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On the Boolean graph $BG_2(G)$ of a graph G

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Abstract: *Let G be a simple (p, q) graph with vertex set V(G) and edge set E(G).* $B_{G,\ INC,\ \bar{L}(G)}(G)$ *is a graph with vertex set* $V(G) \cup E(G)$ *and two vertices are adjacent if and only if they correspond to two adjacent vertices of G, a vertex and an edge incident to it in G or two non-adjacent edges of G. For simplicity, denote this graph by* $BG_2(G)$*, Boolean graph of G-second kind. In this paper, some properties of* $BG_2(G)$ *are studied.*

Key words: *Boolean graph* $BG_2(G)$*.*

1.Introduction

Let G be a finite, simple, undirected (p, q) graph with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [10], Buckley and Harary [7].

The **girth** of a graph G, denoted g(G), is the length of a shortest cycle (if any) in G; the **circumference c(G)** is the length of any longest cycle. The distance d(u, v) between two vertices u and v in G is the minimum length of a path joining them if any; otherwise $d(u, v) = \infty$. A shortest u-v path is called a *u-v* **geodesic**. A graph G is **geodetic**, if for every pair of vertices (u, v) there exists a unique shortest path connecting them in G.

Let G be a connected graph and u be a vertex of G. The **eccentricity** e(v) of v is the distance to a vertex farthest from v. Thus, $e(v) = \max\{d(u, v) : u \in V\}$. The **radius** r(G) is the minimum eccentricity of the vertices, whereas the **diameter** diam(G) is the maximum eccentricity. For any connected graph G, $r(G) \leq diam(G) \leq 2r(G)$. v is a central vertex if e(v) = r(G). The **center C(G)** is the set of all central vertices. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a **peripheral vertex** if e(v) = diam(G). The **periphery** *P*(G) is the set of all such vertices. For a vertex v, each vertex at distance e(v) from v is an eccentric node of v.

A graph is **self-centered** if every vertex is in the center. Thus, in a self-centered graph G all nodes have the same eccentricity, so r(G) = diam(G).

An edge $uv \in E(G)$ is a **dominating edge** of G, if all the vertices of G other than u and v are adjacent to either u or v.

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A vertex (point) and an edge are said to *cover* each other, if they are incident. A set of vertices, which cover all the edges of a graph G is called a **(vertex) point cover** *of G*, while a set of lines (edges), which covers all the vertices is a **line cover**. The smallest number of points in any point cover for G is called its **point covering number** and is denoted by α_0(G) or α_0. Similarly, α_1(G) or α_1 is the smallest number of lines in any line cover of G and is called its **line covering number**. Clearly, $\alpha_0(K_p) = p-1$ and $\alpha_1(K_p) = \lfloor (p+1)/2 \rfloor$. A point cover (line cover) is called **minimum**, if it contains α_0 (respectively α_1) elements.

A set of points in G is **independent**, if no two of them are adjacent. The largest number of points in such a set is called the **point independence number** of G and is denoted by β_o(G) or β_o. Analogously, an independent set of lines (matching) of G has no two of its lines adjacent and the maximum cardinality of such a set is the **line independence number** β_1(G) or β_1, $\beta_o(K_p) = 1$ and $\beta_1(K_p) = \lfloor p/2 \rfloor$. A set of independent edges covering all the vertices of a graph G is called a **1-factor or a perfect matching** of G.

A **coloring** of a graph is an assignment of colors to its vertices so that no two adjacent vertices have the same color. The **chromatic number** χ(G) is defined to be the minimum n for which G has n coloring.

The minimum number of complete subgraphs of G needed to cover the vertices of G is known as the **clique cover number of G** and is denoted θ(G).

The maximum number of mutually adjacent vertices, that is the size of the largest complete subgraphs of G is known as the **clique number of G** and is denoted ω(G).

A graph G is **Berge** [16] if it does not contain odd cycles of length at least five or their respective complement as induced subgraphs. A graph is **perfect** if $\beta_o(H) = \theta(H)$ for every induced subgraph H of G. This implies that $\omega(H) = \chi(H)$ for every induced subgraph H. Clearly, every bipartite graph is perfect.

Theorem 1.1 (Gallai) [8] For any connected graph G, $\alpha_0+\beta_0 = p = \alpha_1+\beta_1$.

Theorem 1.2 [8] $\beta_o(L(G)) = \beta_1(G)$, $\alpha_0(L(G)) = q-\beta_1(G)$ and $\alpha_1(L(G)) = \lceil q/2 \rceil$.

Theorem 1.3 [10] $\chi(G) \leq 1+ \Delta(G)$.

Theorem 1.4 [10] For any simple graph G, $\chi(G) = \theta(\overline{G})$ and $\beta_o(G) = \omega(\overline{G})$.

Theorem 1.5 (Hayward) [16] If G is Berge and if it contains neither a cycle of length at least 6 or its complement as an induced subgraph, then G is perfect.

Motivation: The Line graphs, Middle graphs, Total graphs and Quasi-total graphs are very much useful in computer networks. In analogous to line graph [5, 17], total graph [4], [5], middle graph [1, 2] and quasi-total graph, thirty-two graphs can be defined using different adjacency relations. Out of these operations, eight were already studied. Among the remaining twenty-four graph operations, two are defined and analyzed here. All the others have been defined and studied thoroughly and will be submitted elsewhere. This is illustrated below.

$G/\ \overline{G}/K_p/\ \overline{K_p}$	Incident (INC)/ Non-incident (NINC)	$L(G)/\ \overline{L(G)}/K_q/\ \overline{K_q}$

Defining a new graph from a given graph by using the adjacency relation between two vertices or two edges and incident relationship between vertices and edges is known as Boolean operation. It defines new structure from the given graph and adds extra information of the original graph.

