International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 3, No. 3, July –September 2012, pp. 115-135

Domination Parameters of the Boolean Graph $BG_2(G)$ and its Complement

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Abstract: *Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). $B_{G,\,INC,\,\bar{L}(G)}(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two adjacent vertices of G, a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by $BG_2(G)$, Boolean graph of G-second kind. In this paper, the domination number, connected, cycle and total domination, independent domination, global domination and restrained domination of $BG_2(G)$ and its complement are studied.*

Key words: *Boolean graph $BG_2(G)$, domination number, connected, cycle and total domination, independent domination, global domination, restrained domination.*

1. Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [6], Buckley and Harary [4].

Definition 1.1 [6] A set $S \subseteq V$ is said to be a *dominating set* in G, if every vertex in $V-S$ is adjacent to some vertex in S. A dominating set D is an *independent dominating set*, if no two vertices in D are adjacent that is D is an independent set. A dominating set D is a *connected dominating set*, if < D > is a connected subgraph of G. A dominating set D is a *perfect dominating set*, if for every vertex $u \in V(G)-D$, $|N(u) \cap D| = 1$. A dominating set D is a *total dominating set*, if < D > has no isolated vertices. A dominating set D is called an *efficient dominating set*, if the distance between any two vertices in D is at least three. A cycle C of a graph G is called a *dominating cycle* of C, if every vertex in $V-C$ is adjacent to some vertex in G. A set $D \subseteq V(G)$ is a *global dominating set*, if D is a dominating set in G

Received: 12 February, 2012; Revised: 19 May, 2012; Accepted: 28 May, 2012

and $\overline{G}$. A total dominating set D of a graph is a *total global dominating set*, if D is also a total dominating set of $\overline{G}$.

A set $D \subseteq V(G)$ is a *restrained dominating set*, [7] if every vertex in $V-S$ is adjacent to a vertex in S and other vertex in $V-S$.

Definition 1.2 [7] A set S of vertices is said to be *irredundant*, if for every vertex $v \in S$, $p_n[v, S] = N[v]-N[S-\{v\}] \neq \phi$, that is, every vertex $v \in S$ has a private neighbor. The *irredundance number ir(G)* equals the minimum cardinality of a maximal irredundant set in G.

Definition 1.3 The *domination number* γ of G is defined to be the minimum cardinality of a dominating set in G. Similarly, one can define the perfect domination number γ_p, connected domination number γ_c, total domination number γ_t, independent domination number γ_i, efficient domination number γ_e, cycle domination number γ_o, global domination number γ_g, total global domination number γ_{tg}, restrained domination number γ_r.

Definition 1.4 A vertex (point) and an edge are said to *cover* each other, if they are incident. A set of vertices, which cover all the edges of a graph G is called a (*vertex*) *point cover of G*, while a set of lines (edges), which covers all the vertices is a *line cover*. The smallest number of points in any point cover for G is called its *point covering number* and is denoted by $\alpha_0(G)$ or α_0. Similarly, $\alpha_1(G)$ or α_1 is the smallest number of lines in any line cover of G and is called its *line covering number*. Clearly, A point cover (line cover) is called *minimum*, if it contains α_0 (respectively α_1) elements.

Definition 1.5 A set of points in G is *independent*, if no two of them are adjacent. The largest number of points in such a set is called the *point independence number* of G and is denoted by $\beta_o(G)$ or β_o. Analogously, an independent set of lines (matching) of G has no two of its lines adjacent and the maximum cardinality of such a set is the *line independence number* $\beta_1(G)$ or β_1, A set of independent edges covering all the vertices of a graph G is called a *1-factor* or a *perfect matching* of G.

Theorem 1.1 [7] $ir(G) \leq \gamma(G) \leq \gamma_i(G) \leq \beta_o(G)$. If $\gamma(G) \geq 2$, $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$.

Cockayne and Hedetniemi [5] defined the *domatic number d(G)* of a graph to be the maximum number of elements in a partition of V(G) into dominating sets. G is *domatically full* if $d(G) = 1+\delta(G)$.

Theorem 1.2 [7] $\gamma_{tg}(G) < \gamma_t(G)+2$.

Theorem 1.3 [7] Let D be a γ_g-set of G such that < D > has no isolates and diam(G) = 3, then $\gamma_{tg}(G) \leq \gamma_g(G)+2$.

Theorem 1.4 [9] (Sampathkumar et al.) $\gamma(G) \leq n_o(G) \leq \alpha_0(G)$.

<u>**Motivation:**</u> The Line graphs, Middle graphs, Total graphs and Quasi-total graphs are very much useful in computer networks. In analogous to line graph, total graph, middle graph and quasi-total graph, thirty-two graphs can be defined using different adjacency relations. Out of these operations, eight were already studied. Among the remaining twenty-four graph operations, four are defined and analyzed in [3]. All the others have been defined and studied thoroughly and will be submitted elsewhere. This is illustrated below.

G/ $\overline{G}/K_p$/ $\overline{K_p}$	Incident (INC)/ Non-incident (NINC)	L(G)/ $\overline{L(G)}/K_q$/ $\overline{K_q}$

Defining a new graph from a given graph by using the adjacency relation between two vertices or two edges and incident relationship between vertices and edges is known as Boolean operation. It defines new structure from the given graph and adds extra information of the original graph.

In Management and in social networks, the incident and non-incident relations of vertices and edges are used to define various networks. So these are very much applicable in socio-economical problems. In some cases, it is possible to retrieve the original graph from the Boolean graphs in polynomial time. So these graph operations may be used in graph coding or coding of some grouped signal. Also, it is possible to study the structure of original graphs through these Boolean graph operations. This motivates the study for the exploration of various Boolean operations and study of their structural properties.

In [3, 12] Boolean graph $BG_2(G)$ is defined as follows:

Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G). $B_{G,\ INC,\ \overline{L(G)}}(G)$ [3] is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by $BG_2(G)$, Boolean graph of G-second kind. With an immediate consequence of the definition of $BG_2(G)$, if G is a (p, q) graph, whose vertices are $v_1, v_2, \ldots, v_p$ having degrees d_i, and edges e_{ij}, note that $BG_2(G)$ has p+q vertices and $(q^2+7q-\Sigma d_i^2)/2$ edges with $\deg v_i = 2d_i$; $\deg e_{ij} = q+3-(d_i+d_j)$. Also, G and $\overline{L(G)}$ are induced subgraphs of $BG_2(G)$.

In this paper, domination parameters of $BG_2(G)$ and its complement are studied.

2. Domination Parameters of $BG_2(G)$ and $\overline{BG_2}(G)$

Domination number of $BG_2(G)$ and $\overline{BG_2}(G)$

First domination parameters of $BG_2(G)$ and $\overline{BG_2}(G)$ are studied and bounds for them are found out.

Theorem 2.1 $\gamma(BG_2(G)) = 1$ if and only if $G = K_{1,n}$.

Proof: Radius of $BG_2(G) = 1$ if and only if $G = K_{1,n}$.

Hence, $\gamma(BG_2(G)) = 1$ if and only if $G = K_{1,n}$.

Next two theorems give the bound of $\gamma(BG_2(G))$ in terms of $\gamma(G)$ and degree of vertices of G.

Theorem 2.2 $\gamma(BG_2(G)) \leq \gamma(G)+2$.

Proof: Let $D \subseteq V(G)$ be a minimal dominating set of G with cardinality $\gamma(G)$. Let $u \in D$ be such that u is adjacent to $v \in V(G)$. (This is always possible, since G is non-trivial). Let $e = uv \in E(G)$, where $u, v \in V(G)$. Consider $D' = D \cup \{v, e\} \subseteq V(BG_2(G))$. D' dominates every point vertex of $BG_2(G)$, since D dominates G. The line vertex e dominates all line vertices not adjacent to e in G. All the line vertices adjacent to e in G are dominated by u and v in $BG_2(G)$. Hence, $\gamma(BG_2(G)) \leq |D'| = \gamma(G)+2$. This proves the theorem.

