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Antimagic Labelings of Graphs

N. Sridharan[1] and R. Umarani[2]

[1]Department of Mathematics, Alagappa University, Karaikudi

[2]umaprinci@yahoo.com

Abstract: *Hartsfield and Ringel [1] introduced antimagic labeling. In this paper, we investigate the antimagic labeling of* $C_n \odot P_3$ *for* $n \geq 3$ *;* $K_2 \odot C_n$ *for* $n \geq 3$ *;* C_n^+ *for* $n \geq 3$ *;* $S_1C_n^+$ *; the generalized Peterson graph P(n,k); gear graphs ; Helm* H_n *for all* $n \geq 3$ *; flower* F_n *for* $n \geq 3$ *; shell graph H(n,n-3) for* $n \geq 4$ *; Banana tree BT* (n_1,n_2) *for all* $2 \leq n_1 \leq n_2$*; friendship graph* $C_3^{(t)}$ *for all* $t \geq 2$*; fan graph* F_n *for all* $n \geq 2$ *; Lantern* $K_2 + \overline{K_n}$ *(for* $n \geq 2$*) and triangular snakes.*

1. Introduction

Kotzig and Rosa [3] defined a magic labeling of a graph G(V,E) as a bijection f from $V \cup E$ to $\{1,2,...,|V \cup E|\}$ such that for all edge xy, f(x) + f(y) +f(xy) is constant.

They proved that (1) $k_{m,n}$ has a magic labeling for all m,n. (2) C_n has a magic labeling for all $n \geq 3$. (3) nP_2 has a magic labeling if and only if n is odd. (4) K_n has a magic labeling if and only if n = 1,2,3,4,5 and 6. Balakrishnan and Sampath Kumar [4] proved that the join of K_n and two disjoint copies of K_2 is magic if and only if n = 3. Hartsfield and Ringel [1] introduced antimagic graphs. A graph with q edges is called antimagic if its edges can be labeled with 1,2, …….q. so that the sum of the labels of the edges incident to each vertex are distinct. Paths P_n $(n \geq 3)$ cycles C_n and K_n $(n \geq 3)$are antimagic. Hartsfield and Ringel Conjectured that every tree except P_2 is antimagic. . For an extensive survey on graph labeling we refer to Gallian[2].

In this paper, we investigate the antimagic labeling of $C_n \odot P_3$ for $n \geq 3$; $K_2 \odot C_n$ for $n \geq 3$; C_n^+ for $n \geq 3$; $S_1C_n^+$; the generalized Peterson graph P(n,k); gear graphs ; Helm H_n for all $n \geq 3$; flower F_n for $n \geq 3$; shell graph H(n,n-3) for $n \geq 4$; Banana tree BT (n_1,n_2) for all $2 \leq n_1 \leq n_2$; friendship graph $C_3^{(t)}$ for all $t \geq 2$; fan graph F_n for all $n \geq 2$; Lantern $K_2 + \overline{K_n}$ (for $n \geq 2$) and triangular snakes.

2. Main Results

Let C_n be the cycle with n vertices and P_3 be the path on 3 vertices. We obtain $C_n \odot P_3$ from C_n and n copies of P_3 by joining i^{th} vertex of C_n to every vertex of i^{th} copy

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of P_3 . $C_n \odot P_3$ has 4n vertices and 6n edges. In the following theorem, we prove that $C_n \odot P_3$ is antimagic.

Theorem 2.1 : The graph $C_n \odot P_3$ is antimagic, for all $n \geq 3$.

Proof : Let $G = C_n \odot P_3$;

$V(G) = \{v_i, a_i, b_i, c_i \ / 1 \leq i \leq n\}$ and

$E(G) = \{a_i b_i, b_i c_i, v_i a_i, v_i b_i, v_i c_i \ / 1 \leq i \leq n\}$

$\cup \ \{v_n v_1, v_i v_{i+1} \ / 1 \leq i \leq n-1\}$

We define $f : E(G) \longrightarrow \{1,2,\ldots\ldots,6n\}$ as follows :

$f(v_i v_{i+1}) = 6n - i + 1$, if $i = 1,2,\ldots, n-1$

$f(v_n v_1) = 5n + 1$; and for all i, $1 \leq i \leq n$,

$f(v_i a_i) = i, f(v_i b_i) = 2n + i, f(v_i c_i) = n + i, f(a_i b_i) = 3n + i$ and $f(b_i c_i) = 4n + i$,

The induced map f* on V is given by

$$\left.\begin{aligned} f^*(a_i) &= 3n + 2i \\ f^*(b_i) &= 9n + 3i \\ f^*(c_i) &= 5n + 2i \end{aligned}\right\} \text{for all } 1 \leq i \leq n$$

$f^*(v_i) = 15n + i + 3$ for all $2 \leq i \leq n$ and $f^*(v_1) = 14n + 4$.

