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Lower Bound for Domination Number of any Graph in terms of Maximum Degree of that Graph

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Abstract: *The graphs considered in this paper are finite, simple, connected and undirected. In this paper, first we derive the lower bound for all graphs with maximum degree* Δ *, which are free from cycles of length three and four, from which we deduce the lower bound for domination number for all graphs.*

Keywords: *Domination number, maximum degree, lower bound.*

1. Introduction

The graphs considered in this paper are finite, simple, connected and undirected. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively and let p and q denote the cardinality of those sets respectively. The degree of a vertex in a graph G is denoted by $deg_G(v)$. The minimum and maximum degree of a graph is denoted by δ and Δ respectively. Let C_3 and C_4 denote respectively the cycles of length three and four.

The concept of domination was introduced by Ore [2]. A set $D \subseteq V(G)$ is called a dominating set if every vertex v in V is either an element of D or is adjacent to an element of D. A dominating set D is a minimal dominating set if D-{v} is not a dominating set for any $v \in D$. The domination number γ (G) of a graph G equals the minimum cardinality of a dominating set in G. A set of vertices is said to be a γ-set, if it is a dominating set with cardinality γ (G).

Many works have been done in the upper bound for domination number [3] and even some conjectures are given in [3]. But only very few lower bounds are there for $\gamma(G)$. In this paper, we derive a lower bound for γ(G), and compare the new lower bound found in this paper with the familiar lower bound $\left\lceil \frac{p}{1+\Delta(G)} \right\rceil$ and conclude where our result is best possible.

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2.Prior Results

Following theorem gives very simple lower and upper bound of $\gamma(G)$ in terms of p and $\Delta(G)$.

Theorem 2.1: [1,4] For any graph G, $\left\lceil \frac{p}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq n-\Delta(G)$.

3.Main Results

Following theorem provides the lower bound for all graphs without cycles of length three and four.

Theorem 3.1:

If G is a graph without C_3 and C_4, then

$$\left\lfloor \frac{(2\Delta-3)+\sqrt{(3-2\Delta)^2+8p(2\Delta-1)}}{2(2\Delta-1)} \right\rfloor \leq \gamma(G).$$

Proof:

Let G be a graph without cycles of length three and four. Let D be a γ -set for G. Then in D, the following two conditions will hold:

Claim (i): Any two adjacent vertices in D have no common neighbour in V-D.

If possible, let (u, v) be a pair of adjacent vertices in D, which have a common neighbour w in V-D. Then <u, v, w> form a triangle in G , which is a contradiction to our assumption of G.

Claim (ii): Any two non-adjacent vertices in D will have at most one common neighbour in V-D.

If possible, let (u, v) be a pair of non-adjacent vertices in D, which have two common neighbours say x and y in V-D, then <u, x, v, y> is a C_4 in G, contradiction to our assumption of G.

For any two vertices u and v (either adjacent or non-adjacent), deg(u) + deg(v)-1 vertices will be dominated by D. Let K_1 be the set of all pairs of adjacent vertices and let K_2 be the set of all pairs of non-adjacent vertices. Let $|K_1| = k_1$ and $|K_2| = k_2$.

$$\sum_{(u,v)\in K_1}\left(\deg_G(u)+\deg_G(v)-1-\deg_{<D>}(u)-\deg_{<D>}(v)\right)+$$
$$\sum_{(u,v)\in K_2}\left(\deg_G(u)+\deg_G(v)-1-\deg_{<D>}(u)-\deg_{<D>}(v)\right)+\gamma \geq p \;. \quad \text{---------} (1)$$

But $\deg_G(u) \leq \Delta$ for any vertex u in G.

Case 1: Suppose <D> is independent.

Then $\deg_{<D>}(u) = \deg_{<D>}(v) = 0$. From (1), we have,

$\sum_{(u,v)\in K_1}(2\Delta-1)+\sum_{(u,v)\in K_2}(2\Delta-1)+\gamma \geq p$, which implies, $(2\Delta-1)(k_1+k_2)+\gamma \geq p$.

