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Changing and Unchanging of Distance Closed Domination Number in Graphs

T. N. Janakiraman[1], P. J. A. Alphonse[2] and V. Sangeetha[3]

[1,3]*Department of Mathematics,*

National Institute of Technology, Tiruchirapalli, India.

E-mail:janaki@nitt.edu, sangeetha77_2005@yahoo.co.in

[2]*Department of Computer Applications,*

National Institute of Technology, Tiruchirapalli, India.

E-mail: alphonse@nitt.edu

Abstract: *In a graph G = (V, E), a set S⊂V(G) is a distance closed set of G if for each vertex u∈S and for each w∈V – S, there exists at least one vertex v ∈S such that $d_{<S>}(u, v) = d_G(u, w)$ and S is called a dominating set of G if every vertex in V(G) – D is adjacent to some vertex in S. Also, S is said to be a distance closed dominating set of G if (i) <S> is distance closed and (ii) S is a dominating set. The concept of distance closed domination in graphs is studied in [10] and it is useful in communication networks. In this paper, we have studied the changing and unchanging of distance closed domination number in graphs and this concept is more applicable in a fault tolerance networks.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self-centered graph, neighborhood, induced sub graph, unique eccentric point graph, ciliates, distance closed dominating set, distance closed domination number.*

1. Introduction

Graphs discussed in this paper are undirected and simple. Unless otherwise stated the graphs which we consider are connected and simple graphs only. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $deg_G(v)$. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max \{d_G(u, v): \forall u \in V(G)\}$. If there is no confusion, we simply use the notion deg (v), d(u, v) and e(v) to denote degree, distance and eccentricity respectively for the connected graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted by r(G) and d(G) respectively. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r self-centered* graph. Such graphs are 2-connected graphs. Some structural properties are studied in [2] and [3]. A vertex u is said to be an eccentric vertex of v in a graph G, if d (u, v) = e(v) in that graph. In general, u is called an *eccentric vertex*, if it is an eccentric vertex of some vertex. For v∈V(G), the *neighborhood*

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$N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood* of v. A set S of edges in a graph is said to be independent if no two of the edges in S are adjacent. An edge e = (u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v.

The concept of distance in graph plays a dominant role in the study of structural properties of graphs in various angles using related concept of eccentricity of vertices in graphs. The study of structural properties of graphs using distance and eccentricity started with the study of center of tree and propagated in different directions in the study of structural properties of graphs such as unique eccentric point graphs, k-eccentric point graphs, self-centered graphs, graphs realizing given eccentricity sequence, radius, diameter and eccentric critical graphs and Hamiltonian properties in iterated line graphs. A ciliate $C_{p,\,q}$ is a graph obtained from p disjoint copies of the path p_{q+1} by linking together one end point of each in a cycle C_p. Also these ciliates are the only graphs that are radius critical (graphs in which removal of every vertex changes the radius of the given graph). The structural and eccentricity properties of various graph operations and iterated graph operations are given in references [2], [3], [4], [7] [9], [12] and [14].

The concept of domination in graphs was introduced by Ore [13]. A set $D \subseteq V(G)$ is called dominating set of G if every vertex in V(G) – D is adjacent to some vertex in D. D is said to be a minimal dominating set if D – {v} is not a dominating set for any $v \in D$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of dominating sets. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on the dominating sets. The sub graph of a graph G whose vertex set is S and whose edge set is the set of those edges of G that have both ends in S is called the induced sub graph of G induced by S and is denoted by <S>. A dominating set D is called *connected* (*independent*) dominating set if the induced sub graph <D> is connected (independent). D is called a *total dominating set* if every vertex in V(G) is adjacent to some vertex in D. The list of survey of domination theory papers are in [6], [11], [15], [16] and [17].

The new concepts such as ideal sets, distance preserving sub graphs, eccentricity preserving sub graphs, super eccentric graph of a graph, pseudo geodetic graphs are introduced and structural properties of those graphs are studied in [9]. Janakiraman and Alphonse [1] introduced and studied the concept of weak convex dominating sets, which mixes the concept of dominating set and distance preserving set.

The changing and unchanging terminology was first suggested by Harary [8]. It is useful to partition the vertex set or the edge set of a graph G into three sets according to how their addition or removal affects the domination number. The changing and unchanging of a given domination number is more applicable in a fault tolerance network. The behavior of a network in the presence of a fault can be analyzed by determining the effect that removing an edge (link failure) or a vertex (processor failure) from its underlying graph G has on the fault tolerance criterion. For example, a γ-set in G represents a minimum set of processors that can communicate directly with all other processors in the system. Also the networks can be made fault tolerant by providing redundant communication links (adding edges). The concept of changing and unchanging invariant of graphs is studied in [5], [8], [18], [19] and [20].

In this paper, we have studied the changing and unchanging the above distance closed domination number in graphs.

2. Prior results

The concept of ideal set is defined and studied in the doctoral thesis of Janakiraman [9] and the concept of ideal sets in graph theory is due to the related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to the multiplicative closure property with the elements of that ring. Similarly, the ideal set in a graph is defined with respect to the distance property between the ideal set and the vertices of the graph. Thus, the ideal set of a graph G is defined as follows:

Let I be a vertex subset of G. Then I is said to be an *ideal set* of G if

(i) For each vertex $u \in I$ and for each $w \in V - I$, there exists at least one vertex $v \in I$ such that $d_{<I>}(u, v) = d_G(u, w)$.

(ii) I is the minimal satisfying (i).

Also, a graph G is said to be a 0-ideal graph if it has no non-trivial ideal set other than G. The ideal set without the minimality condition is taken as a distance closed set in the present work. Hence, the distance closed set of a graph G is defined as follows:

A vertex subset S of G is said to be a *distance closed set* of G if for each vertex $u \in S$ and for each $w \in V - S$, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$. For example, in the graph given in Figure 2.1, S = {4, 1, 2, 5} is a distance closed set.