Theorem 2.3 Let G be a graph with $diam(G) \leq 2$ and $G \neq K_{1,n}$. Then $\gamma(BG_2(G)) \leq \min\{2deg_G v, q+3-(deg_G u+deg_G v)\}$, where $e = uv \in E(G)$.

Proof: Since $G \neq K_{1,n}$ and $diam(G) \leq 2$, $BG_2(G)$ is self-centered with diameter 2. Therefore, neighborhood of every vertex in $BG_2(G)$ is a dominating set for $BG_2(G)$ and $deg\ v = 2deg_G v$ and $deg\ e = q+3-(deg_G u+deg_G v)$ in $BG_2(G)$, where $e = uv \in E(G)$. Therefore, $\gamma(BG_2(G)) \leq \min\{2deg_G v, q+3-(deg_G u+ deg_G v)\}$.

Corollary 2.3 If $G \neq K_{1,n}$, $diam(G) \leq 2$ and G has a pendant vertex, then, $\gamma(BG_2(G)) = 2 = \gamma_c(BG_2(G))$.

Proof: $G \neq K_{1,n}$. Hence, $\gamma(BG_2(G)) \neq 1$. Consider the pendant vertex $u \in V(G)$, and let $v \in V(G)$ be its adjacent vertex in G. $e = uv \in E(G)$. Now, in $BG_2(G)$, $D = \{v, e\}$ is a connected dominating set. Thus, $\gamma(BG_2(G)) = 2 = \gamma_c(BG_2(G))$.

Next theorem gives the bound of $\gamma(BG_2(G))$ in terms of $\delta(G)$.

Theorem 2.4 If G is a graph with $diam(G) \leq 2$, $\gamma(BG_2(G)) \leq \delta(G)+1$.

Proof: Consider $u \in V(G)$ with degree $\delta(G)$. Consider $N_G(u) \subseteq V(G)$. In $BG_2(G)$, elements in $N_G(u)$ dominates all the point vertices and line vertices incident with u (in G). The line vertices in $BG_2(G)$, which are not dominated by elements of $N_G(u)$ are nothing but the edges of G in $< N_2(u) >$. Now, consider an edge e incident with u in G. $D = N(u) \cup \{e\} \subseteq V(BG_2(G))$ dominates $BG_2(G)$.

Therefore, $\gamma(BG_2(G)) \leq \delta(G)+1$.

Remark 2.1 (1) If $diam(G) \leq 2$ and $< N_2(u) >$ in G is totally disconnected, then $\gamma(BG_2(G)) \leq \delta(G)$.

(2) Let $S \subseteq E(G)$ be a point cover for G. Then $S \subseteq V(BG_2(G))$ is a dominating set for $BG_2(G)$.

(3) The set of all point vertices is a dominating set for $BG_2(G)$.

(4) The set of all line vertices is a dominating set for $BG_2(G)$, if and only if G has no isolated vertex.

(5) The set of all point vertices is a minimum dominating set for $BG_2(G)$, if and only if $G = \overline{K_n}$.

(6) The set of all line vertices is a minimum dominating set for $BG_2(G)$ if and only if $G = nK_2$.

Following two theorems relate $\gamma(BG_2(G))$ with $\alpha_0(G)$.

Theorem 2.5 (1) $D \subseteq V(G)$ is a dominating set for $BG_2(G)$ if and only if D is a point cover for G. Also, $\gamma(BG_2(G)) \leq \alpha_0(G)$.

(2) $D \subseteq E(G)$ is a dominating set for $BG_2(G)$ if and only if D is a line cover which dominates $\overline{L(G)}$.

Proof of (1): Suppose $D \subseteq V(G)$ is a dominating set for $BG_2(G)$. In G, every edge is incident with some element in D. This D is a point cover for G. On the other hand, suppose $D \subseteq V(G)$ is a point cover for G. In G, every edge in G is incident with some vertex in D. Hence. In $BG_2(G)$, D dominates every point vertices and line vertices. Thus, D is dominating set for $BG_2(G)$.

Proof of 2: $D \subseteq E(G)$ is a dominating set for $BG_2(G)$ if and only if D is a dominating set for $\overline{L}(G)$ and every point vertex in G is incident with some element in D, that is D covers all the point vertices of G and D is a line cover for G and dominates $\overline{L}(G)$. This proves 2.

Theorem 2.6 If $\gamma(G) = \alpha_0(G)$, then $\gamma(BG_2(G)) = \gamma(G)$.

Proof: $\gamma(G) = \alpha_0(G)$. Hence, there exists a dominating set D, which is also a point cover with cardinality $\gamma(G)$. This implies that D is a dominating set for $BG_2(G)$. Therefore, $\gamma(BG_2(G)) = \gamma(G)$.

Remark 2.2 (1) $\gamma(BG_2(G)) \geq \gamma(G)$.

(2) The converse of Theorem 2.6 need not be true. That is, $\gamma(BG_2(G) = \gamma(G)$ need not imply that $\gamma(G) = \alpha_0(G)$.

Remark 2.3 (1) If G is a disconnected graph having K_2 as a component, then $\gamma(BG_2(G)) = \gamma(G)$.

(2) If there exists a $\gamma(G)$ dominating set D such that $e \in E(G)$ is not adjacent to all edges in $< V(G) - D >$ then $\gamma(BG_2(G)) \leq \gamma(G)+1$.

(3) If $\alpha_0(G) = \gamma(G)+1$, then also $\gamma(BG_2(G)) \leq \gamma(G)+1$.

(4) If G has at least one $\gamma(G)$ dominating set, which is not independent then $\gamma(BG_2(G)) \leq \gamma(G)+1$.

Theorem 2.7 If $\gamma(BG_2(G)) = \gamma(G)+2$, then G satisfies all the following conditions:

(1) $\alpha_0(G) \geq \gamma(G)+2$.

(2) G has no component as K_2.

(3) Every $\gamma(G)$ dominating set of G is independent.

(4) There exists no edge $e \in E(G)$, which is not adjacent to all other edges in $< V(G) - D >$, where D is a $\gamma(G)$ dominating set of G.

Proof of (1): If $\alpha_0(G) \leq \gamma(G)+1$, $\gamma(BG_2(G)) \leq \gamma(G)+1$. Hence (1) follows.

Proof of (2): If G has a component as K_2, $\gamma(BG_2(G)) = \gamma(G)$.

Proof of (3): If a $\gamma(G)$ dominating set D contains an edge, then $\gamma(BG_2(G)) \leq \gamma(G)+1$.

Proof of (4): If there exists an edge $e \in E(G)$ such that e is not adjacent to edges of $< V(G) - D >$, then $\gamma(BG_2(G)) \leq \gamma(G)+1$. Hence the theorem follows.

Remark 2.4 Even if every $\gamma(G)$ dominating set is independent, $\gamma(BG_2(G))$ may be equal to $\gamma(G)+1$. Example, $G = C_5$.

Remark 2.5 A line vertex in $BG_2(G)$ can dominate at most two point vertices in $BG_2(G)$. Hence, if $S \subseteq V(BG_2(G))$ is a dominating set of $BG_2(G)$, another dominating set D can be formed with same cardinality $|S|$ such that D contains at most one line vertex.

Following theorems give the characterizations of G for which $\gamma(BG_2(G)) = \gamma(G)$, $\gamma(BG_2(G)) = \gamma(G)+1$ and $\gamma(BG_2(G)) = \gamma(G)+2$.

Theorem 2.8 $\gamma(BG_2(G) = \gamma(G)$ if and only if any one of the following is true:
(1) There exists a $\gamma(G)$ dominating set D of G, which is a point cover. (2) G has K_2 as a component. (3) G has a pendant vertex v such that $e = uv \in E(G)$ and there exists a $\gamma(G)$ dominating set D of G containing u and $\{N(u)-v\}$.

