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Eccentricity Properties of Super Duplicate Graphs

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Abstract: *For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. Let $V'(G) = \{v' : v \in V(G)\}$ be a copy of V(G). The Super duplicate graph $D^*(G)$ of G is the graph whose vertex set is $V(G) \cup V'(G)$ and edge set is $E(\overline{G}) \cup \{u'v, uv' : uv \in V(G)\}$, where $\overline{G}$ is the complement of G. In this paper, radius and diameter of $D^*(G)$ are determined. Also sufficient conditions are obtained for $D^*(G)$ to be self-centered with radius three.*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. *Eccentricity* of a vertex $u \in V(G)$ is defined as $e_G(u) = \max \{d_G(u, v): v \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of vertex v in G as e(v) and d(u, v) to denote the distance between two vertices u, v in G respectively. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called a *self-centered* graph with radius r, equivalently G is r-self-centered. A vertex u is said to be an eccentric point of v in a graph G, if d(u, v) = e(v). In general, u is called an *eccentric point*, if it is an eccentric point of some vertex. We also denote the i^{th} neighborhood of v as $N_i(v)=\{u \in V(G) : d_G(u , v) = i\}$ and denote the cardinality of the set H as $|H|$. If $|N_{e(v)}(v)|$ is m for each point $v \in V(G)$, then G is called an *m- eccentric point graph*. If m = 2, we call the graph G as *bi-eccentric point graph*.

The *Super duplicate graph* $D^*(G)$ of G is the graph whose vertex set is $V(G) \cup V'(G)$ and edge set is $E(\overline{G}) \cup \{u'v, uv' : uv \in V(G)\}$, where $\overline{G}$ is the complement of G. The concept of super duplicate graph of a given graph defines Boolean function of a graph based on the adjacency of the vertices of the given graph. As the concept of the distance in graphs plays a dominant role in the study of the structure of the graphs, the related concept of eccentricity was felt much more reliable to have a view on structural properties of graphs. Hence any study of eccentricity properties of graphs and graph operations on a

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given graph is worth while contribution in the field of communication network system in computer science.

In this paper, radius and diameter of D*(G) are determined. Also sufficient conditions are obtained for D*(G) to be self-centered with radius three. The definitions and details not furnished in this paper may found in [2].

2. Prior Results

In this section, we list some results with indicated references, which will be used in the subsequent main results.

Theorem 2.1[3]: Let G be a graph with $\delta(G) \geq 1$. Then D*(G) is disconnected if and only if G is a complete bipartite graph

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Theorem 2.2 [1]: The following three statements are equivalent.

(i). Both G and $\overline{G}$ are self-centered with radius 2;

(ii). G is self-centered with radius 2 having no dominating edge; and

(iii). Neither G nor $\overline{G}$ contain an dominating edge.

3. Main results

In the following, the eccentricity properties of D*(G) are obtained. First, the graphs G for which r(G) = 1 are considered.

Theorem 3.1: For any graph G with r(G) = 1 and $\delta(G) \geq 2$, D*(G) is self-centered with radius 3.

Proof: Since r(G) = 1 and $\delta(G) \geq 2$, each vertex of G lies on a triangle. Also D*(G) is connected.

(i). Let $w \in V(G)$ be such that $e_G(w) = 1$ then $d_{D^*(G)}(w, u) = 2$ for all $u \in V(G)$. Also $d_{D^*(G)}(w, w') = 3$; and $d_{D^*(G)}(w, u') = 1$ for all $u \in V(G)$, $u \neq w$. Hence, $e_{D^*(G)}(w) = 3$.

(ii). Let $u \in V(G)$ be such that $e_G(u) = 2$. Then $d_{D^*(G)}(u, v) = 2$ for all $v \in V(G)$, $u \neq v$, and

$d_{D^*(G)}(u, u') = 3$; and

$d_{D^*(G)}(u, v') = 1$, if $(u, v) \in E(G)\}$

≤ 3, if $(u, v) \notin E(G)\}$

Hence $e_{D^*(G)}(u) = 3$.

(iii). Let u, v$\in$V(G) be such that u', $v'$$\in$V(D*(G)). Since r(G) = 1, $d_{D^*(G)}(u', v') = 2$. From (i), (ii) and (iii), we find that $e_{D^*(G)}(v) = 3$ for all v$\in$V(D*(G)). Hence D*(G) is self-centered with radius 3.