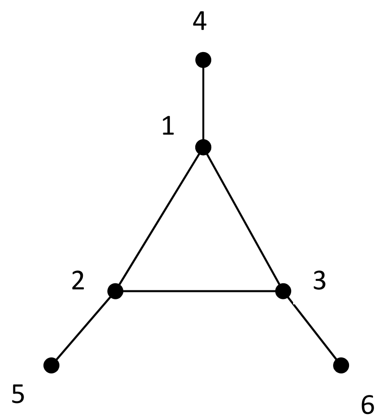

Figure 2.1 - ***An example of distance closed set***

Thus, the distance closed dominating set of a graph G is defined as follows:

A subset $S \subseteq V(G)$ is said to be a *distance closed dominating* (*D.C.D*) *set*, if

(i) <S> is distance closed;

(ii) S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the *distance closed domination number* of G and is denoted by γ_{dcl}.

Clearly from the definition, $1 \le \gamma_{dcl} \le p$ and a graph with $\gamma_{dcl} = p$ is called a 0-distance closed dominating graph. Also if S is a D.C.D set of G, then the complement V – S need not be a D.C.D set of G. The definition and the extensive study of the above said distance closed domination in graphs are studied in [10]. The following results given in [10] are used to prove many results in the present work.

Theorem 2.1[10]: If T is a tree with number of vertices $p \ge 2$, then $\gamma_{dcl}(T) = p - k + 2$, where k is the number of pendant vertices in T.

Theorem 2.2[10]: If G is a 2-self centered graph with a dominating edge, then $\gamma_{dcl}(G) = 4$.

Theorem 2.3[10]: Let G be a 2-self centered graph having no dominating edge. If v is a vertex with degree δ in G, then $\gamma_{dcl}(G) \le 2\delta + 1$.

Theorem 2.4 [10]: If a graph G is connected and $d(G) \ge 3$, then $\gamma_{dcl}(\overline{G}) = 4$.

3. Main results

In this paper, we have studied the structure of graphs by using changing and unchanging parameters of the distance closed domination number in a graph G.

3.1 Changing and unchanging of vertex removal (CVR and UVR):

If we remove a vertex from a graph G, then the vertex set of G is partitioned into three sets according to how its removal changes\unchanges the distance closed domination number. Also removal of a vertex can increase the distance closed domination number by more than one, but can decrease it by at most one. In some graphs, there are vertices whose removal will maintain (unchange) the distance closed domination number. Thus, we have the following 3 sets.

Let $VD^0 = \{v \in V(G) \mid \gamma_{dcl}(G - v) = \gamma_{dcl}(G)\}$ (UVR)

$VD^+ = \{v \in V(G) \mid \gamma_{dcl}(G - v) > \gamma_{dcl}(G)\}$ (CVR)

$VD^- = \{v \in V(G) \mid \gamma_{dcl}(G - v) < \gamma_{dcl}(G)\}$

Then $V = VD^0 \cup VD^+ \cup VD^-$ and the following are the results of some special classes of graphs without proof.

1. For any complete graph K_p, $VD^0 = V(G)$, $p \ge 3$.
2. For any complete bipartite graph $K_{m, n}$, $VD^0 = V(G)$, $p \ge 6$ and $m, n \ge 3$. Also if $m, n = 2$ then $V(G) = VD^-$,where $m + n = p$.

3. For a Petersen graph G (2-self centered), $V = VD^-$.
4. For any cycle C_{2n}, $VD^- = V(G)$ and for C_{2n+1}, $VD^0 = V(G)$.

Proposition 3.1.1: Let G be a graph with $p \geq 3$. If G has $\gamma_{dcl}(G) = 2$ and having at least three vertices with degree $p - 1$, then $V = VD^0$.

Proof: Let G be a graph with $p \geq 3$ and let $\gamma_{dcl}(G) = 2$. If G has at least three vertices with eccentricity equal to 1, then G is either a complete graph or a graph with radius 1 and diameter 2. In both the cases, removal of any vertex u will not affect the D.C.D number of G. Hence, $V = VD^0$.

Proposition 3.1.2: Let G be a graph with $p \geq 3$. If G has $\gamma_{dcl}(G) = 2$ and having exactly two vertices with degree $p - 1$, then $V = VD^0 \cup VD^+$, where $VD^0 = \{v \in V(G) \mid e(v) = 2\}$, $VD^+ = \{v \in V(G) \mid e(v) = 1\}$ and $|VD^+| = 2$.

Proof: Since $\gamma_{dcl}(G) = 2$ and G has exactly two vertices with eccentricity equal to 1, G is of radius 1 and diameter 2. If we remove any vertex v with eccentricity 2, then it will not affect the D.C.D number of G. Thus $v \in VD^0$, for every v with $e(v) = 2$ in G. Also, if we remove any vertex v with eccentricity 1, then it will increase the D.C.D number of G to 3. Thus $v \in VD^+$, for every v with $e(v) = 1$ in G and $|VD^+| = 2$.

Hence, $V = VD^0 \cup VD^+$.

Proposition 3.1.3: If G is a graph with $p \geq 4$ and $\gamma_{dcl}(G) = 3$, then $V = VD^0 \cup VD^+ \cup VD^-$, where

$VD^0 = \{v \in V(G) \mid e(v) = 2$ and v is not a unique eccentric point of any other vertex in G$\}$.

$VD^- = \{v \in V(G) \mid e(v) = 2$ and v is a unique eccentric point of at least one vertex of G$\}$.

$VD^+ = \{v \in V(G) \mid e(v) = 1\}$ and $|VD^+| = 1$.

Proof: Since $\gamma_{dcl}(G) = 3$, G is a graph with radius 1 and diameter 2 and G has exactly one vertex with eccentricity 1 (otherwise $\gamma_{dcl}(G) = 2$), say v. Then clearly removal of the vertex v from G will increase the D.C.D number of G. Therefore, $\gamma_{dcl}(G - v) > \gamma_{dcl}(G)$. Hence, $v \in VD^+$ and $|VD^+| = 1$ (as v is the only such vertex in G).

