) must equal to p-1 in G+xy. Hence either deg(x) or deg(y) must be equal to p-2 in G. Proof of the converse is trivial.

Lemma 5.1: Let G be a 2 – SCD critical graph then $\Delta \leq$ p-2 and $\delta \geq$ p-3.

Lemma 5.2: Any 2 –SCD critical graph has diameter equal to two.

Proof: Let x and y be any two non-adjacent vertices of G. Then in G + xy either {x} or {y} will form a dominating set. Without loss of generality, assume that {x} will form a generating set. Then all the neighbors of $N_1(y)$ will be adjacent with x in G+ xy and hence in G also. Therefore, diameter of G must be equal to 2 as it cannot be equal to one.

Theorem 5.2: Any 2 –SCD critical graph is a block.

Proof: Let v be a cut vertex of G and v_1 be the pendent vertex joined with v. Let {u, v} be a dominating edge of G. Then there exists at least one vertex $u_1 \in N_1(u)$ such that $d(u_1,v_1) \geq 3$, a contradiction to the above lemma 5.2. Hence G is a block.

Corollary 5.1: There exists no graph G for which both G and $\overline{G}$ are 2 –SCD critical.

Proof: Without loss of generality, assume that G is a 2 –SCD critical graph. Then by Theorem 5.1, there exists at least one vertex of degree (p - 2). Then degree of that vertex is one in $\overline{G}$. Therefore, by Theorem 5.2, G cannot be 2–SCD critical.

Theorem 5.3: In a 2 – SCD critical graph, all the vertices of degree p-3 will form a clique.

Proof: Let u and v be any two vertices of degree n–3 in a 2– SCD critical graph G. Let u and v be non-adjacent. Now join u and v. Then clearly either {u} or {v} cannot be a dominating set in G + uv, as deg (u) = deg (v) = n – 2, which is a contradiction to G is 2 - SCD critical. Hence the set of vertices of degree p – 3 will form a clique.

Theorem 5.4: Let G be a 2 – S.C.D critical graph. Let v be any vertex of degree n – 2 and let A = {u $\in N_1(v)$ / deg(u) = n- 2} and B = {u $\in N_1(v)$ / deg(u) = n – 3}. Then the following properties hold good:

(i) any vertex of degree n – 2 in $N_2(v)$ must be adjacent to both A and B.

(ii) any vertex in $N_2(v)$ must be adjacent to all of B.

(iii) any vertex of degree n – 3 in $N_2(v)$ must be non-adjacent to exactly one vertex of B.

(iv) $N_2(v)$ is complete.

Theorem 5.5: Let G be a 2 – S.C.D critical graph. Let v be any vertex of degree n – 3 and let A = {u$\in N_1$ (v) / deg(u) = n – 2} and B = {u$\in N_1$ (v) / deg (u) = n – 3}. Then any vertex of $N_2(v)$ must be adjacent to all of A and B.

Theorem 5.6: Let G be a graph on n vertices. G is 2-S.C.D critical and unique eccentric point graph $\Leftrightarrow$ G is (n – 2) – regular graph.

Theorem 5.7 : Any 2 – S.C.D critical graph on n vertices has at least four (n – 2) degree vertices.

Theorem 5.8 : Let G be a 2-S.C.D critical graph, then $2(q–4) \geq (n–4)(n+1)$.

Theorem 5.9 : For any 2–S.C.D critical graph $2q \leq n(n-2)$.

Proposition 5.3: γ_{wc} = 3 if and only if γ_{sc} =3.

Proposition 5.4: If G is 3-S.C.D critical if and only if G is a 3-W.C.D critical.

Lemma 5.3: The diameter of a 3 - S.C.D critical graph is at most 3.

Theorem 5.10: If u is a cut vertex of the 3 - S.C.D critical graph G, then u is adjacent to a pendant vertex of G.

Corollary 5.2: If G is a 3-S.C.D critical graph with $\delta \geq 2$, then G is a block.

Theorem 5.11: If G is a k- SCD critical graph, then no two endpoints of G have a common neighbor.

Proof: Proof is trivial. Let us suppose there exists two end vertices x, y have a common neighbor z. Then G + xy have the same domination number k, which is a contradiction to G is k-S.C.D critical. Hence the proof.

Theorem 5.12: Any k – SCD critical graph has at most k pendant vertices.

Proof: If G has more than k pendant vertices, then from Theorem 5.11, G would have more than k cut vertices. But any dominating set must contain all the cut vertices and hence G cannot be k dominating graph, which is a contradiction to G is k - S.C.D. critical graph. Hence G has at most k pendant vertices.

Corollary 6.3: Any 3- S.C.D critical graph has at most 2 - cut vertices.

Corollary 6.4: Any two cut vertices of a 3 - S.C.D critical graph is non-adjacent.

Corollary 6.5: Any 3- S.C.D critical graph has at most one cut vertex.

Theorem 5.13: If G is a connected 3 - S.C.D critical graph with a cut vertex, then V(G) = A $\cup$ B such that A has only one vertex which is adjacent to only one vertex of B and <B> is a self centered graph of diameter 2.

Remark 5.1:

If v is a cut vertex of a 3-S.C.D critical graph G and u is pendant vertex of G, whose support is v, then

(i) In G-u, $N_1(v)$ is complete

(ii) $N_2(v)$ is complete

(iii) Each vertex $v_i^{'} \in N_1(v)$ in G-u has exactly one vertex $v_i^{''}$ in $N_2(v)$ such that $v_i^{'} v_i^{''} \notin E(G)$.

This implies q = 1 + $\binom{\deg(v)-1}{2} + \binom{p-1-\deg(v)}{2}$.

Theorem 5.14: If S is an independent set of r vertices in the connected 3-S.C.D critical graph G, then all the vertices of S has degree greater than or equal to r-1.

Corollary 6.6: If S={x_1,x_2,…x_r} is an independent set or r-vertices in the connected 3 - S.C.D critical graph G, then for each x_i there exists r-1 different strong convex dominating sets G_j^i (j=1 to r-1) such that $\bigcap_{j=1}^{r-1} G_j^i = x_i$.

Theorem 5.15: Diameter of k-S.C.D critical graph is at most k.

Conjecture 5.1: Every 2-connected 3-S.C.D critical graph is 2self-centered.

Conjecture 5.2: Every 2-connected 3-S.C.D critical graph must be a member of the family ({u}+K_m+K_n+K_l+{v}) $\cup$ {uv}, for some m, n and l.

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