Theorem 3.2: Let G be any graph other than a star with r(G) = 1 and δ(G) = 1. Then D*(G) is bi-eccentric with radius 3.

Proof: Since G is not a star, G has at least one triangle.

(i). If v$\in$V(G) is such that deg(v) = 1, then $e_{D^*(G)}(v) = e_{D^*(G)}(v') = 4$.

(ii). If u$\in$V(G) is such that e(u) = 1, then $e_{D^*(G)}(u') = 4$, since $d_{D^*(G)}(v', u') = 4$. Thus, the pendant vertices in V(G) and its corresponding vertices in V'(G) have eccentricity 4 in D*(G). Also the vertex in V'(G), whose corresponding vertex in V(G) having eccentricity 1, has eccentricity 4 in D*(G) and all the remaining vertices in D*(G) have eccentricity 3. Hence, D*(G) is bi-eccentric with radius 3.

Theorem 3.3: If both G and $\overline{G}$ are self-centered with radius 2, then D*(G) is self-centered with radius 3.

Proof: It is to be observed that V(D*(G)) = V(G)$\cup$$V'$(G).

(i). Since $\overline{G}$ is an induced sub graph of D*(G), $d_{D^*(G)}(v_i, v_j) \leq 2$, for all $v_i, v_j \in V(G)$.

(ii). If $v_i \in V(G)$, then $e_G(v_i) = 2$. Let $(N_j(v_i)/G)$ denote j^{th} neighbor set of v_i in G. Since both G and $\overline{G}$ are self-centered with radius 2, G has no dominating edge by Theorem 2.2. Hence, there exists a vertex $v_i^{(2)}$ in $(N_2(v_i)/G)$ such that $v_i^{(2)}$ is not adjacent to at least one vertex, say $v_i^{(1)}$ in $(N_1(v_i)/G)$. Then $v_i\, v_i^{(2)}\, v_i^{(1)}\, v_i'$ is geodesic in D*(G) and hence $d_{D^*(G)}(v_i, v_i') = 3$.

(iii). If $(v_i, v_j) \in E(G)$, then $d_{D^*(G)}(v_i, v_j') = 1$. Assume $(v_i, v_j) \notin E(G)$. Then $d_G(v_i, v_j) = 2$. If there exists at least one vertex $v_i^{(2)}$ in $(N_2(v_i)/G)$, adjacent to v_j, then $v_i\, v_i^{(2)}\, v_j'$ is geodesic in D*(G) and hence $d_{D^*(G)}(v_i, v_j') = 2$. If not, $d_{D^*(G)}(v_i, v_j') = 3$ as in (ii).

From (i), (ii) and (iii), it follows that $e_{D^*(G)}(v_i) = 3$.

(iv). Let $(v_i, v_j) \in E(G)$. If (v_i, v_j) lies on a triangle, then $d_{D^*(G)}(v_i', v_j') = 2$. If not, as in (ii), $d_{D^*(G)}(v_i', v_j') = 3$. If $(v_i, v_j) \notin E(G)$, then $d_G(v_i, v_j) = 2$ and hence there exists a vertex, say $v_k \in V(G)$ adjacent to both v_i and v_j and $v_i'\, v_k\, v_j'$ is geodesic in D*(G). Thus, $d_{D^*(G)}(v_i', v_j') = 2$.

From (ii), (iii) and (iv), it follows that $e_{D^*(G)}(v_i') = 3$ for all $v_i \in V(G)$. Hence, D*(G) is self-centered with radius 3.

In the following, the eccentricity properties of a super duplicate graph, when the given graph is bi-eccentric with radius 2 are discussed.

Theorem 3.4: Let G be bi-eccentric with radius 2 and $\delta(G) \geq 2$ and $v \in V(G)$ be such that $e(v) = 3$. If $|N_3(v)| = 1$, $<N_2(v)>$ is totally disconnected and each vertex in $N_1(v)$ is adjacent to each in $N_2(v)$, then $D^*(G)$ has radius 3 and diameter 5.