Also, if we remove any vertex u with eccentricity 2 and u is a unique eccentric point of at least one vertex of G, then G – u has at least one vertex with eccentricity 1 other than u. Therefore, $\gamma_{dcl}(G - v) = 2$. Hence $\gamma_{dcl}(G - v) < \gamma_{dcl}(G)$ and hence $u \in VD^-$ .Suppose that if u is a vertex with $e(u) = 2$ and u is not a unique eccentric point of any other vertex in G, then removal of u will not affect the D.C.D number of G. Hence $v \in VD^0$ and hence $V = VD^0 \cup VD^+ \cup VD^-$.

Theorem 3.1.1: If G is a $(p - 2)$ regular graph, then $V = VD^-$.

Proof: If G is a (p – 2) regular graph, then G is a self-centered graph of diameter 2. Also, G is a 2-connected unique eccentric point graph and every vertex and its eccentric point lie on a C_4. Therefore, $\gamma_{dcl}(G) = 4$. If we remove any vertex v, then the eccentricity of v^1 will be reduced to 1, where v^1 is the eccentric vertex of v. Thus, the set of vertices $\{u, v^1, w\}$ forms a D.C.D set of G – v, where u and w are non adjacent. Hence, $\gamma_{dcl}(G - v) = 3 = \gamma_{dcl}(G) - 1 < \gamma_{dcl}(G)$, for every $v \in V(G)$ and hence $V = VD^-$.

Theorem 3.1.2: Let G be a graph with diameter 3 and radius 2 and $\gamma_{dcl}(G) = 4$. If a vertex $v \in VD^-$, then

(i) v is non adjacent to at least one vertex of degree p – 2 in G.

(ii) d(G – v) < d(G).

(iii) v must be the peripheral vertex.

Proof: Let G be a graph with diameter 3 and radius 2 and $\gamma_{dcl}(G) = 4$.

(i) Let $v \in VD^-$. Then $\gamma_{dcl}(G - v) < \gamma_{dcl}(G)$. That is $\gamma_{dcl}(G - v)$ is either 3 or 2, which only means that G – v has at least one vertex of degree p – 2, say u. Suppose that, if v is adjacent to u, then d(u) = p – 1 in G, a contradiction to G is of diameter 3. Hence, v is not adjacent to u.

(ii) By (i), v is not adjacent to at least one vertex of degree p – 2 in G, say u. That is, d(u) = p – 2 in <G – v>. Hence, the induced sub graph of G – v, <G – v> is of diameter 2 and hence d(G – v) = 2 < d(G).

(iii) By (ii) d(G – v) = 2. Suppose that, if e(v) = 2, then the diameter of G is also equal to 2, a contradiction to G is of diameter 3. Hence, e(v) = 3 and hence v must be the peripheral vertex.

Proposition 3.1.4: If G is an r-self centered and a unique eccentric point graph, then $V = VD^-$.

Proof: Let G be a self centered graph of diameter r. Let $v \in V(G)$ and let u be the vertex of G such that v is the unique eccentric point of u. Then clearly $e(u \mid G) > e(u \mid G - v)$. Hence, $\gamma_{dcl}(G - v) < \gamma_{dcl}(G)$ and hence $v \in VD^-$. This is true for every vertex $v \in V(G)$. Therefore, $V = VD^-$.

Proposition 3.1.5: Let G be a 2-self-centered graph with $\delta \geq 3$. In G, if for every vertex $u \in V(G)$ both $<N_1(u)>$ and $<N_2(u)>$ are independent, then $V = VD^0$.

Proof: Let G be a 2-self centered graph with $\delta \geq 3$. If for every vertex $u \in V(G)$ both $<N_1(u)>$ and $<N_2(u)>$ are independent, then G is a bipartite graph. Hence, $\gamma_{dcl}(G - v) = 4 = \gamma_{dcl}(G)$ for every vertex $v \in V(G)$ and hence $V = VD^0$.

Proposition 3.1.6: If G is a (p – 3) regular graph with $p \geq 5$, then $V = VD^0$.

Proof: Let G be a (p – 3) regular graph with p ≥ 5. Then every vertex v of G must lie on a C_5. For otherwise d(v) ≥ p – 3, a contradiction to G is (p – 3) regular. Hence, G must be 2-self centered and $\gamma_{dcl}(G) = 4$. Also for every $v \in V(G)$, (G – v) is 2-self centered and the D.C.D set of (G – v) is a dominating cycle of length 4 or a P_4. Therefore, $\gamma_{dcl}(G - v) = 4$. Hence, every vertex v of G is in VD^0 and hence $V = VD^0$.

Proposition 3.1.7: If G is a ciliate on p vertices, then $V = VD^-$.

Proof: In a ciliate, we know that $\gamma_{dcl}(G) = p$. Since the ciliates are radius critical, if we remove any vertex v, then $\gamma_{dcl}(G - v) = p - 1$. Hence, $\gamma_{dcl}(G - v) < \gamma_{dcl}(G)$ for every vertex $v \in V(G)$ and hence $V = VD^-$.

Theorem 3.1.3: If T is a tree with p vertices and k number of pendant vertices and if a vertex $v \in VD^0$, then v must satisfy any one of the following conditions.

(i) v is a pendant vertex whose support vertex is of degree greater than or equal to 3.

(ii) v is a support vertex with degree equal to 3 and its adjacent vertices other than the pendant vertex are of degree equal to 2.

(iii) v is a vertex of degree 2 and it is adjacent to two vertices u and w such that d(u) = 2 and d(w) ≥ 3.

Proof: Let T be a tree with p vertices and k number of pendant vertices.

(i) Since the distance closed domination number of any tree T is p – k + 2, removal of any pendant vertex v, whose support vertex is of degree greater than or equal to 3 will maintain the distance closed domination number. That is, $\gamma_{dcl}(T - v) = \gamma_{dcl}(T)$. Hence, $v \in VD^0$.

