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Super Graceful Labeling for H - Class of Graphs

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Abstract: Let G be a *(p, q)* - graph. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., p + q\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ is said to be a super graceful labeling. A graph G is called a super graceful graph if it admits a super graceful labeling. In this paper, we show that the graphs H graph, $H \odot mK_1$ and $H^{\otimes}_{(n)}$ are super graceful graphs.

Keywords : Graceful labeling, Super graceful labeling and Super graceful graphs.

AMS Subject Classification (2010): 05C78

1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A path of length n is denoted by P_n. The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph G obtained by taking one copy of G_1 (which has P_1 points) and P_1 copies of G_2 and then joining the i^{th} point of G_1 to every point in the i^{th} copy of G_2 [2]. The concept of graceful labeling has been introduced by Rosa [3] in 1967.

A function f is a graceful labeling of a graph G with p vertices and q edges if f is an injection from the vertices of G to the set $\{1,2,...,q\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. The gracefulness of graphs motivates us to define a new type of labeling, called "Super graceful labeling"[6], [7].

Let G be a (p, q) - graph. A bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., p + q\}$ such that $f(uv) = |f(u) - f(v)|$ for every edge $uv \in E(G)$ is said to be a super graceful labeling. A graph G is called a super graceful graph if it admits a super graceful labeling.

In this paper, we show that the graphs H graph, $H \odot mK_1$ and $H^{\otimes}_{(n)}$ are super graceful graphs.

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2. Main Results

Definition 2.1. [8]

Let $P_1(u_1, u_2, ..., u_n)$ and $P_2(v_1, v_2, ..., v_n)$ be two paths of length $n-1$. Then we obtain a new graph by joining $u_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$ $\left(u_{\frac{n+1}{2}} \text{ and } v_{\frac{n+1}{2}}\right)$ if n is even (odd). The resultant graph is called as H graph.

Theorem 2.2 H *graph is super graceful.*

Proof. Let $P_1(u_1, u_2, ..., u_n)$ and $P_2(v_1, v_2, ..., v_n)$ be two distinct paths on n vertices $(n \geq 3)$ in H graph. Now, $|V(H)| = 2n$, and $|E(H)| = 2n-1$.

We consider the following two cases.

Case i n is odd

Define $f : V(H) \cup E(H) \rightarrow \{1,2,...,4n-1\}$ as follows.

$$f(u_i) = \begin{cases} 4n-i, & 1 \leq i \leq n,\ i \equiv 1 \ (mod\ 2) \\ i-1, & 1 \leq i \leq n-1,\ i \equiv 0 \ (mod\ 2) \end{cases}$$

and

$$f(v_i) = \begin{cases} n-1+j, & 1 \leq j \leq n,\ j \equiv 1 \ (mod\ 2) \\ 3n-j, & 1 \leq j \leq n-1,\ j \equiv 0 \ (mod\ 2) \end{cases}$$

We construct the vertex label sets as follows:

$$Let\ V_1 = \bigcup_{\substack{i=1 \\ i \equiv 1\ (mod\ 2)}}^{n} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i \equiv 1\ (mod\ 2)}}^{n} \{4n-i\} = \{4n-1, 4n-3, ..., 3n\}$$

$$V_2 = \bigcup_{\substack{i=1 \\ i \equiv 0\ (mod\ 2)}}^{n-1} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i \equiv 0\ (mod\ 2)}}^{n-1} \{i-1\} = \{1,3,5,...,n-2\}$$

$$V_3 = \bigcup_{\substack{j=1 \\ j \equiv 1\ (mod\ 2)}}^{n} \{f(v_j)\} = \bigcup_{\substack{j=1 \\ j \equiv 1\ (mod\ 2)}}^{n} \{n-1+j\} = \{n, n+2, ..., 2n-1\}\ and$$

$$V_4 = \bigcup_{\substack{j=1 \\ j \equiv 0\ (mod\ 2)}}^{n-1} \{f(v_j)\} = \bigcup_{\substack{j=1 \\ j \equiv 0\ (mod\ 2)}}^{n-1} \{3n-j\} = \{3n-2, 3n-4, ..., 2n+1\}$$

We construct the edge label sets as follows:

$$\text{Let}\ \ E_1 = \bigcup_{\substack{i=1 \\ i \equiv 1\ (mod\ 2)}}^{n-2} \{f(u_i u_{i+1})\} = \bigcup_{\substack{i=1 \\ i \equiv 1\ (mod\ 2)}}^{n-2} \{|\, f(u_i) - f(u_{i+1})\, |\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{|(4n-i)-(i+1-1)|\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{4n-2i\} = \{4n-2, 4n-6, \ldots, 2n+4\}$$

$$E_2 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{f(u_i u_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{|f(u_i)-f(u_{i+1})|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{|(i-1)-(4n-(i+1))|\}\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{|2i-4n|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{4n-2i\} = \{4n-4, 4n-8, \ldots, 2n+2\}$$

$$E_3 = \bigcup_{\substack{j=1 \\ j\equiv 1\,(mod\ 2)}}^{n-2} \{f(v_j v_{j+1})\} = \bigcup_{\substack{j=1 \\ j\equiv 1\,(mod\ 2)}}^{n-2} \{|f(v_j)-f(v_{j+1})|\}$$

$$= \bigcup_{\substack{j=1 \\ j\equiv 1\,(mod\ 2)}}^{n-2} \{|(n-1+j)-(3n-(j+1))|\} = \bigcup_{\substack{j=1 \\ j\equiv 1\,(mod\ 2)}}^{n-2} \{|2j-2n|\}$$

$$= \bigcup_{\substack{j=1 \\ j\equiv 1\,(mod\ 2)}}^{n-2} \{2n-2j\} = \{2n-2, 2n-6, \ldots, 4\}$$

$$E_4 = \bigcup_{\substack{j=1 \\ j\equiv 0\,(mod\ 2)}}^{n-1} \{f(v_j v_{j+1})\} = \bigcup_{\substack{j=1 \\ j\equiv 0\,(mod\ 2)}}^{n-1} \{|f(v_j)-f(v_{j+1})|\}$$

$$= \bigcup_{\substack{j=1 \\ j\equiv 0\,(mod\ 2)}}^{n-1} \{|(3n-j)-(n-1+j+1)|\} = \bigcup_{\substack{j=1 \\ j\equiv 0\,(mod\ 2)}}^{n-1} \{2n-2j\}$$

$$= \{2n-4, 2n-8, \ldots, 2\}$$

and $E_5 = \left\{f\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right)\right\} = \left\{\left|f\left(u_{\frac{n+1}{2}}\right)-f\left(v_{\frac{n+1}{2}}\right)\right|\right\}$

$$= \left\{\left|\left(\left(\frac{n+1}{2}\right)-1\right)-\left(3n-\left(\frac{n+1}{2}\right)\right)\right|\right\} = \{|n+1-1-3n|\} = \{2n\}$$

Case ii n is even

Define $f : V(H) \cup E(H) \to \{1, 2, \ldots, 4n-1\}$ as follows:

$$f(u_i) = \begin{cases} 4n-i, & 1 \le i \le n-1,\ i \equiv 1\ (mod\ 2) \\ i-1, & 1 \le i \le n,\ i \equiv 0\ (mod\ 2) \end{cases}$$

and

$$f(v_j) = \begin{cases} 3n - j, & 1 \leq j \leq n-1,\ j \equiv 1\ (mod\ 2) \\ n - 1 + j & 1 \leq j \leq n,\ j \equiv 0\ (mod\ 2) \end{cases}$$