In Management and in social networks, the incident and non-incident relations of vertices and edges are used to define various networks. So these are very much applicable in socio-economical problems. In some cases, it is possible to retrieve the original graph from the Boolean graphs in polynomial time. So these graph operations may be used in graph coding or coding of some grouped signal. Also, it is possible to study the structure of original graphs through these Boolean graph operations. This motivates the study for the exploration of various Boolean operations and study of their structural properties.

Let G be a (p, q) simple, undirected graph with vertex set V(G) and edge set E(G). The **Boolean graph $B_{G,\ INC,\ \overline{L(G)}}(G)$** has vertex set $V(G) \cup E(G)$ and two vertices in $B_{G,\ INC,\ \overline{L(G)}}(G)$ are adjacent if and only if they correspond to two adjacent vertices of G or to a vertex and an edge incident to it in G or two non- adjacent edges of G. For simplicity, denote this graph by **$BG_2(G)$, Boolean graph of G-second kind[6]**. The vertices of $BG_2(G)$, which are in V(G) are called point vertices and those in E(G) are called line vertices of $BG_2(G)$.

$V(BG_2(G)) = V(G) \cup E(G)$ and $E(BG_2(G)) = [E(T(G)) - E(L(G))] \cup E(\overline{L(G)})$. With an immediate consequence of the definition of $BG_2(G)$, if G is a (p, q) graph, whose vertices are $v_1, v_2, \ldots, v_p$ having degrees d_i, and edges e_{ij}, note that $BG_2(G)$ has p+q vertices and $(q^2+7q-\Sigma d_i^2)/2$ edges with $\deg v_i = 2d_i$; $\deg e_{ij} = q+3-(d_i+d_j)$. Also, G and $\overline{L(G)}$ are induced subgraphs of $BG_2(G)$.

2. Properties of $BG_2(G)$ and $\overline{BG_2(G)}$

First let us see some simple properties of the graph $BG_2(G)$.

Proposition 2.1 (1) Every vertex of $BG_2(G)$ lies in a triangle if G has no isolated vertices.

(2) If $v_i \in V(G)$, and $\deg_G v_i = d_i$ in G, then v_i lies on d_i triangles with distinct edges in $BG_2(G)$.

(3) $BG_2(G)$ has at least q triangles and girth of $BG_2(G)$ is three.

Proof: Proof easily follows from the definition.

Remark 2.1 If G has no isolated vertices, then $\Delta(BG_2(G)) \geq 2\Delta(G)$ and $\delta(BG_2(G)) \geq 2$.

Proposition 2.2 Degree of a vertex in $BG_2(G)$ is two if it is a pendant vertex of G or a line vertex which is adjacent to all other edges in G.

Proof: Let $x \in V(BG_2(G))$ such that deg x = e in $BG_2(G)$. Suppose x is a point vertex, then deg x = $2\deg_G x$ in $BG_2(G)$. This implies that $\deg_G x = 1$, that is, x is pendant in G. Suppose x is a line vertex, then deg x = 2 in $BG_2(G)$ implies that number of edges not adjacent to x in G is zero. Hence, $x \in E(G)$ is adjacent to all the edges of G. The converse is obvious.

Proposition 2.3 (1) $BG_2(G)$ has an isolated vertex if and only if G has an isolated vertex.

(2) $BG_2(G)$ is regular if and only if G is regular and $\deg_G u = (q+3)/4$, $p = 4q/(q+3)$.

(3) $BG_2(G)$ is connected if and only if G has no isolated vertices.

Proof of (1): Assume $BG_2(G)$ has an isolated vertex $x \in V(BG_2(G))$. Since by the definition, every line vertex is adjacent to its incident point vertices, x must be a point vertex. Hence, deg x = 0 in $BG_2(G)$ if and only if $\deg_G x = 0$, if and only if x is an isolated vertex of G.

Proof of (2): $BG_2(G)$ is regular implies degree of every vertex of $BG_2(G)$ is equal. Therefore, $\deg u_i = k$ for all point vertices and $\deg e_{ij} = k$ in $BG_2(G)$, for all line vertices $e_{ij} \in E(G)$. This implies that $2\deg_G u_i = k$ for $u_i \in V(G)$. That is $\deg_G u_i = m$ (= k/2) for all $u_i \in V(G)$. Hence, G is regular of degree m. Also, $\deg e_{ij} = q+3-(\deg_G v_i+\deg_G v_j)$. This implies that m = (q+3)/4, since k = 2m. Since, G is m-regular, this implies p = 4q/(q+3). Thus, $BG_2(G)$ is 2m regular if and only if G is regular of degree m, with m = (q+3)/4 and p = 4q/(q+3).

Proof of (3): Suppose G has an isolated vertex. Then $BG_2(G)$ has an isolated vertex and hence, $BG_2(G)$ is disconnected. On the other hand, suppose G has no isolated vertex. Then G may be connected or disconnected. But in either case, $BG_2(G)$ is connected, since any two line vertices are adjacent in $BG_2(G)$ if they correspond to non-adjacent edges of G. Hence, this proves the result (3).

Proposition 2.4 Let G be a disconnected graph with components G_1 and G_2 (has no isolated vertices). Then the edges of $BG_2(G)$ can be partitioned into $BG_2(G_1)$, $BG_2(G_2)$ and $K_{m,n}$, where m, n denotes the number of edges in G_1, G_2 respectively.

Proof: Since edge e in G_1 is not adjacent to all edges of G_2 and vice versa, the result follows.

Proposition 2.5 $BG_2(G)$ is geodetic if and only if $G = K_{1,m}$, $m \geq 1$.