Proof: Suppose $\gamma(BG_2(G)) = \gamma(G)$. Let $S \subseteq V(BG_2(G))$ be a $\gamma(BG_2(G))$ dominating set of $BG_2(G)$. By the remark, one can find a dominating set D of $BG_2(G)$ with at most one line vertex.

<u>**Case 1:** D has no line vertices</u>.

Since D is a dominating set of $BG_2(G)$, D is a point cover for G.

<u>**Case 2:** D has one line vertex</u>.

D dominates all the point vertices in $BG_2(G)$ also. Let $e \in E(G)$ be a line vertex in D. Let $e = uv \in E(G)$. Clearly, e dominates only two point vertices. Therefore, $D-\{e\} \cup \{u\} = D'$ is a $\gamma(G)$ dominating set for G and e is not adjacent to all edges in $< V(G)-D' >$ in G. Three sub cases arise.

<u>**Sub case 1:** $V(G) - D'$ is independent.</u>

This implies D' is a point cover for G.

<u>**Sub case 2:**</u> <u>e is not adjacent to any other edges of G</u>.

In this case, G has K_2 as a component.

<u>**Sub case 3:**</u> <u>e is adjacent to some other edges of G</u>.

In this case, $BG_2(G)$ is dominated by D implies that e cannot dominate the line vertices corresponding to the edges adjacent to e in G. Hence, there must exist, point vertices which are incident with the edges which are adjacent to e.

Let $e = uv \in E(G)$, $D' = D-\{e\} \cup \{u\}$. Let $u_1, u_2, \ldots, u_k$ be neighbors of u. Then $u_1, u_2, \ldots, u_k$ must be in D'. Thus, D' contains u and N(u).

<u>**Claim:**</u> <u>$deg_G\, v = 1$.</u> Suppose $deg_G\, v > 1$. Let $e_1 = vv^1$ adjacent to e in G. e' is not dominated by e. Therefore, either v must be in D or v^1 must be in D. But already u and N(u) are in

D^1 implies D itself dominates u and v. Therefore, e is not necessary to dominate u or v in $BG_2(G)$. This implies that $|D| \geq \gamma(G)+1$, which is a contradiction. Hence, $\deg_G v$ must be one. That is, G has a pendant vertex v and $e = uv \in E(G)$ and D' containing u and N(u) is a dominating set of G.

Converse is obvious.

Theorem 2.9 $\gamma(BG_2(G)) = \gamma(G)+1$ if and only if there exists a $\gamma(G)$-dominating set D of G, which is not independent and is not a point cover or there exists an edge e in G such that e is not adjacent to all the edges of $< V(G)-D >$ and $\gamma(G) \neq \alpha_0(G)$.

Proof: Let $D \subseteq V(BG_2(G))$ be a $\gamma(BG_2(G))$ dominating set of $BG_2(G)$. Assume that $\gamma(BG_2(G)) = \gamma(G)+1$. By the remark 2.5, there exists D such that D contains at most one line vertex e. Therefore, $D-\{e\} \subseteq V(G)$ is a dominating set of G. D is a $\gamma(BG_2(G))$ dominating set implies that $D-\{e\}$ is not a point cover, and e dominates all the line vertices which are in $V(G)-(D-\{e\})$. Hence, e must be either in $< D >$ or $e \in E(G)$ such that e is not adjacent to all the edges of $< V(G)-D >$.

Converse is obvious.

Next, we study the domination parameters of $\overline{BG_2}(G)$. Following theorem characterize the graphs for which $\gamma(\overline{BG_2}(G)) = 1$.

Theorem 2.10 $\gamma(\overline{BG_2}(G)) = 1$ if and only if G has an isolated vertex.

Proof: If G has an isolated vertex u, then $D = \{u\}$ dominates every vertex in $\overline{BG_2}(G)$. Therefore, $\gamma(\overline{BG_2}(G)) = 1$. On the other hand, suppose $\gamma(\overline{BG_2}(G)) = 1$. Then there exists $x \in V(\overline{BG_2}(G))$ such that x is adjacent to every other vertex in $\overline{BG_2}(G)$. Therefore, x must not be a line vertex and x must be adjacent to every other point vertex in $\overline{BG_2}(G)$. This implies that x is an isolated vertex in G.

Following theorem characterize the graphs for which $\gamma(\overline{BG_2}(G)) = 2$.

Theorem 2.11 If G has no isolated vertices, then $\gamma(\overline{BG_2}(G)) = 2$, if and only if any of the following is true. (1) $diam(G) \geq 3$, (2) $\gamma_i(L(G)) = 2$.

Proof: **Case 1: $D = \{u_1, u_2\} \subseteq V(G)$.** D is a dominating set for $\overline{BG_2}(G)$ if and only if u_1, u_2 are not adjacent in G and there is no $w \in V(G)$ adjacent to both u_1 and u_2. This implies that, $d_G(u_1, u_2) \geq 3$. Hence, $diam(G) \geq 3$.

Case 2: $D = \{e_1, e_2\} \subseteq E(G)$. D is a dominating set for $\overline{BG_2}(G)$ if and only if e_1, e_2 dominate L(G) and e_1, e_2 are not adjacent in G. (otherwise, the common point of incidence is not dominated by D). This is true only when $\gamma_i(L(G)) = 2$.

Case 3: $D = \{u, e\}$, where $u \in V(G)$, $e \in E(G)$.

Sub case 3.1: $D = \{u, e\}$, and u and e are not incident in G.

D is a minimal dominating set for $\overline{BG_2}(G)$ if and only if u dominates the end vertices of e in $\overline{BG_2}(G)$ and e dominates all the line vertices in $\overline{BG_2}(G)$ which are edges incident with u in G.

This is true, if and only if u is not adjacent to the incident vertices of e in G and there is no edge incident with u in G.

Hence, $diam(G) \geq 3$ and u is an isolated vertex. In this case, {u} is a minimal dominating set for $\overline{BG_2}(G)$.

Sub case 3.2: $D = \{u, e\}$, where $e = uv \in E(G)$. This case is not possible since in this case, v is not dominated by u or e in $\overline{BG_2}(G)$. Therefore, $\gamma(\overline{BG_2}(G)) = 2$, if and only if (i) $diam(G) \geq 3$ or $\gamma_i(L(G)) = \gamma(L(G)) = 2$.

Remark 2.6 If $\beta_1(G) > 2$, either $diam(G) \geq 3$ or $diam(G) \leq 2$.

When $diam(G) \geq 3$, $\gamma(\overline{BG_2}(G)) = 2$, $D = \{u_1, u_2\}$, $d_G(u_1, u_2) \geq 3$ dominates $\overline{BG_2}(G)$.

When $diam(G) \leq 2$, $\beta_1(G) > 2$ implies that $\gamma_i(L(G)) \neq 2$ and hence $\gamma(\overline{BG_2}(G)) > 2$.

Therefore, $\gamma(\overline{BG_2}(G)) = 2$ if and only if any one of the following is true. (i) $diam(G) \geq 3$ (ii) $\beta_1(G) = 2$.

In the following we obtain a characterization of graphs G for which $\gamma(\overline{BG_2}(G)) = 3$.

Theorem 2.12 $\gamma(\overline{BG_2}(G)) = 3$ if and only if $diam(G) \leq 2$ and $\beta_1(G) \neq 2$.

Proof: G is non-trivial. Therefore, there exists at least one edge $e \in E(G)$. Let $e = uv$, $uv \in E(G)$, $D = \{u, v, e\} \subseteq V(\overline{BG_2}(G))$ is a dominating set for $\overline{BG_2}(G)$. Thus, $\gamma(\overline{BG_2}(G)) \leq 3$. The theorem follows from Theorems 2.10 and 2.11.