Proof: Let $v_i \in V(G)$ be such that $e(v_i) = 3$ satisfying the conditions given in the theorem. Then there exists a vertex $v_j \in V(G)$ such that $d(v_i, v_j) = 3$ and $d_{D^*(G)}(v_i', v_j') = 5$, since $v_i'\, v_i^{(1)}\, v_j\, v_i\, v_i^{(2)}\, v_j'$ is geodesic in $D^*(G)$, where $v_i^{(1)} \in N_1(v_i)$ and $v_i^{(2)} \in N_2(v_i)$ in G. Also $d_{D^*(G)}(v_i', v) \leq 5$, for all $v \in V(D^*(G))$. Hence, $e_{D^*(G)}(v_i') = 5$. Similarly, $e_{D^*(G)}(v_j') = 5$. Also since $e_G(v_i^{(1)}) = 2$, $e_{D^*(G)}(v_i^{(1)\prime}) = 4$ and $e_{D^*(G)}(v_i) = 3$. It is easy to see that the remaining vertices in $D^*(G)$ have eccentricity k, where $3 \leq k \leq 5$. Thus, $D^*(G)$ has radius 3 and diameter 5.

Remark 3.1: Let G be bi-eccentric with radius 2 and $v \in V(G)$ be such that $e(v) = 3$. If $|N_3(v)| \geq 2$ and $<N_i(v)>$, $(i = 2, 3)$ are totally disconnected and each vertex in $<N_j(v)>$ is adjacent to each in $<N_{j+1}(v)>$, $(j = 1, 2)$, then $D^*(G)$ has radius 3 and diameter 5.

Theorem 3.5: Let G be bi-eccentric with radius 2 and $v \in V(G)$ be such that $e(v) = 3$. If $<N_3(v)>$ contains isolated vertices and $<N_2(v)>$ is not totally disconnected and each vertex in $N_1(v)$ is adjacent to each in $N_2(v)$, then $D^*(G)$ has radius 3 and diameter 4.

Proof: Let $v_i \in V(G)$ be such that $e(v_i) = 3$ satisfying the conditions given in the theorem. Let v_j be an isolated vertex in $<N_3(v_i)>$. Since $<N_2(v_i)>$ is not totally disconnected, $|N_2(v_i)| \geq 2$ and contains at least one edge. Let $N_2(v_i) \subseteq \{v_{i1}^{(2)}, v_{i2}^{(2)}\}$ such that $(v_{i1}^{(2)}, v_{i2}^{(2)})$ be an edge in G. Then $v_i'\, v_{i1}^{(1)}\, (v_{i2}^{(2)})'\, v_{i2}^{(2)}\, v_j'$ is geodesic in $D^*(G)$. Hence, $d_{D^*(G)}(v_i', v_j') = 4$ and $d_{D^*(G)}(v_i', u) \leq 4$, for all u in $V(D^*(G))$. Therefore, $e_{D^*(G)}(v_i') = e_{D^*(G)}(v_j') = 4$. Also, $e_{D^*(G)}(v_i) = 3$ and all the other vertices in $D^*(G)$ have eccentricity 3 or 4. Thus, $D^*(G)$ is bi-eccentric with radius 3 and diameter 4.

Remark 3.2: Let G be bi-eccentric with radius 2. If $<N_3(v)>$ contains no isolated vertices and no vertex in $N_1(v)$ is adjacent to all the vertices in $N_2(v)$ for all $v \in V(G)$ with $e(v) = 3$, then $D^*(G)$ is self-centered with radius 3.

Remark 3.3: Let G be bi-eccentric with radius 2. If $<N_3(v)>$ contains no isolated vertices and each vertex in $N_1(v)$ is adjacent to each in $N_2(v)$ for all $v \in V(G)$ with $e(v) = 3$, then $D^*(G)$ is bi-eccentric with radius 3.

Remark 3.4: There exists no bi-eccentric graph G with radius 2 such that D*(G) is bi-eccentric with radius 4, since the vertex v in V(G) with e(v) = 3 has eccentricity 3 in D*(G).

In the following, the graphs G with $r(G) \geq 3$ for which D*(G) is self-centered with radius 3 are characterized.

Theorem 3.6: Let G be any graph with $r(G) \geq 3$. Then D*(G) is self-centered with radius 3 if and only if G contains no pendant vertices.