Proof: Let G be a graph with a triangle. Then $BG_2(G)$ contains a $K_4 - e$. Thus, $BG_2(G)$ is not geodetic. Again if G has an induced P_4, $BG_2(G)$ has a C_4 and hence $BG_2(G)$ is not geodetic. Also, $BG_2(G)$ is geodetic implies G is geodetic, since G is an induced subgraph of $BG_2(G)$. Therefore, $BG_2(G)$ is geodetic if G is geodetic and must not contain a triangle or a P_4. Hence, G must be any one of K_2, $K_{1,n}$, $n \geq 2$. Hence the result is proved.

Proposition 2.6 Let G be a self-centered graph with radius 2. Then every vertices of $BG_2(G)$ lie on $C_3, C_4, \ldots, C_8$ (may not be induced).

Proof: Since G is self-centered, it is two connected. Also, every vertex of $BG_2(G)$ lies on a triangle. Let $u \in V(BG_2(G))$ be a point vertex $u \in V(G)$. Now, let $v \in V(G)$ be an eccentric node of u in G. Therefore, $d_G(u, v) = 2$ and there are at least two paths from u to v, that is u and v lie on a C_4 or C_5 in G. Let $u\, u_1\, v\, v_1 u$ be a C_4 in G. Let $e_1 = uu_1$, $e_2 = u_1v$, $e_3 = vv_1$, $e_4 = uv_1 \in E(G)$. In $BG_2(G)$, one can clearly see that u lies on $C_3, C_4, \ldots, C_8$. Similarly, it can be proved that any line vertex of $BG_2(G)$ lies on some $C_3, C_4, \ldots, C_8$. Hence the theorem is proved.

Remark 2.2 If a point vertex in G lies on C_5, then in $BG_2(G)$ it lies on some $C_3, C_4, \ldots, C_{10}$. If e is any edge in G, which lies in a C_5 in G, then the line vertex e lies in $C_3, C_4, \ldots, C_{10}$ in $BG_2(G)$.

Proposition 2.7 If G is a self-centered graph with radius r, then every vertex of $BG_2(G)$ lies on cycles of length 3, 4, 5, ..., 4r in $BG_2(G)$.

Proof: If $u \in V(G)$ lies on a cycle of length n in G, then as in the previous proportion it can be proved that u lies on a cycle of length 3, 4, ..., 2r in $BG_2(G)$. Similarly, if an edge e in G is an edge on a cycle of length n in G, then in $BG_2(G)$ the line vertex e lies on some cycle $C_3, C_4, \ldots, C_{2n}$ such that C_{k+1} contains exactly $(k-1)$ edges of C_k. Also, if G is a self-centered graph with radius r, then every vertex of G lies on a cycle of length 2r or 2r+1 in G. This proves the proposition.

Now, some properties of the graph $\overline{BG_2(G)}$ can be seen.

$\overline{BG_2(G)}$, the complement of $BG_2(G)$ is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two non-adjacent vertices of G, to two adjacent edges of G or to a vertex and an edge not incident to it in G. Hence, $E(\overline{BG_2(G)}) = (E(\overline{T(G)}) - E(\overline{L(G)})) \cup E(L(G))$. $\overline{G}$ and L(G) are induced subgraphs of $\overline{BG_2(G)}$. Degree of a point vertex in $\overline{BG_2(G)} = p+q-1-2\deg_G u$ and degree of a line vertex e_{ij} in $\overline{BG_2(G)} = p-4+d_i+d_j$. Number of edges of $\overline{BG_2(G)}$ is $p(p-1)/2+q(p-4)+(\sum d_i^2)/2$.

Proposition 2.8 $\overline{BG_2}(G)$ has an isolated vertex if and only if G = $K_{1,n}$, n $\geq$ 1.

Proof: Let G = $K_{1,n}$. Then obviously $\overline{BG_2}(G)$ has an isolated vertex, which is the central vertex of $K_{1,n}$.

On the other hand, assume $\overline{BG_2}(G)$ has some isolated vertices.

Case 1: Suppose this isolated vertex of $\overline{BG_2}(G)$ is a point vertex u $\in$ V(G).
Then u must be adjacent to every other vertex in G and is incident with all edges in G. This gives G = $K_{1,n}$.

Case 2: Suppose this isolated vertex of $\overline{BG_2}(G)$ is a line vertex e.

Then G has only two vertices u, v, where e = uv $\in$ E(G). This implies that G = K_2. Hence the proposition is proved.

Proposition 2.9 $\overline{BG_2}(G)$ has a pendant vertex if and only if G = $K_{1,n} \cup K_1$, $K_{1,n}$+x or K_3.

Proof: Let u $\in$ V($\overline{BG_2}(G)$) be a pendant vertex .
Case 1: u is a point vertex. Then there are only two possibilities (a) u has no non-incident edges in G and has only one non-adjacent vertex in G. In this case, G = $K_{1,n} \cup K_1$. (b) u has a non-incident edge and all other vertices are adjacent to u in G. In this case, G = $K_{1,n}$+x or K_3.
Case 2: e = u is a line vertex. In this case, e has only one non-incident vertex and has no adjacent edges in G. Hence, G = $K_2 \cup K_1$. This proves the result.

Proposition 2.10 Let G be a connected graph. Then girth of $\overline{BG_2}(G)$ = 3, if G $\neq$ $K_{1,2}$ or C_4, girth of $\overline{BG_2}(K_{1,2})$ = 4, and girth of $\overline{BG_2}(C_4)$ = 4.

Proof: Case 1: Δ(G) $\geq$ 3. Then the edges incident at that vertex u such that $\deg_G u$ = Δ(G), form a triangle in $\overline{BG_2}(G)$. Therefore, girth ($\overline{BG_2}(G)$) = 3.