(ii) Let v be a support of a pendant vertex u of T with degree equal to 3. Since the adjacent vertices of v are of degree 2, T – v has three components T_1, T_2 and T_3 each has p_1, p_2 and p_3 ($|p_3| = 1$) vertices and k_1, k_2 and k_3 number of pendant vertices respectively.

$$\begin{aligned}\text{Therefore, } \gamma_{dcl}(T - v) &= \gamma_{dcl}(T_1) + \gamma_{dcl}(T_2) + \gamma_{dcl}(T_3) \\ &= p - 2 - (k + 1) + 5 \\ &= p - k + 2 = \gamma_{dcl}(T).\end{aligned}$$

Hence, $v \in VD^0$.

(iii) If v is a vertex of degree 2 and it is adjacent to two vertices u and w such that d(u) = 2 and d(w) ≥ 3, then T – v has two components T_1 and T_2 such that $|T_1| + |T_2| = p - 1$ and the number of pendant vertices must be increased by one in T – v. Also, $|T_1| = p_1$, $|T_2| = p_2$ and k_1, k_2 are the number of pendant vertices in T_1 and T_2 respectively.

$$\begin{aligned}\text{Therefore, } \gamma_{dcl}(T - v) &= \gamma_{dcl}(T_1) + \gamma_{dcl}(T_2) \\ &= (p - 1) - (k + 1) + 4 \\ &= p - k + 2 = \gamma_{dcl}(T).\end{aligned}$$

Hence, $v \in VD^0$.

Proposition 3.1.8: Let T be any tree. If v is a vertex with $d(v) \geq 4$ in T, then $v \in VD^+$.

Proof: Let T be any tree and let v be a vertex with $4 \leq d(v) = m$. Also, let k be the number of pendant vertices in T.

Case (i): ***T is a star $K_{1, m}$***

If T is a star $K_{1, m}$ and $4 \leq d(v) = m$, where $p = m + 1$ and $k = m = p - 1$, then $T - v$ has m number of isolated vertices and trivially $v \in VD^+$.

Case (ii): ***v is a support vertex of (m – 1) pendant vertices and u is a vertex adjacent to v with $d(u) \geq 2$***

If v is a support vertex of (m – 1) pendant vertices and u is a vertex adjacent to v with $d(u) \geq 2$, then we have the following two sub cases.

Sub case (a): ***$d(u) = 2$***

If $d(u) = 2$, then $T - v$ has $(m - 1)$ isolated vertices and $k - (m - 2)$ pendant vertices.

Thus, $\gamma_{dcl}(T - v) = (p - 1) - [k - (m - 2)] + 2$

$\geq p - k + 3 = \gamma_{dcl}(T) + 1$

Hence, $\gamma_{dcl}(T - v) > \gamma_{dcl}(T)$.

Sub case (b): ***$d(u) \geq 3$***

If $d(u) \geq 3$, then $T - v$ has $(m - 1)$ isolated vertices and $k - (m - 1)$ pendant vertices.

Thus, $\gamma_{dcl}(T - v) = (p - 1) - [k - (m - 1)] + 2$

$\geq p - k + 4 = \gamma_{dcl}(T) + 2$

Hence, $\gamma_{dcl}(T - v) > \gamma_{dcl}(T)$.

Case (iii): ***v is a vertex of T with $4 \leq d(v) = m$ and its adjacent vertices are of degree equal to 2***

If v is a vertex of T with $4 \leq d(v) = m$ and its adjacent vertices are of degree equal to 2, then $T - v$ has m components and it has $k + m$ number of pendant vertices.

Thus, $\gamma_{dcl}(T - v) = (p - 1) - (k + m) + 2m$

$\geq p - k + 3 = \gamma_{dcl}(T) + 1$

Hence, $\gamma_{dcl}(T - v) > \gamma_{dcl}(T)$.

Case (iv): ***v is a vertex of T with $4 \leq d(v) = m$ and its adjacent vertices are of degree greater than or equal to 3***

If v is a vertex of T with $4 \leq d(v) = m$ and its adjacent vertices are of degree greater than or equal to 3, then $T - v$ has m components and it has k number of pendant vertices.

Thus, $\gamma_{dcl}(T - v) = (p - 1) - k + 2m$

$\geq p - k + 7 = \gamma_{dcl}(T) + 5$

Hence, $\gamma_{dcl}(T - v) > \gamma_{dcl}(T)$.

Therefore, from all the above cases we have $v \in VD^+$.

Proposition 3.1.9: Let T be any tree. If v is any vertex of degree equal to 2 in T and it is adjacent to two vertices u and w such that d(u) = 2 and d(w) = 2, then $v \in VD^{-}$.

Proof: If v is a vertex of T with d(v) = 2 and it is adjacent to two vertices u and w such that d(u) = 2 and d(w) = 2, then T – v has 2 components and it has k + 2 number of pendant vertices.

Thus, $\gamma_{dcl}(T - v) = (p - 1) - (k + 2) + 4$

$= p - k + 1$

$= \gamma_{dcl}(T) - 1$

Hence, $\gamma_{dcl}(T - v) < \gamma_{dcl}(G)$.

Therefore, $v \in VD^{-}$.

Proposition 3.1.10: If G is an unicyclic graph with p vertices and if G has a cycle of length (p – 1), then we have the following

(i) If p is even, then $V = VD^0 \cup VD^+$.

(ii) If p is odd, then $V = VD^0 \cup VD^-$.

Proof: Let G be a unicyclic graph with p vertices and let G has a cycle of length (p – 1). Since G is a connected and a unicyclic graph, G is of the structure given in Figure 3.1 and we have the following cases.

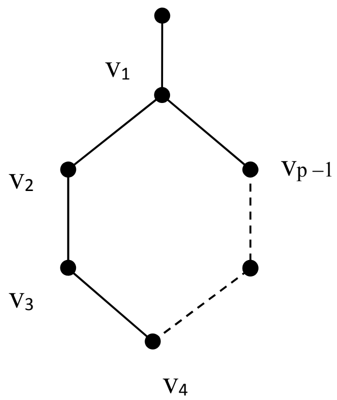

Figure 3.1 - ***A unicyclic graph with a cycle C_{p-1}***

Case (i): ***p is even***

If p is even, then C_{p-1} is an odd cycle of length p – 1. Let us now remove a vertex v from G.