Proof: Assume G is any graph with $r(G) \geq 3$ and has no pendant vertices. Then $\overline{G}$ is self-centered with radius 2 and $d_{\overline{G}}(v_i, v_j) \leq 2$, for all $v_i, v_j \in V(\overline{G})$. Let $v_i, v_j \in V(G)$.

(i). Since $e(v_i) \geq 3$, $d_{D^*(G)}(v_i, v_i') = 3$, since $v_i\ v_i^{(3)}\ v_i^{(1)}\ v_i'$ is geodesic in D*(G), where $v_i^{(1)} \in N_1(v_i)$ and $v_i^{(3)} \in N_3(v_i)$.

(ii). If $(v_i, v_j) \in E(G)$, then $d_{D^*(G)}(v_i, v_j') = 1$. Let $(v_i, v_j) \notin E(G)$. Then $d_G(v_i, v_j) \geq 2$.

(a). If $d_G(v_i, v_j) = 2$, then $d_{D^*(G)}(v_i, v_j') = 2$, since $v_i\, v_i^{(3)}\, v_j'$ is geodesic in D*(G).

(b). If $d_G(v_i, v_j) = 3$, then $d_{D^*(G)}(v_i, v_j') = 2$, since $v_i\ v_i^{(2)}\, v_j'$ is geodesic in D*(G).

(c). If $d_G(v_i, v_j) = m$, for $m \geq 4$, then also $d_{D^*(G)}(v_i, v_j') = 2$, since $v_i\ v_i^{(m-1)}\, v_j'$ is geodesic in D*(G), where $v_i^{(k)} \in N_k(vi)$. If $(v_i, v_j) \notin E(G)$, then $d_{D^*(G)}(v_i, v_j') = 2$.

From (i) and (ii), it follows that $e_{D^*(G)}(v_i) = 3$.

(iii). (a). Let $(v_i, v_j) \in E(G)$. If (v_i, v_j) lies on a triangle in G, then $d_{D^*(G)}(v_i', v_j') = 2$. Let (v_i, v_j) do not lie on a triangle in G. Since G has no pendant vertices, there exists a path in G of length three with middle vertices v_i, v_j. Let that path be $v_l\, v_i\, v_j\, v_m$. Then $d_{D^*(G)}(v_i', v_j')$ = 3, since $v_i'\ v_l\ v_m\ v_j'$ is geodesic in D*(G).

(b).If $d_G(v_i, v_j) = 2$, then $d_{D^*(G)}(v_i', v_j') = 2$.

(c). Let $d_G(v_i, v_j) = 3$. Since $r(G) \geq 3$, there exists at least one vertex $v_k \in V(G)$, adjacent to v_j and not adjacent to $v_i^{(1)} \in N_1(v_i)$ in G. Then $v_i'\ v_i^{(1)}\ v_k\ v_j'$ is geodesic in D*(G) and hence $d_{D^*(G)}(v_i', v_j') = 3$.

(d). Let $d_G(v_i, v_j) = m$, where $m \geq 4$. Then $v_i'\ v_i^{(1)}\ v_i^{(m-1)}\ v_j'$ is geodesic in D*(G) and hence $d_{D^*(G)}(v_i', v_j') = 3$, where $v_i^{(m-1)} \in N_{m-1}(v_i)$.

From (ii) and (iii), it follows that $e_{D^*(G)}(v_i') = 3$. Thus, all the vertices in D*(G) have eccentricity 3 and hence D*(G) is self-centered with radius 3.

Conversely, assume D*(G) is self-centered with radius 3, where $r(G) \geq 3$. Let v be a pendant vertex in G and $u \in V(G)$ be such that $(v, u) \in E(G)$. Then $d_{D^*(G)}(v', u') = 4$ in D*(G), since $v'\ u\ (v^{(3)})'\ v\ u'$ is geodesic in D*(G), where $v^{(3)} \in N_3(v)$. This is a contradiction to the assumption that D*(G) is self-centered with radius 3. Hence, G has no pendant vertices.

Remark 3.5: If $\delta(G) = 1$ and $r(G) \geq 3$, then $D^*(G)$ is bi-eccentric with radius 3.

Theorem 3.7: Let G be any graph with $r(G) = 2$ and $diam(G) = 4$. Then $D^*(G)$ is self-centered with radius 3 if and only if G has no pendant vertices.