Case 2: Δ(G) $\leq$ 2. Then G is either a circle or a path. Clearly, girth of $\overline{BG_2}(P_n)$ = 3, n $\geq$ 4, girth $\overline{BG_2}(P_3)$ = 4.

$$\text{girth of } \overline{BG_2}(C_n) = \begin{cases} 4 \text{ if } n = 4. \\ 3 \text{ if } n = 3. \\ 3 \text{ if } n > 4. \end{cases}$$

Hence, girth of $\overline{BG_2}(G)$ = 3 if G $\neq$ $K_{1,2}$ or C_4 and girth of $\overline{BG_2}(K_{1,2})$ = 4, girth of $\overline{BG_2}(C_4)$ = 4.

Proposition 2.11 If G is disconnected, then girth of $\overline{BG_2}(G)$ is 3 or 4.

Proof: Let G be a disconnected graph. If a component of G contains more than two vertices then girth of $\overline{BG_2}(G)$ is three.

Suppose each component contains at most two vertices. Suppose G has more than two components, then girth of $\overline{BG_2}(G) = 3$. Suppose G has exactly two components and each contains at most two vertices, then $G = K_2 \cup K_2$ or $K_1 \cup K_2$. Girth of $\overline{BG_2}(2K_2) = 4$ and $\overline{BG_2}(K_1 \cup K_2) = K_{1,3}$. Hence the proposition is proved.

Theorem 2.1 Let G be a (p, q) graph such that $\overline{BG_2}(G)$ is connected. Then $\overline{BG_2}(G)$ is geodetic if and only if $G = K_3$ or $K_2 \cup K_1$.

Proof: If $G = K_3$ or $K_2 \cup K_1$, then $\overline{BG_2}(G)$ is clearly geodetic. On the other hand, let $\overline{BG_2}(G)$ be connected and geodetic. If $p \geq 4$, then G has at least four vertices u_1, u_2, u_3 and u_4 and since G is nontrivial, G has at least one edge $e = u_1u_2 \in E(G)$.

Case 1: G has no other edge.

Then in $\overline{BG_2}(G)$, u_3 and u_4 are adjacent to u_1 and u_2. Also u_3, u_4 are adjacent. Hence, u_1, u_2, u_3, u_4 form $K_4 - x$ as induced subgraph of $\overline{BG_2}(G)$.

Case 2: There exists some other edge in G.

(a) If $e_1 = u_1u_2$, $e_2 = u_3u_4 \in E(G)$, then $u_1\, u_3\, u_2\, u_4\, u_1$ form an induced C_4 in $\overline{BG_2}(G)$. (b) If u_1, u_3 are adjacent in G, $e_1 = u_1u_2$, and u_4 not adjacent to u_1, u_2, u_3 in G, then e_1, e_2, u_3, u_4 form an induced $K_4 - x$ in $\overline{BG_2}(G)$. (c) If u_1 and u_3 are not adjacent in G, then u_4 is not adjacent to u_3, then e_1, e_2, u_4, u_3 form an induced $K_4 - x$ in $\overline{BG_2}(G)$. (d) If u_1 and u_3 are not adjacent in G; u_4 is adjacent to u_3; u_4 is not adjacent to u_1 and u_2, then let $e_3 = u_3u_4$ in G. In this case, u_1, u_4, e_3, u_3 form an induced C_4 in $\overline{BG_2}(G)$.

Similarly, in all other cases there exists an induced C_4 or $K_4 - x$ in $\overline{BG_2}(G)$. Therefore, $\overline{BG_2}(G)$ is not geodetic if $p \geq 4$. Now, consider G with $p \leq 3$. G is non-trivial. Therefore, G is any one of K_2, K_3, $K_{1,2}$ or $K_2 \cup K_1$. Among this $\overline{BG_2}(G)$ is disconnected if $G = K_2$ or $K_{1,2}$ and $\overline{BG_2}(K_3)$, $\overline{BG_2}(K_2 \cup K_1)$ are geodetic.

This proves the theorem.

Theorem 2.2 Let G be a non-trivial (p, q) graph such that $\overline{BG_2}(G)$ is connected. Then $\overline{BG_2}(G)$ is two connected if and only if $G \neq K_{1,n} \cup K_1$, K_3 and $K_{1,n}+x$, $n \geq 1$.

Proof: It is clear that G is 2-connected implies $G \neq K_{1,n} \cup K_1$, K_3 and $K_{1,n}+x$. On the other hand, assume $G \neq K_{1,n} \cup K_1$, K_3 and $K_{1,n}+x$. To prove $\overline{BG_2}(G)$ is two connected, it is enough to prove that any two vertices of $\overline{BG_2}(G)$ is connected by at least two edge disjoint paths in $\overline{BG_2}(G)$. Let $x, y \in V(\overline{BG_2}(G))$.

Case 1: x and y are point vertices of $\overline{BG_2}(G)$. Let $x = u$ and $y = v \in V(G)$.

Sub case 1.1: u and v are adjacent in G.

Let $e = uv \in E(G)$. Since, $G \neq K_{1,n} \cup K_1$, $K_{1,n}+x$, there exists another edge $e_1 = u_1v_1$ not adjacent to $uv = e$ in G. Then $u\ e_1\ v$ is a path from u to v in $\overline{BG_2}(G)$. Also, if u, v, u_1, v_1 form a K_4, then $u\ (u_1v)\ (uv_1)\ v$ is another path, where u_1v and $uv_1 \in E(G)$. Otherwise, $u\ u_1\ v$ or $u\ v_1\ v$ or $u\ v_1\ (uu_1)\ v$ is a path. So, in all cases, there exist at least two paths from u to v.