As $3n + 2i < 5n + 2i < 9n + 3i < 14n + 4 < 15n + i + 3$, for all i, $1 \leq i \leq n$, it follows that f* is injective and hence f is an antimagic labeling for $C_n \odot P_3$. An antimagic labeling for $C_9 \odot P_3$ is illustrated in figure 2.1. The graph $K_2 \odot C_n$ obtained from K_2 and two copies of C_n by joining i^{th} vertex (i=1,2) of K_2 to each vertex of i^{th} copy of C_n. It can also be obtained from two copies of wheel W_n by joining central vertices of the two copies by an edge.

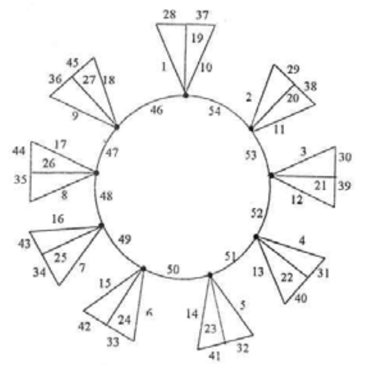

Figure2.1 An antimagic labeling for $C_9 \odot P_3$

Theorem 2.2 : The graph $G = K_2 \odot C_n$ is antimagic, for all $n \geq 3$.

Proof:

Let $V(G) = \{u_1, u_2, v_i, w_i / 1 \leq i \leq n\}$ and

$$E(G) = \{u_1u_2, w_nw_1, v_nv_1\} \cup \{w_iw_{i+1}, v_iv_{i+1} / 1 \leq i \leq n-1\} \cup \{u_1w_i, u_2v_i / 1 \leq i \leq n\}$$

We define $f : E(G) \longrightarrow \{1,2,..4_n,4_{n+1}\}$ as follows:

$f(w_iw_{i+1}) = i$ for $i = 1,2,..n-1$

$f(w_n\ w_1) = n$

$f(u_1\ w_i) = 2n - i + 1$ for $i = 1,2,.., n$

$f(v_1\ v_{i+1}) = 2n + i$ for $i = 1,2,.., n-1$

$f(v_n\ v_1) = 3n$

$f(u_2\ v_i) = 4n - i + 1$ for $i = 1,2,.., n$

$f(u_1\ u_2) = 4n + 1$.

The induced map f* on V(G) is obtained as follows :

$f^*(w_i) = 2n + i$ for $2 \leq i \leq n$

$f^*(w_1) = 3n + 1$; $f^*(v_1) = 9n + 1$

$f^*(v_i) = 8n + i$, for $2 \leq i \leq n$

$f^*(u_1) = \frac{3n^2 + n}{2}$ and $f^*(u_2) = \frac{7n^2 + n}{2}$

If $n \geq 6$, $f^*(u_1) = \frac{3n^2 + n}{2} \geq \frac{18n + n}{2} \geq 9n + 3$

So if $n \geq 6$, we have $2n + i < 3n + 1 < 8n + i < 9n+1 < \frac{3n^2 + n}{2} < \frac{7n^2 + n}{2}$,

for all i, $1 \leq i \leq n$ and hence in this case ($n \geq 6$), f* is injective.

If $n = 5$, $f^*(u_1) = 40 \neq f^*(x)$ for all $x \neq u_1 \in V(G)$ and $f^*(u_2) \geq 18n$.

If $n = 4$, $f^*(u_1) = 26 \neq f^*(x)$ for all $x \neq u_1 \in V(G)$, and

$f^*(u_2) = 58 > 37 = 9n + 1$.

If $n = 3$, $f^*(u_1) = 15 = 5n \neq f^*(x)$ for all $x \neq u_1 \in V(G)$, and $f^*(u_2) > 10n$.

Thus in all the cases, f* is injective and hence *f* is an antimagic labeling for $K_2 \odot C_n$.