We construct the vertex label sets as follows:

$$Let\ V_1^{'} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{4n - i\} = \{4n-1, 4n-3, \ldots, \ldots, 3n+1\}$$

$$V_2^{'} = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n} \{i - 1\} = \{1, 3, 5, \ldots, n-1\}$$

$$V_3^{'} = \bigcup_{\substack{j=1 \\ j\equiv 1\ (mod\ 2)}}^{n-1} \{f(v_j)\} = \bigcup_{\substack{j=1 \\ j\equiv 1\ (mod\ 2)}}^{n-1} \{3n - j\} = \{3n-1, 3n-3, \ldots, 2n+1\}\ and$$

$$V_4^{'} = \bigcup_{\substack{j=1 \\ j\equiv 0\ (mod\ 2)}}^{n} \{f(v_j)\} = \bigcup_{\substack{j=1 \\ j\equiv 0\ (mod\ 2)}}^{n} \{n - 1 + j\} = \{n+1, n+3, \ldots, 2n-1\}$$

We construct the edge label sets as follows:

$$E_1^{'} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{f(u_i u_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{|\ f(u_i) - f(u_{i+1})\ |\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{|\ (4n - i) - (i + 1 - 1)\ |\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n-1} \{4n - 2i\} = \{4n-2, 4n-6, \ldots, 2n+2\}$$

$$E_2^{'} = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{f(u_i u_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{|\ f(u_i) - f(u_{i+1})\ |\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{|\ (i-1) - (4n - (i+1))\ |\} = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{|\ 2i - 4n\ |\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-2} \{4n - 2i\} = \{4n-4, 4n-8, \ldots, 2n+4\}$$

$$E_3^{'} = \bigcup_{\substack{j=1 \\ j\equiv 1\ (mod\ 2)}}^{n-1} \{f(v_j v_{j+1})\} = \bigcup_{\substack{j=1 \\ j\equiv 1\ (mod\ 2)}}^{n-1} \{|\ f(v_j) - f(v_{j+1})\ |\}$$

$$= \bigcup_{\substack{j=1 \\ j\equiv 1\ (mod\ 2)}}^{n-1} \{|\ (3n - j) - (n - 1 + j + 1)\ |\} = \bigcup_{\substack{j=1 \\ j\equiv 1\ (mod\ 2)}}^{n-1} \{2n - 2j\} = \{2n-2, 2n-6, \ldots, 2\}$$

$$E_4^{'} = \bigcup_{\substack{j=1 \\ j\equiv 0\,(mod\ 2)}}^{n-2} \{f(v_j v_{j+1})\} = \bigcup_{\substack{j=1 \\ j\equiv 0\,(mod\ 2)}}^{n-2} \{|\,f(v_j) - f(v_{j+1})\,|\}$$

$$= \bigcup_{\substack{j=1 \\ j\equiv 0\,(mod\ 2)}}^{n-2} \{|\,(n-1+j)-(3n-(j+1))\,|\} = \bigcup_{\substack{j=1 \\ j\equiv 0\,(mod\ 2)}}^{n-2} \{|\,2j-2n\,|\}$$

$$= \bigcup_{\substack{j=1 \\ j\equiv 0\,(mod\ 2)}}^{n-2} \{2n-2j\} = \{2n-4, 2n-8, \ldots, 4\} \ and$$

$$E_5^{'} = \left\{ f\left(u_{\frac{n}{2}} v_{\frac{n+2}{2}} \right) \right\} = \left\{ \left| f\left(u_{\frac{n}{2}} \right) - f\left(v_{\frac{n+2}{2}} \right) \right| \right\} = \left\{ \left| \left(\frac{n}{2} - 1 \right) - \left(3n - \left(\frac{n+2}{2} \right) \right) \right| \right\}$$

$$= \left\{ \left| \left(\frac{n+n+2}{2} \right) - 1 - 3n \right| \right\} = \{|\, n+1-1-3n\,|\} = \{2n\}$$

In both the cases, we observe that the entire vertices labeled sets are having odd values and the edges labeled sets are having even values and they are distinct. Their union is {1, 2,..., 4n − 1}. Therefore, *f* is a super graceful labeling and hence, *H* graph is super graceful.

Example 2.3. Super graceful labeling of H graphs H_9 and H_{10} are given in Figure 1 and Figure 2 respectively.

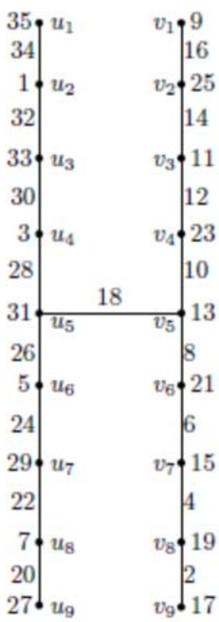

Fig. 1

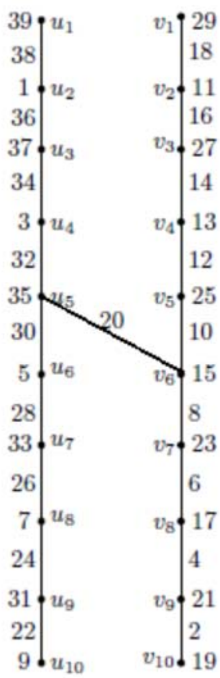

Fig.2

Theorem 2.4 H $\odot$ mK_1 *is a super graceful graph for all m.*

Proof. Let $P_1(u_1, u_2, ..., u_n)$ and $P_2(v_1, v_2, ..., v_n)$ be two distinct paths on n vertices in H graph. Let $u_{i,j}$ and $v_{i,j}$, $1 \leq j \leq m$, be the pendent vertices attached at u_i and v_i respectively, $1 \leq i \leq n$.

Let G = H $\odot mK_1$. Then $|V(G)| = 2n(m+1), |E(G)| = 2n(m+1) - 1$ and $|V(G)| \cup |E(G)| = 4n(m+1) - 1$. We consider the following two cases

Case (i) n is odd

Define $f: V(G) \cup E(G) \rightarrow \{1, 2, ..., 4n(m+1) - 1\}$ as follows:

$$f(u_i) = \begin{cases} (4n+1-i)(m+1)-1, & 1 \leq i \leq n, i \equiv 1 \ (mod\ 2) \\ i(m+1)-1, & 1 \leq i \leq n, i \equiv 0 \ (mod\ 2) \end{cases}$$

$$f(v_i) = \begin{cases} (n+i)(m+1)-1, & 1 \leq i \leq n, i \equiv 1 \ (mod\ 2) \\ (3n+1-i)(m+1)-1, & 1 \leq i \leq n, i \equiv 0 \ (mod\ 2) \end{cases}$$

For $1 \leq i \leq n$ and $i \equiv 1 \ (mod 2)$, $f(u_{ij}) = (i-1)(m+1) + (2j-1), 1 \leq j \leq m$

For $1 \leq i \leq n$ and $i \equiv 0 \ (mod\ 2)$, $f(u_{ij}) = (4n+2-i)(m+1) - (2j+1), 1 \leq j \leq m$