Sub case 1.2: u and v are not adjacent in G.

u and v are adjacent in $\overline{BG_2}(G)$. If $e_1, e_2 \in E(G)$ and e_1 is incident with u, e_2 is incident with v such that e_1 and e_2 are adjacent in G, then $u\ e_2\ e_1\ v$ is a path in $\overline{BG_2}(G)$. If there exists e_3 not incident with u and v, then $u\ e_3\ v$ is a path. If $G = 2K_2$, then also u and v are joined by at least two paths.

Case 2: $x = e_1$ and $y = e_2$ are line vertices.

Sub case 2.1: e_1 and e_2 are adjacent in G.

In $\overline{BG_2}(G)$, e_1e_2 is a path. Again, $e_1\ w\ e_2$ is a path, where w is a vertex not incident with e_1 and e_2 in G.

Sub case 2.2: e_1 and e_2 are not adjacent in G.

Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$. Consider the induced subgraph formed by u_1, v_1, u_2, v_2, e_1, e_2 in $\overline{BG_2}(G)$. In all cases, there exist at least two paths from e_1 to e_2.

Case 3: $x = u$ is a point vertex and $y = e$ is a line vertex.

Sub case 3.1: e is incident with u in G.

If $e_G(u) \neq 1$ in G, then $u\ w\ e$ is a path in $\overline{BG_2}(G)$. If $e_G(u) = 1$ in G, $u\ e_k\ e$ is a path in $\overline{BG_2}(G)$, where e_k is not incident with u and adjacent to e in G. Since G has at least four vertices (by the given conditions), there exist at least two paths of this type.

Sub case 3.2: e is not incident with u in G.

ue is a path in $\overline{BG_2}(G)$. Since $G \neq K_{1,n}+x$, there exists at least one more path from u to e in $\overline{BG_2}(G)$. This proves the theorem.

Following two theorems give the necessary and sufficient conditions for $BG_2(G)$ and $\overline{BG_2}(G)$ to have a dominating edge.

Theorem 2.3 $BG_2(G)$ has a dominating edge if and only if G satisfies any one of the following: (1) G has a dominating edge e such that all other edges are adjacent to e. (2) $r(G) = 1$ and G has a pendant vertex. (3) $G = 2K_2$.

Proof: Suppose $BG_2(G)$ has a dominating edge.

Case 1: u_1u_2, where u_1, u_2 in V(G) is a dominating edge of $BG_2(G)$.

In this case, all point vertices and line vertices are adjacent to u_1 or u_2 in $BG_2(G)$. This implies all vertices are adjacent to u_1 or u_2 and all edges are incident with u_1 or u_2 in G. This proves that G satisfies (1).

Case 2: ue, where $u \in V(G)$ and $e \in E(G)$ is a dominating edge of $BG_2(G)$.

ue is a dominating edge in $BG_2(G)$ implies that e is incident with u in G, all edges not incident with u are not adjacent to e in G and all other point vertices are adjacent to u in G. Hence, if e = uv, v must be a pendant vertex in G. Thus G is a graph with $e_G(u) = 1$ and v is pendant in G. This proves (2).

Case 3: e_1e_2, (where $e_1, e_2 \in E(G)$) is a dominating edge of $BG_2(G)$.
By the assumption, e_1 and e_2 are non-adjacent edges of G and all point vertices are incident with e_1 or e_2 in G. Hence, $G = 2K_2$. This proves (3).

Proof of the converse is obvious.

Theorem 2.4 $\overline{BG_2(G)}$ has a dominating edge if and only if diameter of G is greater than 2.

Proof: Assume $\overline{BG_2(G)}$ has a dominating edge.

Case 1: uv is a dominating edge in $\overline{BG_2(G)}$, where u and v are in V(G).
uv is a dominating edge in $\overline{BG_2(G)}$, where u and v are adjacent in $\overline{BG_2(G)}$, implies that u and v are not adjacent in G and all other vertices are adjacent to u or v in $\overline{BG_2(G)}$. This implies, in G, there is no vertex adjacent to both u and v in G. Hence, $d_G(u, v) \geq 3$.

Case 2: ue is a dominating edge in $\overline{BG_2(G)}$, where u is in V(G) and e is in E(G).
Since ue is a dominating edge in $\overline{BG_2(G)}$, e is not incident with u in G. If there exists $e_1 = uu_1$, incident with u in G, then e_1 must be adjacent to e and hence, u_1 is not dominated by u or e in $\overline{BG_2(G)}$. Thus, u must be an isolated vertex of G. Therefore, diam(G) > 2.

Case 3: e_1e_2 is a dominating edge in $\overline{BG_2(G)}$, where e_1, e_2 are in E(G).
By assumption, e_1, e_2 are adjacent edges of G. Let $e_1 = uu_1$, $e_2 = uu_2$. Then in $\overline{BG_2(G)}$, u is not dominated by e_1 or e_2. Therefore, this case is not possible.

This proves the theorem.

Using the following lemmas and theorems, we prove that, $BG_2(G)$ is not perfect when $p \geq 5$ and r(G) > 1.

Lemma 2.1 $BG_2(G)$ contains P_4 if and only if G contains any one of P_4, C_4, P_5, $K_{1,2}$, $K_{1,2} \cup K_2$, $2K_2$, K_3, $K_3 \cup K_2$, $K_4 - x$ or $K_{1,3}+x$ as induced subgraphs.

Proof: Suppose $BG_2(G)$ contains P_4 as induced subgraph.

Case 1: All the four vertices of P_4 in $BG_2(G)$ are point vertices.
This gives G contains P_4 as induced subgraph.

Case 2: All the four vertices of P_4 in $BG_2(G)$ are line vertices.
Let $e_1\ e_2\ e_3\ e_4$ be a P_4 in $BG_2(G)$, where e_1, e_2, e_3, e_4 are line vertices. P_4 is induced. In this case, P_5 is an induced subgraph of G.