Remark 2.7 Any dominating set of $\overline{G}$ containing at least three elements is a dominating set for $\overline{BG_2}(G)$. If $G \neq K_{1,n}$, then the set of all independent edges of G is a dominating set for $\overline{BG_2}(G)$. If $G \neq K_{1,n}$, K_3, then a line cover for G is a dominating set for $\overline{BG_2}(G)$.

Theorem 2.13 For any non-trivial simple graph G,

$$3 \leq \gamma(BG_2(G)) + \gamma(\overline{BG_2}(G)) \leq \gamma(G) + 5.$$

Proof: G has an isolated vertex if and only if $\gamma(\overline{BG_2}(G)) = 1$. In this case, $\gamma(BG_2(G)) \geq 2$. $\gamma(BG_2(G)) = 1$ if and only if $G = K_{1,n}$. In this case, $\gamma(\overline{BG_2}(G)) \geq 2$, since $\overline{BG_2}(G)$ has an isolated vertex. Thus in all cases, $\gamma(BG_2(G))+\gamma(\overline{BG_2}(G)) \geq 3$. Also, in all cases, $\gamma(\overline{BG_2}(G)) \leq 3$ and $\gamma(BG_2(G)) \leq \gamma(G)+2$. Therefore, $\gamma(BG_2(G))+\gamma(\overline{BG_2}(G)) \leq \gamma(G)+5$. Hence, $3 \leq \gamma(BG_2(G))+\gamma(\overline{BG_2}(G)) \leq \gamma(G)+5$.

Example 2.1 Let $G = K_{1,n} \cup K_1$, $\gamma(\overline{BG_2}(G)) = 1$ and $\gamma(BG_2(G)) = 2$.

Therefore, $\gamma(BG_2(G))+\gamma(\overline{BG_2}(G)) = 3$.

Example 2.2

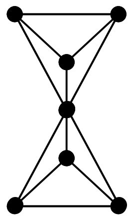

Fig:2.1 G

$\gamma(G) = 1$.

Here, $\gamma(BG_2(G)) = \gamma(G)+2$ and

$\gamma(\overline{BG_2}(G)) = 3$, since diam(G) = 2 and $\beta_1(G) \neq 2$.

Therefore, $\gamma(BG_2(G))+\gamma(\overline{BG_2}(G)) = \gamma(G)+5$.

Hence, the bounds in the theorem are sharp.

Connected, Cycle and total domination of $BG_2(G)$ and $\overline{BG_2}(G)$:

Following theorems deal with the connected domination, cycle domination and total domination parameters of $BG_2(G)$ and $\overline{BG_2}(G)$.

Theorem 2.14 If $\gamma(G) = \gamma_c(G)$, then $\gamma_c(BG_2(G)) = \gamma(G)$ or $\gamma(G)+1$.

Proof: $\gamma(G) = \gamma_c(G)$. Therefore, G has a connected dominating set D with cardinality $\gamma(G)$. Hence, $D_1 = D \cup \{e\}$, where $e = uv \in E(G)$, $u, v \in D$ is a connected dominating set for $BG_2(G)$. Thus, $\gamma_c(BG_2(G)) = \gamma(G)$ or $\gamma(G)+1$.

Remark 2.8 $\gamma_t(BG_2(G)) = \gamma_t(G)$ or $\gamma_t(G)+1$ if $\gamma(G) = \gamma_t(G)$.

Theorem 2.15 If G has a cycle dominating set, then $BG_2(G)$ also has a cycle dominating set and $\gamma_o(BG_2(G)) \leq \gamma_o(G)+1$.

Proof: Let D be a minimum cycle dominating set for G. $D \cup \{\text{edge in} < D >\}$ is a cycle dominating set for $BG_2(G)$. Also, $D \cup \{e\}$, where e is an edge in $< D >$ is a cycle dominating set (not induced) for $BG_2(G)$. Therefore, $\gamma_o(BG_2(G)) \leq \gamma_o(G)+1$.

Remark 2.9 If $\gamma(G) = \alpha_0(G)$, $\gamma_o(BG_2(G)) \leq \gamma_o(G)$.

Theorem 2.16 Let G be a graph without isolated vertices. Then $\gamma(\overline{BG_2}(G)) = \gamma_c(\overline{BG_2}(G)) = \gamma_t(\overline{BG_2}(G)) = 2$ if and only if $diam(G) \geq 3$.

Proof: Suppose $diam(G) \geq 3$. Let $u_1, u_2 \in V(G)$ such that $d_G(u_1,u_2) \geq 3$. $D = \{u_1, u_2\}$ is a connected dominating set for $\overline{BG_2}(G)$. On the other hand, if $\gamma_c(\overline{BG_2}(G)) = 2$, $\gamma(\overline{BG_2}(G)) = 2$. (G has no isolated vertices). This gives $diam(G) \geq 3$ or $\beta_1(G) = 2$ by Remark 2.6. But $\beta_1(G) = 2$ and $diam(G) \leq 2$ means $\gamma(\overline{BG_2}(G)) = 2$, $D = \{e_1, e_2\}$, where e_1, e_2 are independent edges in G is a dominating set for $\overline{BG_2}(G)$. But D is not connected. Hence, $\gamma_c(\overline{BG_2}(G)) = 2$ if and only if $diam(G) \geq 3$.

Theorem 2.17 Let G be a graph without isolated vertices. $\gamma_c(\overline{BG_2}(G)) = \gamma_t(\overline{BG_2}(G)) = 3$ if and only if $diam(G) \leq 2$, $G \neq K_{1,n}$.

Proof: Let $diam(G) \leq 2$, $G \neq K_{1,n}$, K_3. $G \neq K_{1,n}$. Therefore, $\beta_1(G) > 1$.

<u>**Case 1:** $\beta_1(G) = 2$.</u>

If $p \geq 5$, $D = \{e_1, e_2, w\}$, where e_1 and e_2 are independent edges and w not incident with both e_1 and e_2 is a connected dominating set. Therefore, $\gamma_c(\overline{BG_2}(G)) = 3$. If $p \leq 4$, $diam(G) \leq 2$, which implies that G is connected, (Also, $G \neq K_{1,n}$ implies $\overline{BG_2}(G)$ is connected) and there exists an edge, adjacent to both e_1 and e_2 in G. Thus, $D = \{e_1, e_2, e\}$ is a connected dominating set for $\overline{BG_2}(G)$.

Hence, $\gamma_c(\overline{BG_2}(G)) = 3$. If $G = K_3$, then also $\gamma_c(\overline{BG_2}(G)) = \gamma(\overline{BG_2}(G)) = 3$.

<u>**Case 2:** $\beta_1(G) > 2$.</u>

Since $\beta_1(G) > 2$, there exists at least three independent edges and hence $p \geq 6$.

<u>**Sub case 2.1**: diam(G) = 2.</u>

Since $p \geq 6$ and $diam(G) = 2$, there exists at least two non-adjacent vertices $u_1, u_2 \in V(G)$ and an edge $e \in E(G)$, not incident with u_1 and u_2. Take, $D = \{u_1, u_2, e\}$. Clearly, D is a cycle dominating set in $\overline{BG_2}(G)$.

Therefore, $\gamma_c(\overline{BG_2}(G)) = 3 = \gamma_t(\overline{BG_2}(G)) = \gamma_o(\overline{BG_2}(G))$.

<u>**Sub case 2.2**: diam(G) = 1.</u>

In this case, $G = K_n$. Take $D = \{e_1, e_2, e\}$, where e_1, e_2 are independent edges in G and e is adjacent to both e_1 and e_2. D is a connected domination set for $\overline{BG_2}(G)$. Therefore, $\gamma_c(\overline{BG_2}(G)) = 3 = \gamma_t(\overline{BG_2}(G))$.