(a) If d(v) = 1, then $(G - v) = C_{p-1}$ and $\gamma_{dcl}(G - v) = \gamma_{dcl}(G)$. Hence, $v \in VD^0$.

(b) If d(v) = 2 and it is adjacent to a vertex of degree 3, then $\gamma_{dcl}(G - v) = \gamma_{dcl}(G) + 1 > \gamma_{dcl}(G)$. Hence, $v \in VD^+$.

(c) If d(v) = 2 and it is adjacent to vertices of degree 2, then $\gamma_{dcl}(G - v) = \gamma_{dcl}(G)$. Hence, $v \in VD^0$.

Therefore, $V = VD^0 \cup VD^+$.

Case (ii): ***p is odd***

If p is odd, then C_{p-1} is an even cycle of length p – 1. Let us now remove a vertex v from G.

(a) If d(v) = 1, then G = C_{p-1} and $\gamma_{dcl}(G - v) = \gamma_{dcl}(G)$. Hence, $v \in VD^0$

(b) If d(v) = 2 and it is adjacent to a vertex of degree 3, then $\gamma_{dcl}(G - v) = \gamma_{dcl}(G)$. Hence, $v \in VD^0$.

(c) If d(v) = 2 and it is adjacent to vertices of degree 2, then $\gamma_{dcl}(G - v) = \gamma_{dcl}(G) - 1 < \gamma_{dcl}(G)$. Hence, $v \in VD^-$.

Therefore, $V = VD^0 \cup VD^-$.

3.2 Changing and unchanging of edge removal (CER and UER):

Now we classify the edges of a graph G in such a way that their removal affects or doesn't affect the D.C.D number of G. Clearly, removal of an edge from G cannot decrease the distance closed domination number (except ciliates) and increases by at the most 2. Let us define

$ED^0 = \{e \in E(G) \mid \gamma_{dcl}(G - e) = \gamma_{dcl}(G)\}$ (UER)

$ED^+ = \{e \in E(G) \mid \gamma_{dcl}(G - e) > \gamma_{dcl}(G)\}$ (CER)

$ED^- = \{e \in E(G) \mid \gamma_{dcl}(G - e) < \gamma_{dcl}(G)\}$

Clearly, $E = ED^0 \cup ED^+ \cup ED^- = ED^0 \cup ED^+$ (since $ED^- = \Phi$ for any graph). The following are the results of some special classes of graphs without proof.

1. For any complete graph K_p, $E = ED^+$, $p \geq 3$.
2. For any complete bipartite graph $K_{m,n}$, $E = ED^0$, $p \geq 4$ where p = m + n.
3. For a Petersen graph G (2-self centered), $E = ED^0$.
4. For any even cycle C_{2n}, $E = ED^0$.
5. For any odd cycle C_{2n+1}, $E = ED^+$.

Proposition 3.2.1: Let G be a graph with radius 1 and diameter 2 and let

A = {e = uv | e(u) = e(v) = 1};

B = {e = uv | e(u) = e(v) = 2} and;

C = {e = uv | [e(u), e(v)] = (1, 2) or (2, 1)}.

Then we have the following:

(i) If $|A| \leq 3$, then $E = ED^0 \cup ED^+$.

(ii) If $|A| \geq 6$, then $E = ED^0$.

Proof: If G is a graph with radius 1 and diameter 2, then $\gamma_{dcl}(G) = 2$ or 3 according to the number of vertices with eccentricity 1 in G.

Case (i): ***G has exactly one vertex with eccentricity 1***

If G has exactly one vertex with eccentricity 1, then $|A| = \Phi$ and $\gamma_{dcl}(G) = 3$. Now, let us remove an edge e from G. Then

(a) If $e \in C$, then $\gamma_{dcl}(G - e) = 4 = \gamma_{dcl}(G) + 1 > \gamma_{dcl}(G)$. Hence, $e \in ED^+$.

(b) If $e \in B$, then $\gamma_{dcl}(G - e) = 3 = \gamma_{dcl}(G)$. Hence, $e \in ED^0$.

Therefore, $E = ED^0 \cup ED^+$.

Case (ii): ***G has exactly two vertices with eccentricity 1***

If G has exactly two vertices with eccentricity 1, then $|A| = 2C_1 = 2$ and $\gamma_{dcl}(G) = 2$. Now, let us remove an edge e from G. Then

(a) If $e \in A$, then $\gamma_{dcl}(G - e) = 4 = \gamma_{dcl}(G) + 2 > \gamma_{dcl}(G)$. Hence, $e \in ED^+$.

(b) If $e \in B$, then $\gamma_{dcl}(G - e) = 2 = \gamma_{dcl}(G)$. Hence, $e \in ED^0$.

(c) If $e \in C$, then $\gamma_{dcl}(G - e) = 3 = \gamma_{dcl}(G) + 1 > \gamma_{dcl}(G)$. Hence, $e \in ED^+$.

Therefore, $E = ED^0 \cup ED^+$.

Case (iii): ***G has exactly three vertices with eccentricity 1***

If G has exactly three vertices with eccentricity 1, then $|A| = 3C_2 = 3$and $\gamma_{dcl}(G) = 2$. Now, let us remove an edge e from G. Then

(a) If $e \in A$, then $\gamma_{dcl}(G - e) = 3 = \gamma_{dcl}(G) + 1 > \gamma_{dcl}(G)$. Hence, $e \in ED^+$.

(b) If $e \in B$, then $\gamma_{dcl}(G - e) = 2 = \gamma_{dcl}(G)$. Hence, $e \in ED^0$.

(c) If $e \in C$, then $\gamma_{dcl}(G - e) = 2 = \gamma_{dcl}(G)$. Hence, $e \in ED^0$.