For $1 \leq i \leq n$ and $i \equiv 1 \ (mod\ 2)$, $f(v_{ij}) = (3n+2-i)(m+1) - (2j+1), 1 \leq j \leq m$

For $1 \leq i \leq n$ and $i \equiv 0 \ (mod\ 2)$, $f(v_{ij}) = (n-1+i)(m+1) + (2j-1), 1 \leq j \leq m$

We construct the vertex label sets as follows:

$$Let\ V_1 = \bigcup_{\substack{i=1 \\ i \equiv 1\ (mod\ 2)}}^{n} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i \equiv 1\ (mod\ 2)}}^{n} \{(4n+1-i)(m+1)-1\}$$

$$= \{4n(m+1)-1, (4n-2)(m+1)-1, ..., (3n+1)(m+1)-1\}$$

$$V_2 = \bigcup_{\substack{i=1 \\ i \equiv 0\ (mod\ 2)}}^{n} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i \equiv 0\ (mod\ 2)}}^{n} \{i(m+1)-1\}$$

$$= \{2(m+1)-1, 4(m+1)-1, ..., (n-1)(m+1)-1\}$$

$$V_3 = \bigcup_{\substack{i=1 \\ i \equiv 1\ (mod\ 2)}}^{n} \{f(v_i)\} = \bigcup_{\substack{i=1 \\ i \equiv 1\ (mod\ 2)}}^{n} \{(n+i)(m+1)-1\}$$

$$= \{(n+1)(m+1)-1, (n+3)(m+1)-1, ..., 2n(m+1)-1\}$$

$$V_4 = \bigcup_{\substack{i=1 \\ i \equiv 0\ (mod\ 2)}}^{n} \{f(v_i)\} = \bigcup_{\substack{i=1 \\ i \equiv 0\ (mod\ 2)}}^{n} \{(3n+1-i)(m+1)-1\}$$

$$= \{(3n-1)(m+1)-1, (3n-3)(m+1)-1, ..., (2n+2)(m+1)-1\}$$

$$V_5 = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{n} \{f(u_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{n} \{(i-1)(m+1)+2j-1\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \{(i-1)(m+1)+1, (i-1)(m+1)+3, ..., (i-1)(m+1)+2m-1\}$$

$$= \{1,3,...,2m-1; 2(m+1)+1, 2(m+1)+3, ..., 2(m+1)+2m-1;$$
$$4(m+1)+1, 4(m+1)+3, ..., 4(m+1)+2m-1; ...;$$
$$(n-1)(m+1)+1, (n-1)(m+1)+3, ..., \quad (n-1)(m+1)+2m-1\}$$

$$V_6 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(u_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{(4n+2-i)(m+1)-(2j+1)\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \{(4n+2-i)(m+1)-3, (4n+2-i)(m+1)-5, ...,$$

$$(4n+2-i)(m+1)-(2m+1)\}$$

$$= \{4\text{n}(\text{m}+1)-3, 4\text{n}(\text{m}+1)-5, ..., 4\text{n}(\text{m}+1)-(2\text{m}+1);$$
$$(4n-2)(m+1)-3, (4n-2)(m+1)-5, ..., (4n-2)(m+1)-(2m+1); ...:$$
$$(3n+3)(m+1)-3, (3n+3)(m+1)-5, ..., (3n+3)(m+1)-(2m+1)\}$$

$$V_7 = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(v_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{(3n+2-i)(m+1)-(2j+1)\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \{(3n+2-i)(m+1)-3, (3n+2-i)(m+1)-5, ...,$$

$$(3n+2-i)(m+1)-(2m+1)\}$$

$$= \{(3n+1)(m+1)-3, (3n+1)(m+1)-5\ , ..., (3n+1)(m+1)-(2m+1);$$
$$(3n-1)(m+1)-3, (3n-1)(m+1)-5, ..., (3n-1)(m+1)-(2m+1); ...;$$
$$(2n+2)(m+1)-3, (2n+2)(m+1)-5, ..., (2n+2)(m+1)-(2m+1)\} \quad \text{and}$$

$$V_8 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(v_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{(n-1+i)(m+1)+(2j-1)\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \{(n-1+i)(m+1)+1, (n-1+i)(m+1)+3, ...$$

$$(n-1+i)(m+1)+(2m-1)\},$$

$$= \{(n+1)(m+1)+1,(n+1)(m+1)+3,...,(n+1)(m+1)+(2m-1);$$
$$(n+3)(m+1)+1,(n+3)(m+1)+3,...,(n+3)(m+1)+(2m-1);...;$$
$$(2n-2)(m+1)+1,(2n-2)(m+1)+3,...,(2n-2)(m+1)+(2m-1)\}$$

We construct the edge label sets as follows:

$$Let\ E_1 = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{f(u_i u_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{|\,f(u_i)-f(u_{i+1})\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{|\,((4n+1-i)(m+1)-1)-((i+1)(m+1)-1)\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{|\,(m+1)(4n-2i)\,|\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{2(m+1)(2n-i)\}$$

$$= \{2(m+1)(2n-1),2(m+1)(2n-3),...,2(m+1)(n+2)\}$$

$$E_2 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{f(u_i u_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{|\,f(u_i)-f(u_{i+1})\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{|\,(i(m+1)-1)-((4n+1-i-1)(m+1)-1)\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{|\,(m+1)(2i-4n)\,|\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{2(m+1)(2n-i)\}$$

$$= \{2(m+1)(2n-2),2(m+1)(2n-4),...,2(m+1)(n+1)\}$$
$$= \{4(m+1)(n-1),4(m+1)(n-2),...,2(m+1)(n+1)\}$$

$$E_3 = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{f(v_i v_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{|\,f(v_i)-f(v_{i+1})\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{|\,((n+i)(m+1)-1)-((3n+1-i-1)(m+1)-1)\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{|\,(m+1)(2i-2n)\,|\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-2} \{2(m+1)(n-i)\}$$

$$= \{2(m+1)(n-1),2(m+1)(n-3),...,4(m+1)\}$$

$$E_4 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{f(v_i v_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-1} \{|\,f(v_i)-f(v_{i+1})\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \{|((3n+1-i)(m+1)-1)-((n+1+i)(m+1)-1)|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \{|(2n-2i)(m+1)|\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \{2(m+1)(n-i)\}$$

$$= \{2(m+1)(n-2), 2(m+1)(n-4), \ldots, 2(m+1)\}$$

$$E_5 = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(u_i u_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{|f(u_i) - f(u_{ij})|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{|((4n+1-i)(m+1)-1)-((i-1)(m+1)+2j-1)|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{(4n+2-2i)(m+1)-2j\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n} \{(4n+2-2i)(m+1)-2, (4n+2-2i)(m+1)-4, \ldots, (4n+2-2i)(m+1)-2m\}$$

$$\begin{aligned} = \{&4n(m+1)-2, 4n(m+1)-4, \ldots 4n(m+1)-2m\} \\ &\bigcup \{(4n-4)(m+1)-2, (4n-4)(m+1)-4, \ldots, (4n-4)(m+1)-2m\} \\ &\bigcup \{(4n-8)(m+1)\{(4n-8)(m+1)-2, (4n-8)(m+1)-4, \ldots, (4n-8)(m+1)-2m\} \\ &\bigcup \ldots \bigcup \{2(n+1)(m+1)-2, 2(n+1)(m+1)-4, \ldots, 2(n+1)(m+1)-2m\} \end{aligned}$$