Case 3: P_4 in $BG_2(G)$ contains one line vertex.

(a) Let $e_1\, v_1\, v_2\, v_3$, where $e_1 \in E(G)$, $v_1, v_2, v_3 \in V(G)$ be a P_4 in $BG_2(G)$. In this case, P_4, $K_4—x$ or $K_{1,3}+x$ is an induced subgraph of G.

(b) Suppose $v_1\, e_1\, v_2\, v_3$ represent P_4 in $BG_2(G)$. This is not possible.

Case 4: P_4 contains two line vertices.

(a) Let $v_1\, e_1\, v_2\, e_2$ be a P_4 in $BG_2(G)$. This is not possible.

(b) Let $v_1\, e_1\, e_2\, v_2$ be a P_4 in $BG_2(G)$. G has $2K_2$, P_4, $K_4—x$ or $K_{1,3}+x$ as induced subgraph. (c) Let $e_1\, e_2\, v_1\, v_2$ be a P_4 in $BG_2(G)$. G has $K_{1,2} \cup K_2$ or $K_3 \cup K_2$ as induced subgraphs of G.

(d) Let $e_1\, v_1\, v_2\, e_2$ be a P_4. This implies that G contains K_3 as an induced subgraph.

Case 5: P_4 in $BG_2(G)$ contains three line vertices.

(a) $v_1\, e_1\, e_2\, e_3$ represents P_4 in $BG_2(G)$. Therefore, G has $K_{1,2} \cup K_2$ or $K_3 \cup K_2$ as subgraphs.

(b) $e_1\, v_1\, e_2\;\; e_3$ represents P_4 in G. In this case, G has P_4, $K_4—x$, $K_{1,3}+x$ or C_4 as induced subgraphs.

Remark 2.3 Let G be a graph without isolated vertices. $BG_2(G)$ has no P_4, if G has no P_4, P_5, $K_4—x$, $K_{1,3}+x$, $K_{1,2} \cup K_2$, $2K_2$, K_3, $K_2 \cup K_3$. Therefore, G is connected, $diam(G) \leq 2$, and G is triangle free. Hence, two cases arise: (1) diam(G) = r(G) = 2. (2) r(G) =1, diam(G) = 2. If (2) is true, $G = K_{1,n}$ since G has no triangles. If (1) is true, G is self-centered with diameter two and has no triangles, P_4 and C_4. This is not possible. Thus, the following theorem is proved.

Theorem 2.5 Let G be a graph without isolated vertices. Then $BG_2(G)$ is free from P_4 if and only if $G = K_{1,n}$.

Lemma 2.2 $BG_2(G)$ has C_4 as induced subgraph if and only if G has C_4, P_4, $K_{1,2} \cup K_2$, $K_3 \cup K_2$, $K_3 \cup K_{1,2}$, $2K_3$ or $2K_{1,2}$, $K_4—e$, $K_{1,3}+e$ or K_4 as induced subgraphs.

Proof: Assume $BG_2(G)$ has C_4 as induced subgraph.

Case 1: All the four vertices of $BG_2(G)$, forming C_4 are point vertices.

This gives G contains C_4 as induced subgraph.

Case 2: C_4 in $BG_2(G)$ contains only one line vertex.

Let $v_1\, e_1\, v_2\, v_3\, v_1$ be a C_4 in $BG_2(G)$, where $v_1, v_2, v_3 \in V(G)$, $e_1 \in E(G)$. This is not possible.

Case 3: C_4 in $BG_2(G)$ contains two line vertices and two point vertices.

Let $v_1\, e_1\, e_2\, v_2\, v_1$ be a C_4 in $BG_2(G)$, where $v_1, v_2 \in V(G)$, $e_1, e_2 \in E(G)$ [Other case is not possible]. Therefore, P_4, C_4, $K_4—e$, K_4 or $K_{1,3}+e$ are induced subgraphs of G.

Case 4: C_4 in $BG_2(G)$ contains three line vertices.

Let $v_1\, e_1\, e_2\, e_3\, v_1$ be a C_4 in $BG_2(G)$, where $v_1, v_2 \in V(G)$ and $e_1, e_2 \in E(G)$. In G, $K_{1,2} \cup K_2$ is a subgraph of G (may not be induced).

Case 5: C_4 in $BG_2(G)$ has all the four vertices as line vertices.

Let $e_1\ e_2\ e_3\ e_4\ e_1$ be a C_4 in $BG_2(G)$. Therefore, G has $2K_{1,2}$ as subgraph. Hence, if G contains more than three vertices (no isolated vertex), then $BG_2(G)$ is free from C_4 only when G = $K_{1,n}$, K_2, nK_2 or K_3.

Theorem 2.6 $BG_2(G)$ is free from C_4 (induced) if and only if G = $K_{1,n}$, nK_2 or K_3.
Proof: Follows from the Lemma 2.2.

Lemma 2.3 $BG_2(G)$ has C_5 as induced subgraph if and only if G has any of P_5, C_5, $K_3 \cup K_2$ or G_1 as subgraph, where G_1 is a connected subgraph of K_5 containing P_3 as induced subgraph.
Proof: Assume $BG_2(G)$ has C_5 as induced subgraph.
Case 1: C_5 in $BG_2(G)$ has all the five vertices as point vertices.
In this case, G has C_5 as induced subgraph.
Case 2: C_5 in $BG_2(G)$ has all the five vertices as line vertices.
Let $e_1\ e_2\ e_3\ e_4\ e_5\ e_1$ be a C_5 in $BG_2(G)$. In this case, G has C_5 as subgraph.
Case 3: C_5 in $BG_2(G)$ has only one line vertex.
Let $v_1\ e_1\ v_2\ v_3\ v_4\ v_1$ be a C_5 in $BG_2(G)$. This is not possible.
Case 4: C_5 in $BG_2(G)$ has two line vertices.
(a) Let $v_1\ e_1\ e_2\ v_2\ v_3\ v_1$ be a C_5 in $BG_2(G)$. In this case, G has P_5, C_5 or any connected subgraph of K_5, having P_3 as induced subgraph is a subgraph of G. Other cases are not possible.
Case 5: C_5 in $BG_2(G)$ has three line vertices.