Therefore, $E = ED^0 \cup ED^+$.

Case (iv): ***G has more than three vertices with eccentricity 1***

If G has more than three vertices with eccentricity 1, then $|A| \geq 4C_2 = 6$ and $\gamma_{dcl}(G) = 2$. In this case, if we remove any edge $e \in E(G) = A \cup B \cup C$, then $\gamma_{dcl}(G - e) = 2 = \gamma_{dcl}(G)$. Hence, $e \in ED^0$ and hence $E = ED^0$.

Therefore from cases (i), (ii) and (iii), $E = ED^0 \cup ED^+$ for $|A| \leq 3$. Also, from case (iv), $E = ED^0$ for $|A| \geq 6$.

Corollary 3.2.1: If G is a graph with radius 1 and diameter 2, then $\gamma_{dcl}(G - uv) = \gamma_{dcl}(G)$ if and only if u and v must have eccentricity equal to 2.

Corollary 3.2.2: If G is a graph with $\gamma_{dcl}(G) = 2$ and $d_{dcl}(G) \geq 2$, then $E = ED^0$.

Corollary 3.2.3: Let G be a graph with $\gamma_{dcl}(G) = 3$. If an edge $e = uv \in ED^+$, then either u or v is of eccentricity equal to 2.

Theorem 3.2.1: Let T be a tree. If$e = uv$ is an edge of T with $d(u) = 1$ and $d(v) = 2$ (or) $d(u) = 2$ and $d(v) = 2$, then $e \in ED^0$.

Proof: We know that, in a tree T every edge $e = uv$ is a cut edge and $T - e$ has two components T_1 and T_2 with $|T_1| = p_1$ and $|T_2| = p_2$, where $p_1 + p_2 = p$.

Case (i): ***e = uv with d(u) = 1 and d(v) =2***

If we remove an edge e = uv with d(u) = 1 and d(v) = 2, then T – e has two components T_1 and T_2 with $|T_1| = 1$ and $|T_2| = p - 1$ and T_2 has k number of pendant vertices.

Therefore, $\gamma_{dcl}(T - e) = 1 + (p - 1) - k + 2 = \gamma_{dcl}(T)$.

Hence, $e \in ED^0$.

Case (ii): ***e = uv with d(u) = 2 and d(v) =2***

If we remove an edge e = uv with d(u) = 2 and d(v) = 2, then T – e has two components T_1 and T_2 with $|T_1| = p_1$ and $|T_2| = p_1$, where $p_1 + p_2 = p$ and T – e has k + 2 number of pendant vertices.

Therefore, $\gamma_{dcl}(T - e) = p - (k + 2) + 4 = \gamma_{dcl}(T)$

Hence, $e \in ED^0$.

Theorem 3.2.2: For any tree T, if an edge e = uv with either d(u) or d(v) greater than or equal to 3, then $e \in ED^+$.

Proof: Let T be a tree with p vertices and k pendant vertices. If we remove an edge e = uv, then T – e has two components T_1 and T_2 with $|T_1| = p_1$ and $|T_2| = p_2$, where $p_1 + p_2 = p$. Thus, we have the following 3 cases.

Case (i): ***d(u) = 1 and d(v) ≥ 3***

If e = uv is an edge with d(u) = 1 and d(v) ≥ 3, then T – e has two components T_1 and T_2 with $|T_1| = 1$ and $|T_2| = p - 1$ and T_2 has k – 1 number of pendant vertices.

Therefore, $\gamma_{dcl}(T - e) = 1 + (p - 1) - (k - 1) + 2$

$= \gamma_{dcl}(T) + 1 > \gamma_{dcl}(T)$

Hence, $e \in ED^+$.

Case (ii): ***d(u) = 2 and d(v) ≥ 3 and vice versa***

If e = uv is an edge with d(u) = 2 and d(v) ≥ 3, then T – e has k + 1 number of pendant vertices.

Therefore, $\gamma_{dcl}(T - e) = p - (k + 1) + 4$

$= \gamma_{dcl}(T) + 1 > \gamma_{dcl}(T)$

Hence, $e \in ED^+$.

Case (iii): ***d(u) ≥ 3and d(v) ≥ 3***

If e = uv is an edge with d(u) ≥ 3and d(v) ≥ 3, then T – e has k number of pendant vertices.

Therefore, $\gamma_{dcl}(T - e) = p - (k) + 4$

$= \gamma_{dcl}(T) + 2 > \gamma_{dcl}(T)$

Hence, $e \in ED^+$.

Proposition 3.2.2: If G is a unicyclic graph with p vertices and if G has a cycle of length (p – 1), then we have the following:

(i) If p is even, then $E = ED^+$.

(ii) If p is odd, then $E = ED^0 \cup ED^+$.

Proof: Let G be a unicyclic graph with p vertices and let G has a cycle of length (p – 1). Then, G is of the structure given in Figure 3.1 and we have the following cases.

Case (i): ***p is even***

If p is even, then C_{p-1} is an odd cycle of length p – 1. Let us now remove an edge e = uv from G.

(a) If d(u) = 3 and d(v) = 1, then $\gamma_{dcl}(G - e) = \gamma_{dcl}(G) + 1 > \gamma_{dcl}(G)$. Hence, $e \in ED^+$.

(b) If d(u) = 3 and d(v) = 2, then $\gamma_{dcl}(G - e) = \gamma_{dcl}(G) + 2 > \gamma_{dcl}(G)$. Hence, $e \in ED^+$.

(c) If d(u) = 2 and d(v) = 2, then $\gamma_{dcl}(G - e) = \gamma_{dcl}(G) + 1 > \gamma_{dcl}(G)$. Hence, $e \in ED^+$.

Therefore, $E = ED^+$.

Case (ii): ***p is odd***

If p is odd, then C_{p-1} is an even cycle of length p – 1. Let us now remove an edge e = uv from G.