$$E_6 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{f(u_i u_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{|f(u_i) - f(u_{ij})|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{|(i(m+1)-1)-((4n+2-i)(m+1)-2j-1)|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{|(i-4n-2+i)(m+1)+2j|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{|\ (2i-4n-2)(m+1)+2j\ |\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{(4n+2-2i)(m+1)-2j\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n-1} \{(4n+2-2i)(m+1)-2, (4n+2-2i)(m+1)-4, ..., (4n+2-2i)-2m\}$$

$$= \{(4n-2)(m+1)-2, (4n-2)(m+1)-4, ..., (4n-2)(m+1)-2m\}$$
$$\bigcup \{(4n-6)(m+1)-2, (4n-6)(m+1)-4, ..., (4n-6)(m+1)-2m\}$$
$$\bigcup \{(4n-10)(m+1)-2, (4n-10)(m+1)-4, ..., (4n-10)(m+1)-2\}, \bigcup ...$$
$$\bigcup \{(2n+4)(m+1)-2, (2n+4)(m+1)-4, ..., (2n+4)(m+1)-2m\}$$

$$E_7 = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(v_i v_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{|\ f(v_i) - f(v_{ij})\ |\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{|\ ((n+i)(m+1)-1) - ((3n+2-i)(m+1)-2j-1)\ |\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{|\ (n+i-3n-2+i)(m+1)+2j\ |\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{|\ (-2n-2+2i)(m+1)+2j\ |\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{(2n+2-2i)(m+1)-2j\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \{(2n+2-2i)(m+1)-2, (2n+2-2i)(m+1)-4, ...,$$
$$(2n+2-2i)(m+1)-2m\}$$

$$= \{2n(m+1)-2, 2n(m+1)-4, ..., 2n(m+1)-2m\}$$
$$\bigcup \{(2n-4)(m+1)-2, (2n-4)(m+1)-4, ..., (2n-4)(m+1)-2m\}$$
$$\bigcup \{(2n-8)(m+1)-2, (2n-8)(m+1)-4, ..., 2(m+1)-2m\} \bigcup ...$$
$$\bigcup \{2(m+1)-2, 2(m+1)-4, ..., 2(m+1)-2m\}$$

$$E_8 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1}\left(\bigcup_{j=1}^{m}\{f(v_i v_{ij})\}\right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1}\left(\bigcup_{j=1}^{m}\{|\,f(v_i)-f(v_{ij})\,|\}\right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1}\left(\bigcup_{j=1}^{m}\{|\,((3n+1-i)(m+1)-1)-((n-1+i)(m+1)+2j-1)\,|\}\right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1}\left(\bigcup_{j=1}^{m}\{|\,(3n+1-i-n+1-i)(m+1)-2j\,|\}\right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1}\left(\bigcup_{j=1}^{m}\{(2n+2-2i)(m+1)-2j\}\right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1}\{(2n+2-2i)(m+1)-2,(2n+2-2i)(m+1)-4,\ldots,(2n+2-2i)(m+1)-2m\}$$

$$=\{(2n-2)(m+1)-2,(2n-2)(m+1)-4,\ldots,(2n-2)(m+1)-2m\}$$
$$\bigcup\{(2n-6)(m+1)-2,(2n-6)(m+1)-4,\ldots,(2n-6)(m+1)-2m\}$$
$$\bigcup\{(2n-10)(m+1)-2,(2n-10)(m+1)-4,\ldots,(2n-10)(m+1)-2m\}$$
$$\bigcup\ldots\bigcup\{4(m+1)-2,4(m+1)-4,\ldots,4(m+1)-2m\}$$

$$E_9 = \left\{f\left(u_{\frac{n+1}{2}}v_{\frac{n+1}{2}}\right)\right\} = \left\{\left|f\left(u_{\frac{n+1}{2}}\right)-f\left(v_{\frac{n+1}{2}}\right)\right|\right\}$$

$$= \left\{\left|\left(\left(\frac{n+1}{2}\right)(m+1)-1\right)-\left(\left((3n+1)-\left(\frac{n+1}{2}\right)\right)(m+1)-1\right)\right|\right\}$$

$$= \{|\,(m+1)(-2n)\,|\} = \{2n(m+1)\}$$

Case ii n is even.

Define $f:V(G)\cup E(G)\rightarrow\{1,2,3,\ldots,4n(m+1)-1\}$ as follows.

$$f(u_i)=\begin{cases}(4n+1-i)(m+1)-1, & 1\le i\le n,\ i\equiv 1\ (mod\ 2)\\ i(m+1)-1, & 1\le i\le n,\ i\equiv 0\ (mod\ 2)\end{cases}$$

and

$$f(v_i) = \begin{cases} (3n+1-i)(m+1)-1, & 1 \le i \le n,\ i \equiv 1\ (mod\ 2) \\ (n+i)(m+1)-1, & 1 \le i \le n,\ i \equiv 0\ (mod\ 2) \end{cases}$$

For $1 \le i \le n$ and $i \equiv 1\ (mod 2)$, $f(u_{ij}) = (i-1)(m+1)+(2j-1), 1 \le j \le m$

For $1 \le i \le n$ and $i \equiv 0\ (mod\ 2)$, $f(u_{ij}) = (4n+2-i)(m+1)-(2j+1), 1 \le j \le m$

For $1 \le i \le n$ and $i \equiv 1\ (mod\ 2)$, $f(v_{ij}) = (n-1+i)(m+1)+(2j-1), 1 \le j \le m$

For $1 \le i \le n$ and $i \equiv 0\ (mod\ 2)$, $f(v_{ij}) = (3n+2-i)(m+1)-(2j+1), 1 \le j \le m$

We construct the vertex label sets as follows:

$$Let\ V_1' = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \{(4n+1-i)(m+1)-1\}$$

$$= \{4n(m+1)-1, (4n-2)(m+1)-1, ..., (3n+2)(m+1)-1\}$$

$$V_2' = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n} \{i(m+1)-1\}$$

$$= \{2(m+1)-1, 4(m+1)-1, ..., n(m+1)-1\}$$

$$V_3' = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \{f(v_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \{(3n+1-i)(m+1)-1\}$$

$$= \{3n(m+1)-1, (3n-2)(m+1)-1, ..., 2(m+1)(n+1)-1\}$$

$$V_4' = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n} \{f(v_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 0\ (mod\ 2)}}^{n} \{(n+i)(m+1)-1\}$$

$$= \{(n+2)(m+1)-1, (n+4)(m+1)-1, ..., 2(m+1)n-1\}$$

$$V_5' = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(u_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{(i-1)(m+1)+(2j-1)\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\ (mod\ 2)}}^{n} \{(i-1)(m+1)+1, (i-1)(m+1)+3, ..., (i-1)(m+1)+(2m-1)\}$$

$$= \{1, 3, ..., 2m-1; 2(m+1)+1, 2(m+1)+3, ..., 2(m+1)+(2m-1); ...;$$
$$(n-2)(m+1)+1, (n-2)(m+1)+3, ..., (n-2)(m+1)+(2m-1)\}$$