Conversely, $\gamma_c(\overline{BG_2(G)}) = 3$ implies, $\gamma(\overline{BG_2(G)}) = 2$ or 3. Suppose, diam(G) ≥ 3, $\gamma(\overline{BG_2(G)}) = 2$. Therefore, $\gamma_c(\overline{BG_2(G)}) = 3$ implies that diam(G) ≤ 2 and $G \neq K_{1,n}$ (since $\overline{BG_2(K_{1,n})}$ is not connected). This proves the theorem.

Theorem 2.18 If G has an isolated vertex, then $\gamma_o(\overline{BG_2(G)}) = 3$.

Proof: Let $u \in V(G)$ be an isolated vertex of G and G non-trivial. $G = K_1 \cup G_1$, where G_1 has at least one edge. If G_1 is not complete, there exists at least two non-adjacent vertices v_1, v_2 in G_1. Hence in $\overline{BG_2(G)}$, $D = \{u, v_1, v_2\}$ forms a cycle dominating set. Therefore, $\gamma_o(\overline{BG_2(G)}) = 3$. If G_1 is complete and is not K_2, there exists an edge $e = xy \in E(G)$ and another vertex z in G_1. In this case, $D = \{e, z, u\}$ forms a cycle dominating set for $\overline{BG_2(G)}$. Hence $\gamma_o(\overline{BG_2(G)}) = 3$.

If $G_1 = K_2$, then $G = K_2 \cup K_1$ and $\overline{BG_2(G)}$ is $K_{1,3}$, which has no cycle domination.

Theorem 2.19 Let G be a graph with $\delta(G) \geq 1$ and $G \neq K_{1,n}$, and K_n, then

(1) $\gamma_o(\overline{BG_2(G)}) = 3$, if any one of the following is true (a) $\gamma(G) > 1$ and $G \neq 2K_2$, (b) $\gamma(G) = 1$ and $\delta(G) = 1$, (c) $\gamma(G) = 1$, $\delta(G) = 2 \neq deg_G\, v$ for all v in the center.

(2) $\gamma_o(\overline{BG_2(G)}) = 4$ if any one of the following is true: (a) $\gamma(G) = 1$, $\delta(G) = 2 = deg_G\, v$ for all $v \in V(G)$ such that $e(v) \neq 1$. (b) $\delta(G) = 3$, $\gamma(G) = 1$ (c) $\delta(G) = 4$, $p = 5$, $\gamma(G) = 1$, (d) $G = 2K_2$.

(3) $\gamma_o(\overline{BG_2(G)}) = 5$ if $\gamma(G) = 1$, $\delta(G) \geq 4$ and $p \geq 6$.

Proof: **<u>Case 1</u>:** <u>$\gamma(G) > 1$.</u>

Suppose $p \geq 5$. Then there exist u, v not adjacent in G and an edge $e \in E(G)$ not incident with u and v such that u and v are not both adjacent to the end vertices of e. Consider, $D = \{u, v, e\}$, D is a cycle dominating set for $\overline{BG_2(G)}$. Therefore, $\gamma_o(\overline{BG_2(G)}) = 3$.

If $p = 4$, $\gamma(G) > 1$, which implies that $G = C_4$ or $G = 2K_2$. If $G = C_4$, consider two adjacent edges e_1, e_2 in G and a vertex u not incident with e_1 and e_2. $D = \{e_1, e_2, u\}$ is a cycle dominating set for $\overline{BG_2(G)}$. Hence, $\gamma_o(\overline{BG_2(G)}) = 3$. If $G = 2K_2$, then $\gamma_o(\overline{BG_2(G)}) = 4$.

<u>Case 2</u>: <u>$\gamma(G) = 1$ and diam(G) = 2.</u>

In this case, $G \neq K_{1,n}$ and $G \neq K_n$. Since $G \neq K_{1,n}$ and $\gamma(G) = 1$, G has at least one triangle. Let $u \in V(G)$ be such that $e_G(u) = 1$.

<u>Sub case 2.1</u>: <u>$\delta(G) = 1$.</u>

Let $w \in V(G)$ be such that $deg_G\, w = 1$. Then G has two adjacent edges e_1 and e_2, one incident with u, other not incident with u and e_1, e_2 are not incident with w. $D = \{e_1, e_2, w\}$ dominates $\overline{BG_2(G)}$ and < D > is C_3 in $\overline{BG_2(G)}$. Therefore, $\gamma_o(\overline{BG_2(G)}) = 3$. (If G has only one triangle formed by the edges e_1, e_2, e_3, $D = \{e_1, e_2, e_3\}$ is also a cycle dominating set).

Sub case 2.2: $\delta(G) = 2$.

If there exists $v \in V(G)$, such that $2 < deg_G\, v < p-1$, as in the previous cases, there exists edges e_1 incident with u, e_2 adjacent to e_1 and not incident with u. Edges incident with w are adjacent to one of e_1 and e_2. In this case, $D = \{e_1, e_2, w\}$ is a cycle dominating set for $\overline{BG_2(G)}$.

Consider the case that $deg_G\, v = 2$ for all $v \neq u$.

In this case, take e_1 and e_2 independent such that e_1 is incident with u and adjacent to the edges incident with w. Let e be an edge adjacent to both e_1 and e_2. $D = \{e_1, e_2, e, w\}$ is a cycle dominating set and there is no cycle dominating set with cardinality less than this. Therefore, $\gamma_o(\overline{BG_2(G)}) = 4$.

Sub case 2.3: $\delta(G) = 3$. In this case, $p \geq 5$ and $G \neq K_n$.

Let $w \in V(G)$ such that $deg_G\, w = 3$. Then there exists (i) an edge e_1 incident with u and not incident with w. (ii) e_2 not adjacent to e_1 (iii) e adjacent to both e_1 and e_2 such that edges incident with w are adjacent to any one of e_1, e_2, e. Then $D = \{w, e, e_1, e_2\}$ is a cycle dominating set for $\overline{BG_2(G)}$ and $\gamma_o(\overline{BG_2(G)}) = 4$.

Sub case 2.4: $\delta(G) \geq 4$.

Since $G \neq K_n$, $p \geq 6$. Here, one cannot find two independent edges e_1, e_2 and an edge e adjacent to e_1 and e_2 such that edges incident with w is adjacent to any one of e_1, e_2 or e. Therefore, $D = \{e_1, e_2, e, w\}$ is not a dominating set. But, $D_1 = \{w, e_1, e_2, e_3, e_4\}$, where e_1 and e_2 are not adjacent; e_3 adjacent to both e_1 and e_2; e_4 adjacent to all e_1, e_2, e_3 is a cycle dominating set for $\overline{BG_2(G)}$. Here, e_1 is an edge incident with w and all edges incident with w are dominated by e_1 in $\overline{BG_2(G)}$. Therefore, $\gamma_o(\overline{BG_2(G)}) = 5$. If $p \geq 6$ and $\delta(G) \geq 4$, one can always find such a dominating set. This proves the theorem.

Theorem 2.20 (1) $D = \{u, v, w\} \subseteq V(G)$ is a cycle dominating set for $\overline{BG_2(G)}$ if and only if D is independent in G and D dominates $\overline{G}$.

(2) $D = \{e_1, e_2, e_3\} \subseteq E(G)$ is a cycle dominating set for $\overline{BG_2(G)}$ if and only if e_1, e_2, e_3 form a triangle in G and all other edges are adjacent to any one of e_1, e_2, e_3 in G.

Proof of (1): $D = \{u, v, w\} \subseteq V(G)$ is a cycle dominating set for $\overline{BG_2(G)}$ if and only if D forms a triangle in $\overline{BG_2(G)}$ and dominates $\overline{BG_2(G)}$. This is true, if and only if D is independent in G and there exists no point vertex, which is adjacent to all these point vertices. This proves (1).