Proof: Assume G is a graph with $r(G) = 2$ and $diam(G) = 4$. Let $v_i, v_j \in V(G)$.

(i)(a). If $(v_i, v_j) \in E(G)$, since $diam(G) = 4$ there exists at least one vertex $v_k \in V(G)$ (say), not adjacent to both v_i and v_j. Thus, $d_{D^*(G)}(v_i, v_j) = 2$.

(b). If $(v_i, v_j) \notin E(G)$, since $\overline{G}$ is an induced sub graph of $D^*(G)$, $d_{D^*(G)}(v_i, v_j) = 1$.

(ii)a. Let $e_G(v_i) = 2$. If v_i lies on a triangle in G, then $d_{D^*(G)}(v_i, v_i') = 3$. If v_i does not lie on a triangle in G, then there exists at least one vertex $v_i^{(2)} \in N_2(v_i)$, not adjacent to a vertex $v_i^{(1)} \in N_1(v_i)$, since diam(G) is 4. Thus, $v_i\ v_i^{(2)}\ v_i^{(1)}\ v_i'$ is geodesic in $D^*(G)$ & hence $d_{D^*(G)}(v_i, v_i')=3$.

(b). If $e_G(v_i) \geq 3$, then as in (i) of Theorem 3.6., $d_{D^*(G)}(v_i, v_i') = 3$.

(iii)(a). If $(v_i, v_j) \in E(G)$, then $d_{D^*(G)}(v_i, v_j') = 3$.

(b). Assume $d_G(v_i, v_j) = 2$ and $e_G(v_i) = 2$. If there exists a vertex $v_j^{(1)}$ in $N_1(v_j)$ not adjacent to v_i, then $d_{D^*(G)}(v_i, v_j') = 2$, since $v_i\ v_j^{(1)}\ v_j'$ is geodesic in $D^*(G)$. If not, since $diam(G) = 4$, there exists at least one vertex $v_i^{(2)} \in N_2(v_i)$ and $v_i^{(1)} \in N_1(v_i)$ such that $(v_i^{(2)}, v_i^{(1)}) \notin E(G)$ and $(v_i^{(1)}, v_j) \in E(G)$. Then $v_i\, v_i^{(2)}\, v_i^{(1)}\, v_j'$ is geodesic and hence $d_{D^*(G)}(v_i, v_j') =3$.

(c). Assume $d_G(v_i, v_j) = 2$ and $e_G(v_i) \geq 3$. Then $d_{D^*(G)}(v_i, v_j') = 2$.

(d). Let $d_G(v_i, v_j) \geq 3$. Then $d_{D^*(G)}(v_i, v_j') = 2$.

From (i), (ii) and (iii), it follows that $e_{D^*(G)}(v_i) = 3$.

(iv). Since G has pendant vertices, as in (iii) of Theorem 3.6.,

$d_{D^*(G)}(v_i', v_j') = 2$, if (v_i, v_j) lies on a triangle or $d_G(v_i, v_j) = 2$ and
= 3, otherwise.

Thus, $e_{D^*(G)}(v_i') = 3$ by (ii), (iii) and (iv). Hence, each vertex in $D^*(G)$ has eccentricity 3 and $D^*(G)$ is self-centered with radius 3. The proof of the converse part follows from that of Theorem 3.6.

Remark 3.6: From Theorem 3.1.., 3.3., 3.6., 3.7. and Remark 3.2., it follows that $D^*(G)$ is self-centered with radius 3 if one of the following is true.

1. $r(G) =1$ and $\delta(G) \geq 2$;
2. Both G and $\overline{G}$ are self-centered with radius 2;
3. $r(G) \geq 3$ and $\delta(G) \geq 2$;
4. $r(G) = 2$, $diam(G) = 4$ and $\delta(G) \geq 2$; and

5. $r(G) = 2$, $diam(G) = 3$ and <$N_3(v)$> contains no isolated vertices and no vertex in $N_1(v)$ is adjacent to all the vertices in $N_2(v)$ for all $v \in V(G)$ with $e_G(v) = 3$.

Note 3.1: In all the above cases, the complement $\overline{D^*(G)}$ of $D^*(G)$ is self-centered with radius 2.

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