$$V_6' = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(u_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{(4n+2-i)(m+1)-(2j+1)\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \{(4n+2-i)(m+1)-3, (4n+2-i)(m+1)-5, \dots, (4n+2-i)(m+1)-(2m+1)\}$$

$$= \{4n(m+1)-3, 4n(m+1)-5, \dots, 4n(m+1)-(2m+1); (4n-2)(m+1)-3, (4n-2)(m+1)-5, \dots, (4n-2)(m+1)-(2m+1); \dots; (3n+2)(m+1)-3, (3n+2)(m+1)-5, \dots, (3n+2)(m+1)-(2m+1)\}$$

$$V_7' = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(v_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{(n-1+i)(m+1)+(2j-1)\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \{(n-1+i)(m+1)+1, (n-1+i)(m+1)+3, \dots, (n-1+i)(m+1)+(2m-1)\}$$

$$= \{n(m+1)+1, n(m+1)+3, \dots, n(m+1)+(2m-1); (n+2)(m+1)+1, (n+2)(m+1)+3, \dots, (n+2)(m+1)+(2m-1); \dots; 2(m+1)(n-1)+1, 2(m+1)(n-1)+3, \dots, 2(m+1)(n-1)+(2m-1)\}$$

and
$$V_8' = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(v_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m} \{(3n+2-i)(m+1)-(2j+1)\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \{(3n+2-i)(m+1)-3, (3n+2-i)(m+1)-5, \dots, (3n+2-i)(m+1)-(2m+1)\}$$

$$= \{3n(m+1)-3, 3n(m+1)-5, \dots, 3n(m+1)-(2m+1); (3n-2)(m+1)-3, (3n-2)(m+1)-5, \dots, (3n-2)(m+1)-(2m+1); \dots; 2(m+1)(n+1)-3, 2(m+1)(n+1)-5, \dots, 2(m+1)(n+1)-(2m+1)\}$$

We construct the edge label sets as follows:

Let
$$E_1' = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-1} \{f(u_i u_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-1} \{|f(u_i) - f(u_{i+1})|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-1} \{|\,((4n+1-i)(m+1)-1)-((i+1)(m+1)-1)\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-1} \{|\,(m+1)(4n-2i)\,|\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-1} \{2(m+1)(2n-i)\}$$

$$= \{2(m+1)(2n-1), 2(m+1)(2n-3), \ldots, 2(m+1)(n+1)\}$$

$$E_2^{'} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-2} \{f(u_i u_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-2} \{|\,f(u_i)-f(u_{i+1})\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-2} \{|\,(i(m+1)-1)-((4n+1-i-1)(m+1)-1)\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-2} \{|\,(m+1)(2i-4n)\,|\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-2} \{2(m+1)(2n-i)\}$$

$$= \{4(m+1)(n-1), 4(m+1)(n-2), \ldots, 2(m+1)(n+2)\}$$

$$E_3^{'} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-1} \{f(v_i v_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-1} \{|\,f(v_i)-f(v_{i+1})\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-1} \{|\,((3n+1-i)(m+1)-1)-((n+1+i)(m+1)-1)\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n-1} \{|\,2(m+1)(n-i)\,|\} = \{2(m+1)(n-1), 2(m+1)(n-3), \ldots, 2(m+1)\}$$

$$E_4^{'} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-2} \{f(v_i v_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-2} \{|\,f(v_i)-f(v_{i+1})\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n-2} \{|\,((n+i)(m+1)-1)-((3n+1-1-i)(m+1)-1)\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-2} \{2(m+1)(n-i)\} = \{2(m+1)(n-2), 2(m+1)(n-4), \ldots, 4(m+1)\}$$

$$E_5^{'} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{f(u_i u_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{|\, f(u_i) - f(u_{ij})\,|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{|\, ((4n+1-i)(m+1)-1) - ((i-1)(m+1)+2j-1)\,|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{|\, (4n+1-i-i+1)(m+1)-2j\,|\}) \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{(4n+2-2i)(m+1)-2j\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \{(4n+2-2i)(m+1)-2, (4n+2-2i)(m+1)-4, \ldots,$$

$$(4n+2-2i)(m+1)-2m\}$$

$$= \{4n(m+1)-2, 4n(m+1)-4, \ldots, 4n(m+1)-2m\}$$
$$\bigcup \{(4n-4)(m+1)-2, (4n-4)(m+1)-4, \ldots, (4n-4)(m+1)-2m\}$$
$$\bigcup \{(4n-8)(m+1)-2, (4n-8)(m+1)-4, \ldots, (4n-8)(m+1)-2m\} \bigcup \ldots$$
$$\bigcup \{(2n+4)(m+1)-2, (2n+4)(m+1)-4, \ldots, (2n+4)(m+1)-2m\}$$

$$E_6^{'} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(u_i u_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{|\, f(u_i) - f(u_{ij})\,|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{|\, (i(m+1)-1) - ((4n+2-i)(m+1)-2j-1)\,|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{|\, (i-4n-2+i)(m+1)+2j\,|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{(4n+2-2i)(m+1)-2j\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \{(4n+2-2i)(m+1)-2, (4n+2-2i)(m+1)-4, \ldots, (4n+2)(m+1)-2m\}$$

$$= \{(4n-2)(m+1)-2, (4n-2)(m+1)-4, \ldots, (4n-2)(m+1)-2m\} \bigcup \{(4n-6)(m+1)-2, (4n-6)(m+1)-4, \ldots, (4n-6)(m+1)-2m\} \bigcup \{(4n-10)(m+1)-2, (4n-10)(m+1)-4, \ldots, (4n-10)(m+1)-2m\} \bigcup \ldots \bigcup \{2(n+1)(m+1)-2, 2(n+1)(m+1)-4, \ldots, 2(n+1)(m+1)-2m\}$$

$$E_7' = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{f(v_i v_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{| f(v_i) - f(v_{ij}) |\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{| ((3n+1-i)(m+1)-1) - ((n-1+i)(m+1)+2j-1) |\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{| (3n+1-i-n+1-i)(m+1)-2j |\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \left(\bigcup_{j=1}^{m} \{(2n+2-2i)(m+1)-2j\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-1} \{(2n+2-2i)(m+1)-2, (2n+2-2i)(m+1)-4, \ldots, (2n+2-2i)(m+1)-2m\}$$

$$= \{2n(m+1)-2, 2n(m+1)-4, \ldots, 2n(m+1)-2m\} \bigcup \{(2n-4)(m+1)-2, (2n-4)(m+1)-4, \ldots, (2n-4)(m+1)-2m\} \bigcup \{(2n-8)(m+1)-2, (2n-8)(m+1)-4, \ldots, (2n-8)(m+1)-2m\} \bigcup \ldots \bigcup \{4(m+1)-2, 4(m+1)-4, \ldots, 4(m+1)-2m\}$$

$$E_8' = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(v_i v_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{| f(v_i) - f(v_{ij}) |\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{|\,((n+i)(m+1)-1)-((3n+2-i)(m+1)-2j-1)\,|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{|\,(n+i-3n-2+i)(m+1)+2j\,|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{(2n+2-2i)(m+1)-2j\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \{(2n+2-2i)(m+1)-2, (2n+2-2i)(m+1)-4, \ldots, (2n+2-2i)(m+1)-2m\}$$