Proof of (2): $D = \{e_1, e_2, e_3\} \subseteq E(G)$ is a cycle dominating set of $\overline{BG_2}(G)$ if and only if e_1, e_2, e_3 are adjacent to each other and dominates all other point vertices and line vertices in $\overline{BG_2}(G)$. This is true only when D forms a triangle in G and all other edges of G are adjacent to any one of e_1, e_2, e_3. This proves (2).

Remark 2.10 Any cycle dominating set of $\overline{G}$ is a cycle dominating set for $\overline{BG_2}(G)$.

Theorem 2.21 $\gamma_o(\overline{BG_2}(K_n)) = \begin{cases} 3 & \text{if } n = 3, 4 \\ 4 & \text{if } n = 5 \\ 5 & \text{if } n \geq 6. \end{cases}$

Proof: <u>When n = 3 or 4</u>, any three edges forming triangle in G dominates $\overline{BG_2}(G)$ and $< D > = C_3$. Therefore, $\gamma_o(\overline{BG_2}(K_n)) = 3$ if n = 3, 4.

<u>When n = 5</u>. Consider any two independent edges e_1, e_2 of G, e_3 adjacent to both e_1 and e_2, w is a vertex, not incident with e_1 and e_2. $D = \{e_1, e_2, e_3, w\}$ is a cycle dominating set for $\overline{BG_2}(G)$. Therefore, $\gamma_o(\overline{BG_2}(K_n)) = 4$, if n = 5.

<u>When $n \geq 6$</u>. Take any four vertices v_1, v_2, v_3, v_4. Consider the edges e_1, e_2, e_3 incident with v_1 and $e_4 = v_3v_4$. Let $D = \{e_1, e_2, e_3, e_4, v_2\}$. e_1 and e_2 dominates all other point vertices of $\overline{BG_2}(G)$. v_2 dominates all the line vertices which are edges not incident with v_2, edges incident with v_2 are dominated by e_1. D is a dominating set for $\overline{BG_2}(G)$. Also, $< D >$ has a Hamiltonian cycle.

Therefore, $\gamma_o(BG_2(K_n)) = 5$ if $n \geq 6$.

Remark 2.11 For any graph $G \neq K_{1,n}$, $\gamma_o(\overline{BG_2}(G)) \leq 5$.

Independent domination of $BG_2(G)$ and $\overline{BG_2}(G)$:

This sub section deals with the independent domination of $BG_2(G)$ and $\overline{BG_2}(G)$.

Theorem 2.22 Let G be a graph with a $\gamma(G)$ independent dominating set D. If there exists $e \in E(G)$ such that e is not adjacent to all other edges of $<V(G)-D>$, then $\gamma_i(BG_2(G)) \leq \gamma(G)+1$.

Proof: D is an independent $\gamma(G)$ dominating set for G. Hence, $\gamma_i(G) = \gamma(G)$. By the property of e, $D_1 = D \cup \{e\} \subseteq V(BG_2(G))$ is an independent dominating set for $BG_2(G)$. Therefore, $\gamma_i(BG_2(G)) = \gamma(G)$ or $\gamma(G)+1$.

Theorem 2.23 Let every $\gamma(G)$ dominating set of G is independent and D be a minimum dominating set of G. D is a dominating set of $BG_2(G)$ if and only if G is bipartite.

Proof: Let D be a $\gamma(G)$ dominating set of G. D is independent. If D is a dominating set for $BG_2(G)$, it dominates every line vertices of $BG_2(G)$, that is D is a point cover for G. Therefore, $V(G)-D$ is independent in G. Both D and $V(G)-D$ are independent implies that G is bipartite.

On the other hand if G is bipartite, every minimal dominating set is independent and is a point cover for G. Hence, D is a minimum $\gamma(BG_2(G))$ dominating set of $BG_2(G)$.

Theorem 2.24 Let D be a minimum independent dominating set for G, with cardinality $\gamma_i(G)$. Then (1) $\gamma_i(BG_2(G)) \leq \gamma_i(G)+k$, where $k \geq 3$ is the minimum degree of $< V(G)-D >$.

(2) $\gamma_i(BG_2(G)) \leq \gamma_i(G)+3$ if $< V(G)-D >$ has triangle or has a vertex of degree two or three or has K_2 as a component.

Proof of (1): Let $v \in V(G)-D$ such that $deg_G\, v = k \geq 3$ in G. Consider the edges $e_1, e_2, \ldots, e_k$ in $< V(G)-D >$ incident with v. $\{e_1, e_2, \ldots, e_k\}$ is an independent set in $V(BG_2(G))$. None of $e_1, e_2, \ldots, e_k$ is incident with elements of D. Hence, $D \cup \{e_1, e_2, \ldots, e_k\}$ is an independent dominating set for $BG_2(G)$. Therefore, $\gamma_i(BG_2(G)) \leq \gamma_i(G)+k$.

Proof of (2): If e_1, e_2, e_3 forms a triangle in $< V(G)-D >$, then $D \cup \{e_1, e_2, e_3\}$ is an independent dominating set for $BG_2(G)$. If $<V(G)-D>$ has a vertex of degree two, then $D \cup \{e_1, e_2\}$ or $\{e_1, e_2, e_3\} \cup D$ forms an independent dominating set for $BG_2(G)$. If $< V(G)-D >$ has K_2 as a component, then $D \cup \{e_1\}$, where e_1 is the edge in K_2 is an independent dominating set for $BG_2(G)$. This proves (2).

Theorem 2.25 Let G be a graph without isolated vertices. Then (1) $\gamma_i(\overline{BG_2(G)}) = 2$ if and only if $\beta_1(G) = 2$. (2) $\gamma_i(\overline{BG_2(G)}) = 3$ if and only if $\beta_1(G) \neq 2$.

Proof of (1): Assume that $\beta_1(G) = 2$. Take $D = \{e_1, e_2\}$, where e_1, e_2 are two independent edges in G. Then D dominates $\overline{BG_2(G)}$ and is independent in $\overline{BG_2(G)}$. Therefore, $\gamma_i(\overline{BG_2(G)}) = 2$. Conversely, $\gamma_i(\overline{BG_2(G)}) = 2$ implies that $\gamma(\overline{BG_2(G)}) = 2$ (since G has no isolated vertices.) $\gamma(\overline{BG_2(G)}) = 2$ if and only if $\beta_1(G) = 2$ or $diam(G) \geq 3$. Also, if $diam(G) \geq 3$, then $\beta_1(G) \neq 2$ implies that $\gamma(\overline{BG_2(G)}) \geq 3$ (Remark 2.6). Therefore, $\gamma_i(\overline{BG_2(G)}) = 2$ implies that $\beta_1(G) = 2$.

Proof of (2): When $\beta_1(G) \neq 2$. Consider $e \in E(G)$ such that $e = uv$, where $u, v \in V(G)$. Take $D = \{u, v, e\} \subseteq V(\overline{BG_2(G)})$. D is an independent dominating set for $\overline{BG_2(G)}$. Therefore, $\gamma_i(\overline{BG_2(G)}) = 3$. On the other hand $\gamma_i(\overline{BG_2(G)}) = 3$ implies that $\gamma_i(\overline{BG_2(G)}) \neq 2$. This implies that $\beta_1(G) \neq 2$. Hence the theorem is proved.

Global domination of $BG_2(G)$:

Dominating sets of $BG_2(G)$ which are also dominating sets of $\overline{BG_2}(G)$ are studied here.

Proposition 2.1 $\gamma(G) \leq \gamma_{gd}(BG_2(G)) \leq \gamma(G)+2$.

Proof: Let D be a dominating set for G. Take $u \in D$, $v \notin D$ and $e = uv \in E(G)$. Then $\{u, v, e\}$ is a dominating set for $\overline{BG_2}(G)$ and $D \cup \{v, e\}$ is a dominating set for $BG_2(G)$. Therefore, $D \cup \{v, e\}$ is a global dominating set for $BG_2(G)$. Hence, $\gamma_{gd}(BG_2(G)) \leq \gamma(G)+2$.