(a) Let $v_1\ e_1\ e_2\ e_3\ v_2\ v_1$ be a C_5 in $BG_2(G)$. Therefore, G has $K_3 \cup K_2$ as subgraph (may not be induced). Other cases are not possible.
Case 6: C_5 in $BG_2(G)$ has four line vertices.
Let $v_1\ e_1\ e_2\ e_3\ e_4\ v_1$ be a C_5 in $BG_2(G)$. In this case, G has a subgraph having P_5 as subgraph. This proves the lemma.

Theorem 2.7 If G is a connected graph with more than four vertices and diam(G) ≥ 2, then $BG_2(G)$ has C_5 as induced subgraph.
Proof: Follows from the Lemma 2.3.

The following theorems are stated without proof, since the proofs are similar to the proof of the previous theorems.

Theorem 2.8 $BG_2(G)$ has C_6 as induced subgraph if and only if G has C_6 or P_6 as subgraphs (containing P_4 as induced) or $C_4 \cup K_2$ or $K_4-e \cup K_2$ as subgraphs (having C_4 or K_4-e as induced).

Theorem 2.9 $BG_2(G)$ has C_7 as induced subgraph if and only if G has C_7 or P_7 as subgraph containing P_5 as induced or $C_5 \cup K_2$ as subgraph, where C_5 has P_4 as induced subgraph.

Remark 2.4 If G has no P_4 as induced subgraph, then $BG_2(G)$ has no induced C_7. In general, $BG_2(G)$ contain C_n, $n \geq 7$ as induced subgraph if and only if G contains C_n or P_n as subgraph or $C_{n-2} \cup K_2$, as subgraph, where C_{n-2} contains P_{n-3} as induced subgraph.

Theorem 2.10 $\overline{BG_2}(G)$ contains C_4 as induced subgraph if and only if G has C_4, $\overline{C_4}$ or $K_{1,2}$ as induced subgraph or $2K_2$, $3K_2$ as subgraphs.

Theorem 2.11 $\overline{BG_2}(G)$ contains C_5 as induced subgraph if and only if G has C_5 as induced subgraph or C_5, P_5, (containing P_3 as induced) or $C_3 \cup K_2$ as subgraphs.

Theorem 2.12 $\overline{BG_2}(G)$ contains C_6 as induced subgraph if and only if G contains $\overline{C_6}$ as induced subgraph or C_6 as subgraph.

Theorem 2.13 $\overline{BG_2}(G)$ contains C_n, $n \geq 6$ as induced subgraph if and only if G contains $\overline{C_n}$ as induced subgraph or C_n as subgraph.

Theorem 2.14 $BG_2(G)$ is not perfect when $p \geq 5$ and $r(G) > 1$.

Next, chromatic number and covering numbers of $BG_2(G)$ and $\overline{BG_2}(G)$ can be found.

Theorem 2.15 Max $\{\chi(G), \chi(\overline{L}(G)\} \leq \chi(BG_2(G)) \leq \chi(G) + \chi(\overline{L}(G))$.

Proof: By the definition of $BG_2(G)$, every independent set of G and $\overline{L}(G)$ are independent in $BG_2(G)$ also. Hence,

$$\chi(BG_2(G)) \leq \chi(G) + \chi(L(G)) \text{ -------------(1).}$$

Now, let S be an independent set in G. Let S' be an independent set in $\overline{L}(G)$. If elements of S' are not incident with elements of S in G, $S \cup S'$ is independent in $BG_2(G)$. Thus,

$$\max\{\chi(G), \chi(\overline{L}(G)\} \leq \chi(BG_2(G)) \text{ --------------(2).}$$

From (1) and (2), the theorem follows.

Theorem 2.16 $\omega(BG_2(G)) = \max\{3, \omega(G), \omega(\overline{L}(G))\}$.

Proof: K_3 is an induced subgraph of $BG_2(G)$. Also, G and $\overline{L}(G)$ are induced subgraphs of $BG_2(G)$ and there exists no adjacent point vertices v_1, v_2 such that v_1 is adjacent to e_1 and

e_2 and v_2 is adjacent to e_1 and e_2 (e_1, e_2 are line vertices). Therefore, $\omega(BG_2(G)) = \max\{3, \omega(G), \omega(\overline{L}(G)\}$.

Theorem 2.17 Suppose G has no isolated vertices. Then $\theta((BG_2(G)) = \min\{q, \theta(G)+\theta(\overline{L}(G))\}$.

Proof: In $BG_2(G)$, each line vertex is adjacent to its incident point vertices, which are adjacent in $BG_2(G)$. Therefore, at most q triangles or K_3's and K_2's are needed to cover the vertices of $BG_2(G)$. Also, in $BG_2(G)$, there is no complete subgraph K_m, $m > 3$ containing point vertices and line vertices. Thus,

$$\theta((BG_2(G)) = \min\{q, \theta(G)+\theta(\overline{L}(G))\}.$$

Theorem 2.18 Max $\{\beta_o(G), \beta_o(\overline{L}(G))\} \leq \beta_o(BG_2(G)) \leq \beta_o(G)+\beta_o(\overline{L}(G))$.