$$= \{(2n-2)(m+1)-2, (2n-2)(m+1)-4, \ldots, (2n-2)(m+1)-2m\}$$
$$\bigcup \{(2n-6)(m+1)-2, (2n-6)(m+1)-4, \ldots, (2n-6)(m+1)-2m\}$$
$$\bigcup \{(2n-10)(m+1)-2, (2n-10)(m+1)-4, \ldots, (2n-10)(m+1)-2m\} \bigcup \ldots$$
$$\bigcup \{2(m+1)-2, 2(m+1)-4, \ldots, 2(m+1)-2m\}$$

$$E_9' = \left\{ f\left(u_{\frac{n}{2}} v_{\frac{n}{2}+1} \right) \right\}, \ when\ \frac{n}{2}\ is\ even. = \left\{ \left| f\left(u_{\frac{n}{2}} \right) - f\left(u_{\frac{n}{2}+1} \right) \right| \right\}$$

$$= \left\{ \left| \left(\frac{n}{2}(m+1)-1 \right) - \left(\left(3n-\frac{n}{2}-1+1 \right)(m+1)-1 \right) \right| \right\} = \left\{ \left| (m+1)\left(\frac{n}{2}-3n+\frac{n}{2} \right) \right| \right\}$$

$$= \{|\,(m+1)(-2n)\,|\} = \{2(m+1)(n)\} = \{2n(m+1)\}\ and$$

$$E_{10}' = \left\{ f\left(u_{\frac{n}{2}} v_{\frac{n}{2}+1} \right) \right\}, \ when\ \frac{n}{2}\ is\ odd. = \left\{ \left| f\left(u_{\frac{n}{2}} \right) - f\left(v_{\frac{n}{2}+1} \right) \right| \right\}$$

$$= \left\{ \left| \left(\left(4n+1-\frac{n}{2} \right)(m+1)-1 \right) - \left(\left(n+\frac{n}{2}+1 \right)(m+1)-1 \right) \right| \right\}$$

$$= \{|\,(m+1)(2n)\,|\} = \{2n(m+1)\}$$

In both the cases, we observe that all the vertex label sets are having odd values and the edge label sets are having even values and their union is $\{1, 2, \ldots, 4n(m+1) - 1\}$.

Therefore, f is a super graceful labeling and hence, $H \odot mK_1$ is a super graceful graph.

Corollary 2.5: By taking $m=1$, in the above theorem $H \odot K_1$ is a super graceful graph.

Example 2.6. Super graceful labelings of $H_5 \odot 3K_1$ and $H_6 \odot 3K_1$ are given in Figure 3 and Figure 4 respectively.

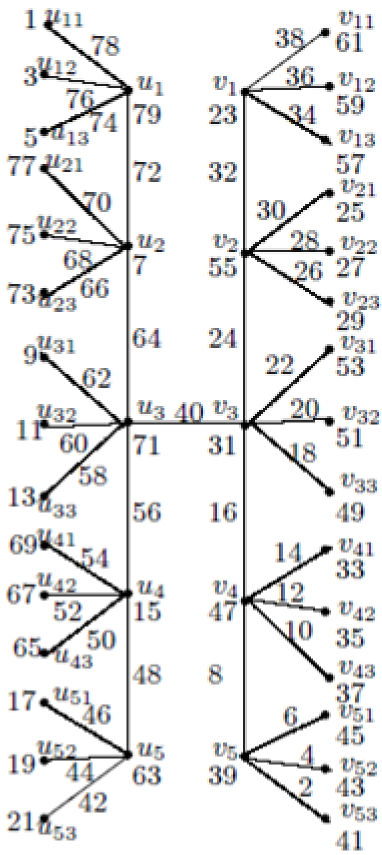

Fig. 3

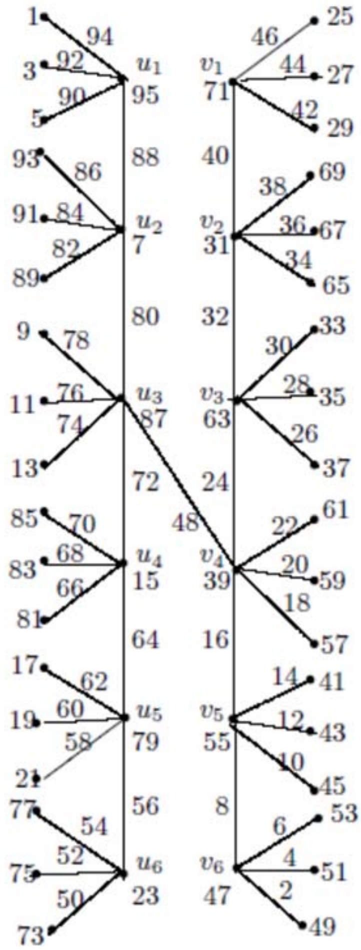

Fig. 4

Definition 2.7. The graph $H_{(n)}^{\otimes}$ is a graph obtained from the H graph by attaching i pendant vertices at each i^{th} vertex on the two paths of n vertices, $1 \le i \le n$.

Theorem 2.8. $H_{(n)}^{\otimes}$ is a super graceful graph when $n \equiv 1 \ (mod\ 2)$.

Proof. Let $P_1(u_1, u_2, ..., u_n)$ and $P_2(v_1, v_2, ..., v_n)$ be the paths of n vertices.

Let $G = H_{(n)}^{\otimes}$. Let u_{ij} and v_{ij}, $1 \le j \le i$ be the pendant vertices attached to u_i and v_i respectively, $1 \le i \le n$.

Now, $|V(G)| = n^2 + 3n, |E(G)| = n^2 + 3n - 1$ and $|V(G) \cup E(G)| = 2n^2 + 6n - 1$.

Define $f : V(G) \cup E(G) \rightarrow \{1, 2, ..., 2n^2 + 6n - 1\}$ as follows.

$$f(u_i) = \begin{cases} 2n^2 + 6n - \left(\dfrac{i^2 + 2i - 1}{2}\right) & 1 \le i \le n, i \equiv 1(mod2) \\ \left(\dfrac{i^2 + 2i - 2}{2}\right) & 1 \le i \le n, i \equiv 0(mod2) \end{cases}$$

and

$$f(v_i) = \begin{cases} n^2 + 3n - \left(\dfrac{i^2 + 2i - 1}{2}\right) & 1 \le i \le n, i \equiv 1(mod2) \\ n^2 + 3n + \left(\dfrac{i^2 + 2i - 2}{2}\right) & 1 \le i \le n, i \equiv 0(mod2) \end{cases}$$

For $1 \le i \le n$ and $i \equiv 1\,(mod2)$, $f(u_{ij}) = \dfrac{i^2 + 1}{2} + 2(j - 1), 1 \le i \le j$

For $1 \le i \le n$ and $i \equiv 0\,(mod2)$, $f(u_{ij}) = 2n^2 + 6n - 1 - \dfrac{i^2}{2} - 2(j - 1), 1 \le i \le j$

For $1 \le i \le n$ and $i \equiv 1\,(mod2)$, $f(v_{ij}) = n^2 + 3n + \left(\dfrac{i^2 + 1}{2}\right) + 2(j - 1), 1 \le i \le j$