As $(k_1+k_2) \leq \gamma(\gamma-1)/2$, we have,

$$(2\Delta-1)\gamma\frac{(\gamma-1)}{2}+\gamma\geq p. \qquad \text{--------- (2)}$$

(2) implies $(2\Delta-1)\gamma^2+(3-2\Delta)\gamma-2p\geq 0$ and in turn, we have,

$$\gamma\geq\frac{(2\Delta-3)\pm\sqrt{(3-2\Delta)^2+8p(2\Delta-1)}}{2(2\Delta-1)}.$$

As $X-\sqrt{X^2+Y} < 0$, $\forall$ $Y \geq 1$, we have,

$$\gamma\geq\left\lfloor\frac{(2\Delta-3)+\sqrt{(3-2\Delta)^2+8p(2\Delta-1)}}{2(2\Delta-1)}\right\rfloor.$$

Case 2: Suppose <D> is complete.

Then $\deg_{<D>}(u) = \deg_{<D>}(v) = \gamma-1$. Also, from (1), we have,

$\sum_{(u,v)\in K_1}[(2\Delta-1)-2(\gamma-1)]+\sum_{(u,v)\in K_2}[(2\Delta-1)-2(\gamma-1)]+\gamma\geq p$, which implies,

$[(2\Delta-1)-2(\gamma-1)](k_1+k_2)+\gamma\geq p$. As $(k_1+k_2)\leq\gamma(\gamma-1)/2$, we have,

$$[(2\Delta-1)-2(\gamma-1)]\gamma\frac{(\gamma-1)}{2}+\gamma\geq p.$$

As $2(\gamma-1)^2\gamma/2$ is positive for $\gamma\geq 2$, we have $(2\Delta-1)\gamma\frac{(\gamma-1)}{2}+\gamma\geq p$.

Also from (2) and subsequent steps, it is clear that

$$\gamma\geq\left\lfloor\frac{(2\Delta-3)+\sqrt{(3-2\Delta)^2+8p(2\Delta-1)}}{2(2\Delta-1)}\right\rfloor,$$ which is the required inequality.

Property A: A set D is said to possess the property A, if the vertices of D do not induce C_3 and C_4, within D and between D and V-D.

Theorem 3.2: If a graph G has at least one γ-set with the property A, then

$$\left\lfloor\frac{(2\Delta-3)+\sqrt{(3-2\Delta)^2+8p(2\Delta-1)}}{2(2\Delta-1)}\right\rfloor\leq\gamma(G).$$

Proof: Let G be a graph and let G has a γ-set D with the property A. Then the vertices of D do not induce C_3 and C_4, within D and between D and V-D. Therefore, by Theorem 3.1, the required inequality is obtained.

Theorem 3.3: For any graph G, $\left\lfloor\frac{(2\Delta-3)+\sqrt{(3-2\Delta)^2+8p(2\Delta-1)}}{2(2\Delta-1)}\right\rfloor\leq\gamma(G)$.

Proof: Let G be a graph.

Case 1: Suppose G has no C_3 and C_4. Then by Theorem 3.1, the required inequality is obtained.

Case 2: Suppose G has C_3 and C_4.

Subcase 1: Suppose G has at least one minimum dominating set with the property A. Then by Theorem 3.2, the required inequality is obtained.

Subcase 2: Suppose G has no such dominating set, that is all the minimum dominating sets of G have C_3 and C_4. We choose one of them, say D and remove those edges which induce C_3 and C_4, from D and hence from G, such that G satisfies the subcase (1). Then, by the above subcase, we get the required inequality. Hence the proof.

4. Analysis

Now, we compare the lower bounds in Theorem 2.1 and in Theorem 3.1.

Let $F_1 = \left\lceil \frac{p}{1+\Delta(G)} \right\rceil$ and $F_2 = \left\lfloor \frac{(2\Delta-3)+\sqrt{(3-2\Delta)^2+8p(2\Delta-1)}}{2(2\Delta-1)} \right\rfloor$.

Now, we prove the following theorem.