The following propositions are stated without proof, since they are easy to follow.

Proposition 2.2 If there exists a global dominating set D of G with cardinality $\gamma(G) > 2$ and is also a point cover for G, then $\gamma_{gd}(BG_2(G)) = \gamma(G)$.

Proposition 2.3 If D is a global dominating independent set for G and is a point cover for G such that $|D| = 2$, then $\gamma_{gd}(BG_2(G)) = 2 = \gamma(G) = |D|$.

Remark 2.12 If $G = K_{1,n} \cup K_{1,m}$, then $\gamma_{gd}(BG_2(G)) = \gamma(G) = 2$.

Proposition 2.4 $\gamma_{gd}(BG_2(G)) \leq \alpha_1(G)+2$.

Proof: Let $D \subseteq E(G)$ be a line cover of G with cardinality $\alpha_1(G)$. Then D or $D \cup \{e\}$ or $D \cup \{u\}$ is a dominating set for $BG_2(G)$. Hence, $\gamma(BG_2(G)) \leq \alpha_1(G)+1$. Again, D is a dominating set for L(G) and hence $D \cup \{e\}$ dominates $\overline{BG_2}(G)$ if $G \neq K_{1,n}$. When $G = K_{1,n}$, $D \cup \{u\}$, where u is the central node of G, dominates $\overline{BG_2}(G)$. Hence, $D \cup \{e, u\}$ is a global dominating set of $BG_2(G)$. Therefore, $\gamma_{gd}(BG_2(G)) \leq \alpha_1(G)+2$.

Following theorem characterize the graphs for which $\gamma_{gd}(BG_2(G)) = 2$.

Theorem 2.26 $\gamma_{gd}(BG_2(G)) = 2$ if and only if G is any one of the following:

(1) $G = K_{1,n} \cup K_{1,m}$. (2) $G = K_2 \cup K_1$. (3) $G = 2K_2$.

Proof: Assume that $\gamma_{gd}(BG_2(G)) = 2$.

Case 1: $D = \{u_1, u_2\} \subseteq V(G)$ is a global dominating set for $BG_2(G)$.

D is a dominating set for $BG_2(G)$. Therefore, all point vertices are adjacent to u_1 or u_2 in G and all the edges in G are incident with either u_1 or u_2 or both.

D is a dominating set for $\overline{BG_2}(G)$. Therefore, all other point vertices are adjacent to either u_1 or u_2 and there is no edge incident with both u_1 and u_2.

Combining all these, it is seen that $G = K_{1,n} \cup K_{1,m}$.

Case 2: $D = \{u, e\} \subseteq V(\overline{BG_2}(G))$, $u \in V(G)$, $e \in E(G)$ is a global dominating set.

D is a dominating set for $BG_2(G)$ implies all other point vertices are incident with e or adjacent to u in G. Let $e = vv_1 \in E(G)$. Suppose G is connected and u and v are not adjacent. Then there exists an edge adjacent to e and not incident with u, which is a contradiction to D dominates $BG_2(G)$. Therefore, v must be adjacent to u. Also, there is no other edge incident with v. But in $\overline{BG_2}(G)$, the edge $e^1 = uv \in E(G)$. The point vertex v is not dominated by u or e, which is a contradiction.

Therefore, in this case G must be disconnected. In $\overline{BG_2}(G)$, e dominates all point vertices not incident with e and u dominates all point vertices not adjacent to u. Suppose there exists an edge e^1, incident with u, in $\overline{BG_2}(G)$, e^1 cannot be dominated by u or e. Therefore, u must be an isolated vertex of G. If e has an adjacent edge e_2 in G, in $BG_2(G)$, e_2 cannot be dominated by e or u. Hence, G must be $K_2 \cup K_1$.

Case 3: $D = \{e_1, e_2\} \subseteq E(G)$ is a global dominating set for $BG_2(G)$.

D dominates $BG_2(G)$ and hence G must be $K_{1,2}$ or $2K_2$. But when $G = K_{1,2}$, $\overline{BG_2}(G)$ cannot be dominated by D. Thus, $\gamma_{gd}(BG_2(G)) = 2$ only when $G = K_{1,n} \cup K_{1,m}$, $K_2 \cup K_1$ or $2K_2$. Converse is obvious. Hence the theorem is proved.

Remark 2.13 (1) Set of all point vertices is a global dominating set for $BG_2(G)$ if and only if $G \neq K_2$.

(2) Set of all line vertices is a global dominating set for $BG_2(G)$ if and only if $G \neq K_{1,n}$.

Restrained domination of $BG_2(G)$ and $\overline{BG_2}(G)$:

In this sub section we study the restrained domination of $BG_2(G)$ and $\overline{BG_2}(G)$.

Theorem 2.27 Let G be a graph with isolated vertices. If there exists a point cover with cardinality $\gamma(G)$, $\gamma_r(BG_2(G)) = \gamma(G)$ or $\gamma(G)+1$.

Proof: D is a point cover in G. Hence, D is a dominating set of $BG_2(G)$ and $V(G)-D$ is independent in G. Consider $V-D = V(BG_2(G))-D$. The point vertices in $V-D$ are adjacent to edges incident with them and vice-versa in $BG_2(G)$. Consider the line vertices in $BG_2(G)$. These can be divided into two parts (1) edges in $< D >$; (2) edges joining vertices of D to vertices of $V(G)-D$. The line vertices of $BG_2(G)$ which are edges in (2) are adjacent to point vertices (in $(V(G)-D)$) in $BG_2(G)$.

<u>Case 1</u>: <u>D is independent</u>.

Then D is a restrained dominating set in $BG_2(G)$. Therefore, $\gamma_r(BG_2(G)) = \gamma(G)$.

<u>Case 2</u>: <u><D> is connected and $|D| = 2$.</u>

Then $D \cup \{e\}$, where e is the edge in $< D >$ is a restrained dominating set. Hence, $\gamma_r(BG_2(G)) \leq \gamma(G)+1$.

<u>Case 3</u>: <u>D is not independent and $|D| \geq 3$</u>.

Each edge in $< D >$ always have some non-adjacent edge in G. Hence, in $V-D$, every line vertex have some adjacent elements. Therefore, D is a restrained dominating set of $BG_2(G)$. Therefore, $\gamma_r(BG_2(G)) = \gamma(G)$.

Theorem 2.28 Let G be a graph without isolated vertices. If $\alpha_0(G) > \gamma(G)$ and D is a $\gamma(G)$ dominating set of G, then $\gamma_r(BG_2(G)) = \gamma(G)$, $\gamma(G)+1$ or $\gamma(G)+2$.

Proof: D is a dominating set for G with cardinality $\gamma(G)$.

<u>Case 1: $\gamma(BG_2(G)) = \gamma(G)$.</u>

In this case, there exists an edge e in G such that $e = uv \in E(G)$, $u \in D$, $v \notin D$, $D_1 = (D-\{u\}) \cup \{e\}$ is a dominating set for $BG_2(G)$. Every point vertex not in D_1 has an adjacent vertex in $V(BG_2(G))-D$ (namely the line incident to it in G) and u and v are adjacent in $BG_2(G)$. Every line vertex e' not in D_1 has an adjacent point vertex in $V(BG_2(G))-D_1$ if e is not in $< D-\{u\} >$. Suppose, e' is an edge in $< D-\{u\}>$, since $\alpha_0(G) > \gamma(G)$ there exists e'' in $V(BG_2(G))-D_1$, which is an edge in $< V(G)-D >$ such that e' and e'' are adjacent elements in $BG_2(G)$. Therefore, D_1 is a restrained dominating set.