For $1 \le i \le n$ and $i \equiv 0\,(mod2)$, $f(v_{ij}) = n^2 + 3n - 1 - \dfrac{i^2}{2} - 2(j - 1), 1 \le i \le j$

We construct the vertex label sets as follows:

Let $V_1 = \bigcup_{\substack{i=1 \\ i \equiv 1\,(mod\ 2)}}^{n} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i \equiv 1\,(mod\ 2)}}^{n} \left\{2n^2 + 6n - \left(\dfrac{i^2 + 2i - 1}{2}\right)\right\}$

$$= \left\{2n^2 + 6n - 1, 2n^2 + 6n - 7, 2n^2 + 6n - 17, ..., \frac{3n^2 + 10n + 1}{2}\right\}$$

$$V_2 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \{f(u_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left\{\frac{i^2+2i-2}{2}\right\} = \left\{3,11,23,...,\frac{n^2-3}{2}\right\}$$

$$V_3 = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \{f(v_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left\{n^2+3n-\left(\frac{i^2+2i-2}{2}\right)\right\}$$

$$= \left\{n^2+3n-1, n^2+3n-7,..., \frac{n^2+4n+1}{2}\right\}$$

$$V_4 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \{f(v_i)\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left\{n^2+3n+\left(\frac{i^2+2i-2}{2}\right)\right\}$$

$$= \left\{n^2+3n+3, n^2+3n+11,..., \frac{3n^2+6n-3}{2}\right\}$$

$$V_5 = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i}\{f(u_{ji})\}\right) = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i}\left\{\left(\frac{i^2+1}{2}\right)+2(j-1)\right\}\right)$$

$$= \left\{1;5,7,9;13,15,17,19,21;...;\frac{n^2+1}{2}, \frac{n^2+1}{2}+2,..., \frac{n^2+1}{2}+2(n-1)\right\}$$

$$V_6 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i}\{f(u_{ij})\}\right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i}\left\{2n^2+6n-1-\frac{i^2}{2}-2(j-1)\right\}\right)$$

$$= \left\{\begin{matrix} 2n^2+6n-3, 2n^2+6n-5; 2n^2+6n-9, 2n^2+6n-11, 2n^2+6n-13, \\ 2n^2+6n-15;...;\frac{3n^2+14n-3}{2}, \frac{3n^2+14n-7}{2}, \frac{3n^2+14n-11}{2},..., \frac{3n^2+10n+5}{2} \end{matrix}\right\}$$

$$V_7 = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i}\{f(v_{ij})\}\right) = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i}\left\{n^2+3n+\left(\frac{i^2+1}{2}\right)+2(j-1)\right\}\right)$$

$$= \left\{\begin{matrix} n^2+3n+1; n^2+3n+5, n^2+3n+7, n^2+3n+9;...; \\ \frac{3n^2+6n+1}{2}, \frac{3n^2+6n+5}{2},..., \frac{3n^2+10n-3}{2} \end{matrix}\right\}$$ and

$$V_8 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m}\{f(v_{ij})\}\right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{m}\left\{n^2+3n-1-\frac{i^2}{2}-2(j-1)\right\}\right)$$

$$= \left\{ \begin{array}{r} n^2 + 3n - 3, n^2 + 3n - 5; n^2 + 3n - 9, n^2 + 3n - 11, n^2 + 3n - 13, \\ n^2 + 3n - 15; ...; \frac{n^2 + 8n - 3}{2}, \frac{n^2 + 8n - 7}{2}, ..., \frac{n^2 + 4n + 5}{2} \end{array} \right\}$$

We construct the edge label sets as follows:

$$Let\ E_1 = \bigcup_{\substack{i=1 \\ i \equiv 1\,(mod\ 2)}}^{n-2} \{f(u_i u_{i+1})\} = \bigcup_{\substack{i=1 \\ i \equiv 1\,(mod\ 2)}}^{n-2} \{| f(u_i) - f(u_{i+1}) |\}$$

$$= \bigcup_{\substack{i=1 \\ i \equiv 1\,(mod\ 2)}}^{n-2} \left\{ \left| \left(2n^2 + 6n - \left(\frac{i^2 + 2i - 1}{2} \right) \right) - \left(\frac{(i+1)^2 + 2(i+1) - 2}{2} \right) \right| \right\}$$

$$= \bigcup_{\substack{i=1 \\ i \equiv 1\,(mod\ 2)}}^{n-2} \left\{ \left| \frac{4n^2 + 12n - 2i^2 - 6i}{2} \right| \right\} = \bigcup_{\substack{i=1 \\ i \equiv 1\,(mod\ 2)}}^{n-2} \{2n^2 + 6n - i^2 - 3i\}$$

$$= \{2n^2 + 6n - 4, 2n^2 + 6n - 18, 2n^2 + 6n - 40, ..., n^2 + 7n + 2\}$$

$$E_2 = \bigcup_{\substack{i=1 \\ i \equiv 0\,(mod\ 2)}}^{n-1} \{f(u_i u_{i+1})\} = \bigcup_{\substack{i=1 \\ i \equiv 0\,(mod\ 2)}}^{n-1} \{| f(u_i) - f(u_{i+1}) |\}$$

$$= \bigcup_{\substack{i=1 \\ i \equiv 0\,(mod\ 2)}}^{n-1} \left\{ \left| \left(\frac{i^2 + 2i - 2}{2} \right) - \left(2n^2 + 6n - \left(\frac{(i+1)^2 + 2(i+1) - 1}{2} \right) \right) \right| \right\}$$

$$= \bigcup_{\substack{i=1 \\ i \equiv 0\,(mod\ 2)}}^{n-1} \left\{ \left| \frac{i^2 + 2i - 2 - 4n^2 - 12n + i^2 + 2i + 1 + 2i + 2 - 1}{2} \right| \right\}$$

$$= \bigcup_{\substack{i=1 \\ i \equiv 0\,(mod\ 2)}}^{n-1} \left\{ \left| \frac{2i^2 + 6i - 4n^2 - 12n}{2} \right| \right\} = \bigcup_{\substack{i=1 \\ i \equiv 0\,(mod\ 2)}}^{n-1} \{| i^2 + 3i - 2n^2 - 6n |\}$$

$$= \bigcup_{\substack{i=1 \\ i \equiv 0\,(mod\ 2)}}^{n-1} \{2n^2 + 6n - i^2 - 3i\}$$

$$= \{2n^2 + 6n - 10, 2n^2 + 6n - 28, 2n^2 + 6n - 54, ..., n^2 + 5n + 2\}$$

$$E_3 = \bigcup_{\substack{i=1 \\ i \equiv 1\,(mod\ 2)}}^{n-2} \{f(v_i v_{i+1})\} = \bigcup_{\substack{i=1 \\ i \equiv 1\,(mod\ 2)}}^{n-2} \{| f(v_i) - f(v_{i+1}) |\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-2} \left\{ \left| \left(n^2 + 3n - \left(\frac{i^2 + 2i - 1}{2} \right) \right) - \left(n^2 + 3n + \left(\frac{(i+1)^2 + 2(i+1) - 2}{2} \right) \right) \right| \right\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-2} \left\{ \left| \frac{2n^2 + 6n - (i^2 + 2i - 1) - 2n^2 - 6n - (i^2 + 2i + 1) - 2i - 2 + 2}{2} \right| \right\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-2} \left\{ \left| \frac{-2i^2 - 6i}{2} \right| \right\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-2} \left\{ \frac{2i^2 + 6i}{2} \right\} = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n-2} \{i^2 + 3i\}$$