Proof: Let $D_1 \subseteq V(G)$ be a set of mutually non-adjacent vertices of G. Let $|D_1| = \beta_o(G)$. Consider $< V(G)-D_1 >$. Let $D_2 \subseteq E(G)$ be the set containing maximum number of mutually adjacent edges in $< V(G)-D_1 >$. Then $D_1 \cup D_2$ is an inde-pendent set of vertices of $BG_2(G)$. Also, $|D_1 \cup D_2| = \beta_o(G)+k$, where $k =|D_2|$. Therefore, $\beta_o(BG_2(G)) \leq \beta o(G)+k$. Also, $\beta_o(BG_2(G)) \geq \max\{\beta(G), \beta_o(\overline{L}(G))\}$. Hence,

$$\max\{\beta_o(G), \beta_o(\overline{L}(G))\} \leq \beta_o(BG_2(G)) \leq \beta_o(G)+\beta_o(\overline{L}(G)).$$

Theorem 2.19 $\alpha_0(BG_2(G)) \leq \alpha_0(G)+\alpha_0(\overline{L}(G))$; $\alpha_0(BG_2(G)) \leq \min\{p+\alpha_0(\overline{L}(G)), q+\alpha_0(G)\}$.

Proof: Let D_1, D_2 be point covers of G and $\overline{L}(G)$ respectively. Then $D_1 \cup D_2$ is a point cover for $BG_2(G)$. Therefore, $\alpha_0(BG_2(G)) \leq \alpha_0(G)+\alpha_0(\overline{L}(G))$. This also implies that

$$\alpha_0(BG_2(G)) \leq \min\{p+\alpha_0(\overline{L}(G)), q+\alpha_0(G)\}.$$

Theorem 2.20 $\alpha_1(BG_2(G)) \leq \alpha_1(G)+\alpha_1(\overline{L}(G))$.

Proof: Since G and $\overline{L}(G)$ are subgraphs of $BG_2(G)$, $\alpha_1(BG_2(G)) \leq \alpha_1(G)+\alpha_1(\overline{L}(G))$.

Case 1: $q \leq p$.

Consider q edges in $BG_2(G)$, each joining a line vertex to a point vertex (distinct). Consider the remaining $p-q$ point vertices. Let $k = \min\{\alpha_1(G_1)\}$, where G_1 is a subgraph of G containing $p-q$ point vertices. Hence, $\alpha_1(BG_2(G)) \leq q+k$.

Case 2: $q > p$.

Consider the p edges joining a point vertex to a line vertex (distinct). Consider the remaining $q-p$ line vertices and let $k_1 = \min\{\alpha_1(G_2)\}$, where $G_2 \subseteq \overline{L}(G)$ containing, $q-p$ line vertices. Therefore, $\alpha_1(BG_2(G)) \leq p+k_1$.

Theorem 2.21 $\beta_1(BG_2(G)) \geq \beta_1(G)+\beta_1(\overline{L(G)})$

$$\beta_1(BG_2(G)) \leq \begin{cases} q + k' \text{ if } q \leq p \\ q + k_1' \text{ if } p < q, \text{ where} \end{cases}$$

$k' = \max(\beta_1(G_1))\}$, G_1 is a subgraph of G containing p−q vertices,

$k_1' = \max\{\beta_1(G_2)$, G_2 is a subgraph of $\overline{L(G)}$ containing q−p vertices.

Proof: Similar to the previous theorem.

The following theorem is stated without proof, since the results are easy to follow.

Theorem 2.22

(1) $\max\{\chi(\overline{G}), \chi(L(G)\} \leq \chi(\overline{BG_2(G)}) \leq \chi(\overline{G})+\chi(L(G))$.

$\chi(\overline{BG_2(G)}) = \min\{\chi(\overline{G}), \chi(L(G)\}$.

2. $\max\{\omega(\overline{G}), \omega(L(G)\} \leq (\omega(\overline{BG_2(G)}) \leq \omega(\overline{G})+\omega(L(G))$.

3. $\max\{\theta(\overline{G}), \theta(L(G)\} \leq \theta(\overline{BG_2(G)}) \leq \theta(\overline{G})+\theta L(G))\}$.

4. $\beta_o(\overline{BG_2(G)}) = \max\{3, \beta_o(\overline{G}), \beta_o(L(G))\}$.

5. $\alpha_0(\overline{G})+\alpha_0(L(G)) \geq \alpha_0(\overline{BG_2(G)})$.

$\alpha_0(\overline{BG_2(G)}) \leq \min\{p+\alpha_0(L(G)), q+\alpha_0(\overline{G})\}$.

6. $\alpha_1(\overline{BG_2(G)}) \leq \min\{\alpha_1(\overline{G})+\alpha_1(L(G)), p+k, q+k_1\}$,

where $k = \min\{\alpha_1(G_1)\}$, where $G_1 \subseteq L(G)$ containing q−p line vertices.

$k_1 = \min\{\alpha_1(G_2)\}$, where $G_2 \subseteq \overline{G}$ containing p−q point vertices.

7. $\beta_1(\overline{BG_2(G)}) \geq \max\{\beta_1(G)+\beta_1(\overline{L(G)}), q+k' \text{ if } q \leq p\}$

$$\beta_1(\overline{BG_2(G)}) \geq \begin{cases} q + k' \text{ if } q \leq p \\ q + k_1' \text{ if } p < q, \end{cases}$$

where $k' = \max\{\beta_1(G_1)\}$, G_1 is a subgraph of G containing p−q vertices, and

$k_1' = \max\{\beta_1(G_2)\}$, G_2 is a subgraph of $\overline{L(G)}$ containing q−p vertices.

Conclusion: Other properties such as eccentricity, traversability, connectivity, characterization, edge partition of $BG_2(G)$ and domination parameters are studied and submitted.

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