(a) If d(u) = 3 and d(v) = 1, then $\gamma_{dcl}(G - e) = \gamma_{dcl}(G) + 1 > \gamma_{dcl}(G)$. Hence, $e \in ED^+$.

(b) If d(u) = 3 and d(v) = 2, then $\gamma_{dcl}(G - e) = \gamma_{dcl}(G) + 1 > \gamma_{dcl}(G)$. Hence, $e \in ED^+$.

(c) If d(u) = 2 and d(v) = 2, then $\gamma_{dcl}(G - e) = \gamma_{dcl}(G)$. Hence, $e \in ED^0$.

Therefore, $E = ED^0 \cup ED^+$.

Proposition 3.2.3: If G is a ciliate, then $E = ED^-$.

Proof: We know that for a ciliate G, $\gamma_{dcl}(G) = p$. Let e = uv be an edge of G. Then, we have the following cases.

Case (i): ***d(u) = 1 and d(v) = 2***

If e = uv is an edge of G with d(u) = 1 and d(v) = 2, then removal of e from G reduce the distance closed domination number of G by 1. That is, $\gamma_{dcl}(G - e) = \gamma_{dcl}(G) - 1 < \gamma_{dcl}(G)$. Hence, $e \in ED^-$.

Case (ii): ***d(u) = 3 and d(v) = 2***

If we remove an edge e = uv with d(u) = 3 and d(v) = 2, then <G – e> has two components G_1 and G_2 in which one of them is a path. Also, the distance closed domination number of <G – e> will be reduced by 1. That is, $\gamma_{dcl}(G - e) = \gamma_{dcl}(G) - 1 < \gamma_{dcl}(G)$. Hence, $e \in ED^-$.

Case (iii): ***d(u) = 2 and d(v) = 2***

If we remove an edge e = uv with d(u) = 3 and d(v) = 2, then <G – e> has two components G_1 and G_2 such that one of them is a path. Also, the distance closed domination number of <G – e> will be reduced by 1. That is, $\gamma_{dcl}(G - e) = \gamma_{dcl}(G) - 1 < \gamma_{dcl}(G)$. Hence, $e \in ED^-$.

Case (iv): ***d(u) = 3 and d(v) = 3***

If we remove an edge e = uv with d(u) = 3 and d(v) = 2 (removing an edge within the cycle), then <G – e> is a tree with $\gamma_{dcl}(G - e) = p - k + 2$, where k is the number of pendant vertices in G – e. Hence, $\gamma_{dcl}(G - e) = \gamma_{dcl}(G) - 2 < \gamma_{dcl}(G)$ and hence $e \in ED^-$.

Therefore in all the cases, every edge e = uv of G is in ED^-. Hence, $E = ED^-$.

3.3 Changing and unchanging of edge addition (CEA and UEA):

Just as deleting an edge can increase the distance closed domination number by at most 2, adding an edge can decrease it by at most 2. In some cases the distance closed domination number is unchanged when an arbitrary edge is added. But there is no graph in which the distance closed domination number is increased after adding an edge.

Let us define

$EA^0 = \{e \notin E(G) \mid \gamma_{dcl}(G + e) = \gamma_{dcl}(G)\}$ (UEA)

$EA^+ = \{e \notin E(G) \mid \gamma_{dcl}(G + e) > \gamma_{dcl}(G)\}$ (CEA)

$EA^- = \{e \notin E(G) \mid \gamma_{dcl}(G + e) < \gamma_{dcl}(G)\}$

Then $E = EA^0 \cup EA^+ \cup EA^- = EA^0 \cup EA^-$ (since $EA^+ = \Phi$ for any graph). The following are the results of some special classes of graphs without proof.

1. For an even cycle, every additional edge e reduces the D.C.D number. Hence, every additional edge eis in EA^-.
2. For an odd cycle, addition of an edge, which induces a C_3, is in EA^0. But the remaining additional edges are in EA^-.
3. For any complete graph $K_{m,n}$ with m + n = p and p ≥ 4, every additional edge e is in EA^-.
4. For a Petersen graph (2-self centered), every additional edge is in EA^-.

Proposition 3.3.1: If G is a graph with $\gamma_{dcl}(G) = 2$, then every additional edge e is in EA^0.

Proof: Since G is a graph with $\gamma_{dcl}(G) = 2$, G must be a complete graph or G is a graph with radius 1 and diameter 2 and also G has more than two vertices with eccentricity 1. Hence, every additional edge between the vertices with eccentricity 2 will not affect the D.C.D number of G and hence every additional edge e is in EA^0.

Proposition 3.3.2: If G is a graph with exactly one vertex with eccentricity 1, then

(i) An additional edge $e = uv \in EA^-$,if at least one of u or v is of degree equal to p – 2

(ii) An additional edge $e = uv \in EA^0$,if both u and v are of degree not equal to p –2

Proof: Since G is a graph with $\gamma_{dcl}(G) = 3$, G must have exactly one vertex with eccentricity equal to 1.

(i) If we add an edge e between the vertices u and v such that at least one of them is of degree equal to p – 2, then $\gamma_{dcl}(G + e) = 2 < \gamma_{dcl}(G)$. Hence, $e \in EA^-$.

(ii) If we add an edge e between the vertices u and v such that both d(u) and d(v) are not equal to p – 2, then $\gamma_{dcl}(G + e) = 3 = \gamma_{dcl}(G)$. Hence, $e \in EA^0$.

Proposition 3.3.3: If G is a (p – 2) regular graph, then every additional edge is in EA^-.

Proof: Since G is a (p – 2) regular graph, it is 2-self centered and $\gamma_{dcl}(G) = 4$. If we add an edge e between a pair of non adjacent vertices of G, then that new edge will form a D.C.D set for (G + e). Hence, $\gamma_{dcl}(G + e) = 2 < \gamma_{dcl}(G)$ and hence $e \in EA^-$, for every additional edge e in G.

Proposition 3.3.4: If G is a 2-self centered graph and if an edge e added between two vertices u and v such that at least one of them is of degree p – 2, then $e \in EA^-$.