Theorem 4.1:

If $\Delta \geq (p/2)-1$, then $F_1 \leq F_2$.

Proof:

Assume $F_1 \leq F_2$.

$$\left\lceil \frac{p}{1+\Delta(G)} \right\rceil \leq \left\lfloor \frac{(2\Delta-3)+\sqrt{(3-2\Delta)^2+8p(2\Delta-1)}}{2(2\Delta-1)} \right\rfloor . \qquad \text{---------} (3)$$

$$\frac{2p(2\Delta-1)-(2\Delta-3)(\Delta+1)}{\Delta+1} \leq \sqrt{(3-2\Delta)^2+8p(2\Delta-1)}$$, since $x \leq \lceil x \rceil$ and $\lfloor x \rfloor \leq x$. Squaring both sides, we get,

$$4p^2(2\Delta-1)^2+(2\Delta-3)^2(\Delta+1)^2-4p(2\Delta-1)(2\Delta-3)(\Delta+1)$$
$$\leq (3-2\Delta)^2(\Delta+1)^2+8p(2\Delta-1)(\Delta+1)^2,$$ which implies,

$$p(2\Delta-1)-(2\Delta-3)(\Delta+1) \leq 2(\Delta+1)^2$$, which in turn impies,

$$2\Delta^2+4\Delta+2 \geq 2p\Delta-p-2\Delta^2+\Delta+3$$ and hence we have,

$$4\Delta^2+(3-2p)\Delta+(p-1) \geq 0. \qquad \text{---------} (4)$$

Thus, for $F_1 \leq F_2$, the inequality (4) has to be satisfied.

Claim : If $\Delta \geq (p/2)-1$, then the inequality (4) is satisfied.

We prove by induction on Δ.

Case 1: Suppose $\Delta = (p/2)-1$. Then,

$$4\Delta^2 + (3-2p)\Delta + (p-1)$$
$$= 4[(p/2)-1]^2 + (3-2p)[(p/2)-1] + (p-1)$$
$$= [(5/2)-2]p + (4-4) = \frac{p}{2} > 0, \ \forall \ p \geq 1$$

Thus the inequality is true for $\Delta = (p/2)-1$.

Case 2: Suppose $\Delta = p/2$.

$$4\Delta^2 + (3-2p)\Delta + (p-1)$$
$$= 4(p/2)^2 + (3-2p)(p/2) + (p-1) = (5/2)p - 1 > 0, \ \forall \ p \geq 1.$$

Thus the inequality is true for Δ = p/2.

Case 3: Assume that the inequality is true for all Δ = p/2 +k, $\forall \ -1 \leq k \leq m$,

which implies, $4[(p/2)+k]^2 + (3-2p)[(p/2)+k] + (p-1) \geq 0, \ \forall \ -1 \leq k \leq m$,

that is, assume $[(5/2)+2k]p + (4k^2 + 3k - 1) \geq 0, \ \forall \ -1 \leq k \leq m$. --------- (5)

Case 4: Suppose Δ = p/2 +(m+1).

$$4\Delta^2 + (3-2p)\Delta + (p-1)$$
$$= 4[(p/2)+(m+1)]^2 + (3-2p)[(p/2)+(m+1)] + (p-1)$$
$$= [(5/2)+2(m+1)]p + 4(m+1)^2 + 3(m+1) - 1.$$

Thus , $4\Delta^2 + (3-2p)\Delta + (p-1)$

$$= [(5/2)+2m]p + (4m^2 + 3m - 1) + (2p + 11m + 7) \geq 0, \ \forall \ m \geq 1$$

[as (2p+11m+7) > 0 , $\forall$ p $\geq$ 1 and m$\geq$ 1 and by (5)]

Thus, the inequality is true for Δ = p/2 +(m+1). Therefore, by induction hypothesis, the inequality (4) is true, $\forall \ \Delta \geq (p/2)-1$.

Therefore, if $\Delta \geq (p/2)-1$, then $F_1 \leq F_2$. Hence the proof.

5.Conclusion

In this paper, we have derived the lower bound for γ and analyzed the cases, for which the bound is the best possible.

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