$$= \{4, 18, 40, \ldots, n^2 - n - 2\}$$

$$E_4 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \{f(v_i v_{i+1})\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \{|\, f(v_i) - f(v_{i+1})\,|\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \left\{ \left| \left(n^2 + 3n + \left(\frac{i^2 + 2i - 2}{2} \right) \right) - \left(n^2 + 3n - \left(\frac{(i+1)^2 + 2(i+1) - 1}{2} \right) \right) \right| \right\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \left\{ \left| \frac{2n^2 + 6n + i^2 + 2i - 2 - 2n^2 - 6n + i^2 + 2i + 1 + 2i + 2 - 1}{2} \right| \right\}$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \left\{ \left| \frac{2i^2 + 6i}{2} \right| \right\} = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n-1} \{i^2 + 3i\} = \{10, 28, 54, \ldots, n^2 + n - 2\}$$

$$E_5 = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{m} \{f(u_i u_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{i} \{|\, f(u_i) - f(u_{ij})\,|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{i} \left\{ \left| \left(2n^2 + 6n - \left(\frac{i^2 + 2i - 1}{2} \right) \right) - \left(\left(\frac{i^2 + 1}{2} \right) + 2(j-1) \right) \right| \right\} \right)$$

$$= \{2n^2 + 6n - 2; 2n^2 + 6n - 12, 2n^2 + 6n - 14, 2n^2 + 6n - 16; \ldots;$$

$$n^2 + 5n, n^2 + 5n - 2, n^2 + 5n - 4, \ldots, n^2 + 3n + 2\}$$

$$E_6 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{i} \{f(u_{ij})\} \right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\,2)}}^{n} \left(\bigcup_{j=1}^{i} \{|\, f(u_i) - f(u_{ij})\,|\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i} \left\{ \left| \left(\frac{i^2+2i-2}{2} \right) - \left((2n^2+6n-1) - \frac{i^2}{2} - 2(j-1) \right) \right| \right\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i} \{ |(2n^2+6n+2)-(i^2+i+2j)| \} \right)$$

$$= \left\{ \begin{matrix} 2n^2+6n-6, 2n^2+6n-8; 2n^2+6n-20, 2n^2+6n-22, 2n^2+6n-24, \\ 2n^2+6n-26; \ldots; n^2+7n, n^2+7n-2, n^2+7n-4, \ldots, n^2+5n+4 \end{matrix} \right\}$$

$$E_7 = \bigcup_{\substack{i=1 \\ i\equiv 1\ mod\ 2}}^{n} \left(\bigcup_{j=1}^{i} \{ f(v_i v_{ij}) \} \right) = \bigcup_{\substack{i=1 \\ i\equiv 1\ mod\ 2}}^{n} \left(\bigcup_{j=1}^{i} \{ | f(v_i) - f(v_{ij}) | \} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod 2)}}^{n} \left(\bigcup_{j=1}^{i} \left\{ \left| \left(n^2+3n - \left(\frac{i^2+2i-1}{2} \right) \right) - \left(n^2+3n + \left(\frac{i^2+1}{2} \right) + 2(j-1) \right) \right| \right\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 1\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i} \{ | i^2+i+2j-2 | \} \right)$$

$$= \{2; 12, 14, 16; 30, 32, 34, 36, 38; \ldots; n^2+n, n^2+n+2, n^2+n+4, \ldots, n^2+3n-2\}$$

$$E_8 = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i} \{ f(v_i v_{ij}) \} \right) = \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i} \{ | f(v_i) - f(v_{ij}) | \} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i} \left\{ \left| \left(n^2+3n + \left(\frac{i^2+2i-2}{2} \right) \right) - \left(n^2+3n-1 - \frac{i^2}{2} - 2(j-1) \right) \right| \right\} \right)$$

$$= \bigcup_{\substack{i=1 \\ i\equiv 0\,(mod\ 2)}}^{n} \left(\bigcup_{j=1}^{i} \{ | i^2+i+2j-2 | \} \right)$$

$$= \{6, 8; 20, 22, 24, 26; \ldots; n^2-n, n^2-n+2, n^2-n+4, \ldots, n^2+n-4\}$$ and

$$E_9 = \left\{ f\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}} \right) \right\} = \left\{ \left| f\left(u_{\frac{n+1}{2}} \right) - f\left(v_{\frac{n+1}{2}} \right) \right| \right\}$$

$$= \left\{ \left| \left(2n^2 + 6n - \left(\frac{\left(\frac{n+1}{2}\right)^2 + 2\left(\frac{n+1}{2}\right) - 1}{2} \right) \right) - \left(n^2 + 3n - \left(\frac{\left(\frac{n+1}{2}\right)^2 + \left(\frac{n+1}{2}\right) - 1}{2} \right) \right) \right| \right\}$$

$$= \{| n^2 + 3n |\} = \{n^2 + 3n\}$$

So far, we have observed that all the vertex label sets and the edge label sets are distinct and their union is $\{1, 2, ..., 2n^2 + 6n - 1\}$. Therefore, f is a super graceful labeling and hence, $H_{(n)}^{\otimes}$ is a super graceful graph, when $n \equiv 1 \pmod 2$

Example 2.9. Super graceful labeling of $H_{(5)}^{\otimes}$ is given in Figure 5.

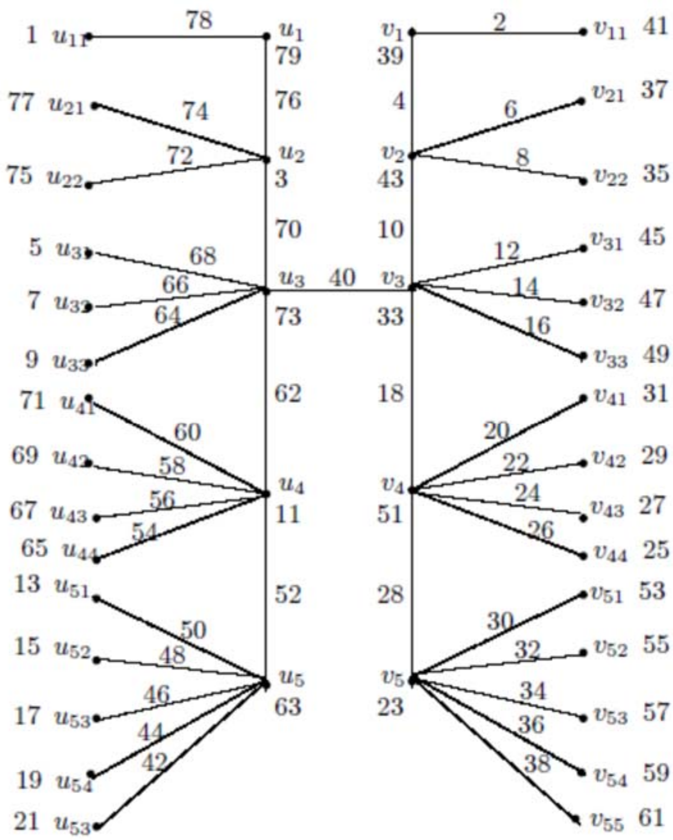

Fig. 5

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