Proof: Since G is 2-self centered, $\gamma_{dcl}(G) \geq 4$. Let u and v be any pair of vertices in G such that at least one of them is of degree p – 2. Now, if we add an edge e between u and v then clearly, it will reduce D.C.D number to 3 or 2. Hence, $\gamma_{dcl}(G + e) < \gamma_{dcl}(G)$ and hence $e \in EA^-$.

Theorem 3.3.1: Let G be a graph with radius 2 and diameter 3. If C is the set of central vertices of G and if <C> is a clique, then we have the following

(i) If $\gamma_{dcl}(G) = 4$, then every additional edge is in EA^0.

(ii) If $\gamma_{dcl}(G) > 4$, then every additional edge is in EA^-.

Proof: Let G be a graph with radius 2 and diameter 3 and let C be the set of central vertices of G with <C> is a clique.

Case (i): $\gamma_{dcl}(G) = 4$

Sub case (a): ***G has a unique pair of peripheral nodes***

Here, if we add an edge e between a pair of vertices u and v such that

(i) If e(u) = 3 and e(v) = 2 or vice versa, then <G +e > is 2-self centered and $\gamma_{dcl}(G + e) = 4$.

(ii) If e(u) = 3 and e(v) = 3, then also <G + e> is 2-self centered and $\gamma_{dcl}(G + e) = 4$. Therefore, $e \in EA^0$, for every $e \notin E(G)$.

Sub case (b): ***G has more than 2 peripheral nodes***

Here if we add an edge e between a pair of vertices u and v such that

(i) If e(u) = 3 and e(v) = 2 or vice versa, then <G + e> is still of diameter 3 and $\gamma_{dcl}(G + e) = 4$.

(ii) If e(u) = 3 and e(v) = 3, then <G + e> is of diameter 3 or 2-self centered according to u and v have same or different eccentric node and $\gamma_{dcl}(G + e) = 4$.

Therefore, $e \in EA^0$, for every $e \notin E(G)$.

Case (ii): $\gamma_{dcl}(G) > 4$

If $\gamma_{dcl}(G) > 4$, then G has more than a pair of peripheral nodes and at least 2 of them are pendant vertices. Here, if we add an edge e between a pair of vertices u and v, then <G + e> is of diameter 3 and $\gamma_{dcl}(G + e) < \gamma_{dcl}(G)$.

Therefore, $e \in EA^-$, for every $e \notin E(G)$.

Theorem 3.3.2: If T is a tree with p vertices and k pendant vertices, then we have the following

(i) Addition of an edge between the pendant vertices which are incident at a common vertex is in EA^0.

(ii) Addition of an edge e between any two vertices u and v with $d(u) \geq 2$ and $d(v) \geq 2$ is in EA^-.

Proof: Let T be a tree with p vertices and k pendant vertices. We know that, for any tree T, $\gamma_{dcl}(T) = p - k + 2$.

(i) If we add an edge e between the pendant vertices which are incident at a common vertex, then $\gamma_{dcl}(G + e) = \gamma_{dcl}(G)$. Hence, $e \in EA^0$.

(ii) Suppose that, if u and v are any two vertices with degree greater than or equal to 2, then $e(u) \neq d$ (diameter of T) and $e(v) \neq d$. Also, addition of an edge e between u and v will reduce the D.C.D number of <T + e> by at least one, as T has a unique shortest path between every pair of vertices. Hence, $e \in EA^-$.

Proposition 3.3.5: For any graph G, addition of an edge between any two pendant vertices which are incident at a common vertex is in EA^0.

Proposition 3.3.6: For any graph G, addition of an edge e between any two central vertices u and v such that they are the eccentric nodes of each other, is in EA^0.

Proof: Let C be the set of central vertices of G and let $u, v \in C$ such that u and v are the eccentric nodes of each other. If we add an edge e between u and v, then clearly it will not affect the eccentricity of any vertex of G. Thus, $\gamma_{dcl}(G + e) = \gamma_{dcl}(G)$.Hence, $e \in EA^0$.

Theorem 3.3.3: Let G be a graph with radius r and diameter d. If G has a unique pair of peripheral nodes and if we add an edge e between the peripheral nodes, then

(a) $e \in EA^0$, if d is odd.

(b) $e \in EA^-$, if d is even.

Proof: Let G be a graph with radius r and diameter d and let G has a unique pair of peripheral nodes.

Case (i): If we add an edge e between the peripheral nodes, then <G + e> is r-self centered and

(a) If d is odd, then $\gamma_{dcl}(G + e) = \gamma_{dcl}(G)$. Therefore, $e \in EA^0$.

(b) If d is even, then $\gamma_{dcl}(G + e) < \gamma_{dcl}(G)$. Therefore, $e \in EA^-$.

Theorem 3.3.4: If G is a ciliate, then every additional edge is in EA^-.

Proof: We know that for a ciliate G, $\gamma_{dcl}(G) = p$. Also, addition of an edge e in G will reduce the D.C.D number of <G + e>. Hence, $\gamma_{dcl}(G + e) < \gamma_{dcl}(G)$ and hence $e \in EA^-$.

Corollary 3.3.1: If G is radius critical, then every additional edge is in EA^-.

4. Conclusion:

In general, the concept of dominating sets in graph theory finds wide applications in different types of communication networks. In particular, the concept of distance closed dominating set has remarkable applications in fault tolerance analysis. The faults and failures in larger networks may be due to the failure of components like nodes, links etc. This is analogous to the notion of studying the structural properties of graphs with vertex deletions and edge deletion respectively. Similarly, the network behavior also changes when new nodes or links are introduced in case of communication network`s such as mobile networks and this is analogous to the study of structural properties of graphs with vertex additions and edge additions. Hence, by the changing and unchanging parameters of the graphs with respect to vertex, edge additions and deletions and thereby we have studied the behavior of the networks in all the above situations. Thus, the above work has lots of applications in the communication network, social and economical network and signal processing.

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