<u>Case 2: $\gamma(BG_2(G)) = \gamma(G)+1$.</u>

In this case, $D_1 = D \cup \{u\}$, where all edges in $< V(G)-D >$ are incident with $u \in V(G)$ is a dominating set for $BG_2(G)$ or $D_1 = D \cup \{e\}$, where $e \in E(G)$ is not adjacent to edges in $< V(G)-D >$ is a dominating set for $BG_2(G)$. In this case also D_1 is restrained in $BG_2(G)$ (as in case 1, it can be proved).

<u>Case 3</u>: <u>$\gamma(BG_2(G)) = \gamma(G)+2$.</u>

In this case, $D \cup \{v, e\}$, $u \in D$, $v \notin D$ and $e = uv \in E(G)$ is a dominating set of $BG_2(G)$, which is also restrained. Hence, $\gamma_r(BG_2(G)) \leq \gamma(G)+2$.

Remark 2.14 (1) Set of all line vertices is restrained in $BG_2(G)$ if and only if G has no isolated vertex.

(2) Set of all point vertices is a restrained dominating set for $BG_2(G)$ if and only if $\overline{L(G)}$ has no isolated vertices.

Theorem 2.29 Let G be a graph with $p \geq 6$ and G has a perfect matching. Then $D \subseteq E(G)$ containing the line vertices corresponding to the edges in the perfect matching is a restrained dominating set for $BG_2(G)$.

Proof: Consider $V(BG_2(G)) - D = V - D$. Every element in $V(BG_2(G)) - D = V - D$ has neighbors in D and in $V - D$. Hence, D is a restrained dominating set for $BG_2(G)$. Therefore, $\gamma_r(BG_2(G)) \leq p/2$, if G has a perfect matching.

Remark 2.15 D is clique dominating set of $BG_2(G)$.

Now, we characterize the graphs for which $\gamma_r(BG_2(G)) = 2$.

Theorem 2.30 $\gamma_r(BG_2(G)) = 2$ if and only if G satisfies any one of the following: (1) $G = K_{1,2}$ or K_3. (2) $\gamma(G) = 2 = \alpha_0(G) \neq \gamma_c(G)$. (3) $G = K_2 \cup G'$ with $r(G') = 1$.

Proof: Let D be a restrained dominating set of $BG_2(G)$ with two elements.

<u>**Case 1:** $D = \{u, v\} \subseteq V(G)$.</u>

D dominates $BG_2(G)$ implies that D is a dominating set of G, which is also a point cover of G. Hence, $\alpha_0(G) = 1$ or 2. If $\alpha_0(G) = 1$, then $\gamma(G) = 1$, which implies that $G = K_{1,n}$. But in this case, $\gamma(BG_2(G)) = 1$. Hence <u>$\alpha_0(G)$ must be 2</u>. If $\gamma(G) = 1$ and $\alpha_0(G) = 2$, then G must be K_3 or $K_{1,n}$+edges incident with v, where u is the central vertex of $K_{1,n}$. But in this case, $e = uv \in E(G)$ is not adjacent to any other element in $V(BG_2(G)) - D$. Hence, D is not restrained. Therefore, $\alpha_0(G) = \gamma(G) \neq \gamma_c(G)$.

<u>**Case 2:** $D = \{e_1, e_2\} \subseteq E(G)$.</u>

D is a dominating set of $BG_2(G)$ if and only if $G = 2K_2$ and D is also restrained.

<u>**Case 3:** $D = \{u, e\}$, where $u \in V(G)$ and $e \in E(G)$.</u>

<u>(i) If e is incident with u in G,</u> then $e = uv \in E(G)$. D is a dominating set of $BG_2(G)$ implies that v is pendant in G. Therefore, v is not adjacent to any other element in $BG_2(G)$, which is a contradiction to D is restrained.

(ii) <u>If e is not incident with u in G,</u> then either e has adjacent edges in G or not. If e has no adjacent edges, $G = K_2 \cup G'$, where $r(G') = 1$. If e has an adjacent edge e_1, then u must dominate e_1 in $BG_2(G)$. Hence, G is of the form $K_{1,2}$ or K_3. In all these cases, D is restrained. Hence the theorem is proved.

Following theorems characterize the graphs for which $\gamma_r(\overline{BG_2}(G)) = 1$, $\gamma_r(\overline{BG_2}(G)) = 2$ and $\gamma_r(\overline{BG_2}(G)) = 3$.

Theorem 2.31 (1) $\gamma_r(\overline{BG_2}(G)) = 1$ if and only if G has an isolated vertex.

(2) $\gamma_r(\overline{BG_2}(G)) = 2$ if and only if diam(G) ≥ 3 or $\beta_1(G) = 2$ with $G \neq K_{1,n}+x$, and G has no isolated vertex.

Proof: **<u>Case 1</u>:** <u>diam(G) $\geq$ 3.</u>

There exists u, v $\in$ V(G) such that $d_G(u, v) \geq 3$. Consider, D = {u, v}. D is a dominating set for $\overline{BG_2}(G)$ and is restrained if G has no isolated vertex. Therefore, $\gamma_r(\overline{BG_2}(G)) = 2$ if diam(G) ≥ 3 and G has no isolated vertex. $\gamma_r(\overline{BG_2}(G)) = 1$ if G has an isolated vertex.

<u>Case 2</u>: <u>$\beta_1(G) = 2$.</u>

Let $e_1, e_2 \in E(G)$ be two independent edges of G. D = $\{e_1, e_2\} \subseteq V(\overline{BG_2}(G))$. If $G \neq K_{1,n}+x$, D is restrained in $\overline{BG_2}(G)$. If $G = K_{1,n}+x$, D = $\{e_1, e_2\}$ is a dominating set but it is not restrained, since the central node u $\in$ V(G) is not adjacent to any element of V$-$D.

Conversely, assume $\gamma_r(\overline{BG_2}(G)) = 2$. This implies that $\gamma(\overline{BG_2}(G)) \leq 2$. But, $\gamma(\overline{BG_2}(G)) = 1$ if and only if G has an isolated vertex. Therefore, if G has no isolated vertices, $\gamma(\overline{BG_2}(G)) = 2$. Let D be a minimum dominating set of $\overline{BG_2}(G)$. By Theorem 2.11 and by the definition of restrained domination, G satisfies the given conditions. This proves the theorem.

Theorem 2.32 $\gamma_r(\overline{BG_2}(G)) = 3$ if and only if any one of the following is true:

(1) diam(G) ≤ 2 and $\beta_1(G) > 2$. (2) $G = K_{1,n}+x$.

Proof: $\gamma_r(\overline{BG_2}(G)) = 3$ implies $\gamma(\overline{BG_2}(G)) \leq 3$. Suppose, $\gamma(\overline{BG_2}(G)) = 1$, G has some isolated vertex; suppose $\gamma(\overline{BG_2}(G)) = 2$, $\gamma_r(BG_2(G)) = 2$ if and only if diam(G) $\geq$ 3 or $\beta_1(G) = 2$ with $G \neq K_{1,n}+x$. Therefore, $\gamma_r(\overline{BG_2}(G)) = 3$ implies G has no isolated vertices and diam(G) ≤ 2 and $\beta_1(G) > 2$ or $G = K_{1,n}+x$. Also, $\gamma_r(\overline{BG_2}(G) \leq 3$, since D = {u, v, e}, where u, v $\in$ V(G), e = uv $\in$ E(G) is a restrained dominating set for $\overline{BG_2}(G))$. Converse is obvious.

Remark 2.16 (1) Set of all point vertices is a restrained dominating set for $\overline{BG_2}(G)$ if and only if L(G) has no isolated vertices.

(2) Set of all line vertices is a restrained dominating set if and only if $\overline{G}$ has no isolated vertices, that is radius of G > 1.

(3) If G has no isolated vertices, any dominating set of $\overline{G}$ ($p \geq 4$) containing at least three elements is a restrained dominating set of $\overline{BG_2(G)}$.

Conclusion: Other properties such as eccentricity, traversability, connectivity, characterization, edge partition of $BG_2(G)$ and other domination parameters are studied and submitted.

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