An antimagic labeling for $K_2 \odot C_7$ is illustrated in the figure 2.2

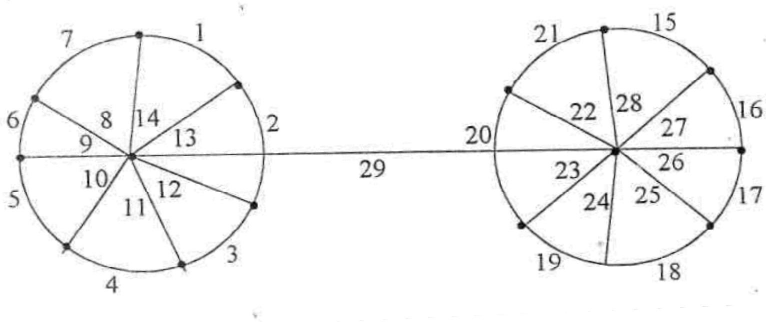

Figure 2.2 An antimagic labeling for $K_2 \odot C_7$

Theorem 2.3 : The graph C_n^+ is antimagic, for all n ≥ 3.

Proof :

Let $v_1 v_2 \ldots v_n v_1$ be the cycle C_n, and let u_i be the pendant vertex attached to the vertex v_i, for all i, $1 \le i \le n$.

Define $f : E(G) \longrightarrow \{1,2,\ldots,2n\}$ as follows:

$f(u_i v_i) = i$ for all $1 \le i \le n$; $f(u_i v_{i+1}) = 2n - i + 1$ for all $1 \le i \le n-1$, and $f(u_1 v_n) = n+1$.

The induced map f* on V(G) is obtained as follows:

$f^*(u_i) = i$ for all i, $1 \le i \le n$; $f^*(v_1) = 3n+2$ and $f^*(v_1) = 4n - i + 3$, for all $2 \le i \le n$

Clearly f* is injective and hence is an antimagic labeling for C_n^+.

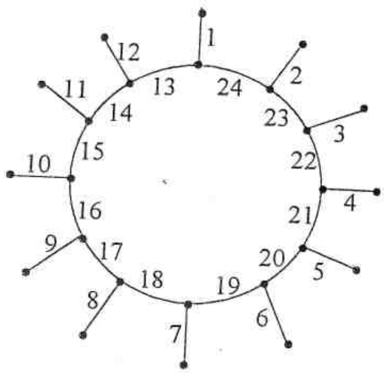

Figure 2.3 Antimagic labeling for C_{12}^+

Theorem 2.4 : The graph $s_1(C_n^+)$, obtained from C_n^+ by subdividing each edge of C_n^+ once, is antimagic.

Proof : Let $u_1 u_2 .. u_n$ be the cycle C_n and $v_1, v_2, \ldots, v_n$ be the pendant vertices of C_n^+, u_i being adjacent to u_i. Subdivide the edge $u_i\ u_{i+1}$ by introducing a new vertex y_i (for i

$1 \le i \le n-1$), the edge u_1u_n by introducing a new vertex. y_n, and the edge $u_i\ v_i$, ($1 \le i \le n$), by introducing a new vertex x_i. Let the resulting graph $S_1(C_n^+)$ be G.

Define $f : E(G) \longrightarrow \{1,2,\ldots,4n\}$ as follows :

$f(x_i v_i) = i$, for $1 \le i \le n$

$f(u_i x_i) = n + i$, for $1 \le i \le n$

$f(u_1 y_n) = 3n$

$f(y_{i-1} u_i) = 3n - i + 1$, for $2 \le i \le n$

$f(u_i y_i) = 4n - i$, for $1 \le i \le n-1$

$f(u_n y_n) = 4n$

The induced map f* is obtained as follows:

$$\left.\begin{aligned} f^*(v_i) &= i \\ f^*(x_i) &= n + 2i \end{aligned}\right\} \text{for } 1 \le i \le n$$

$$\left.\begin{aligned} f^*(u_i) &= 8n - i + 1 \\ f^*(y_i) &= 7n - 2i \end{aligned}\right\} \text{for } 1 \le i \le n-1$$

$f^*(y_n) = 7n$ and $f^*(u_n) = 8n + 1$.

As $i < n + 2j < 7n-2k < 7n < 8n-s+1 < 8n+1$, for all $1 \le i, j \le n$; $1 \le k,s \le n-1$, it follows that f* is injective.

Thus f is an antimagic labeling for $S_1(C_n^+)$.

An antimagic labeling for $S_1(C_9^+)$ is illustrated in Figure 2.4.

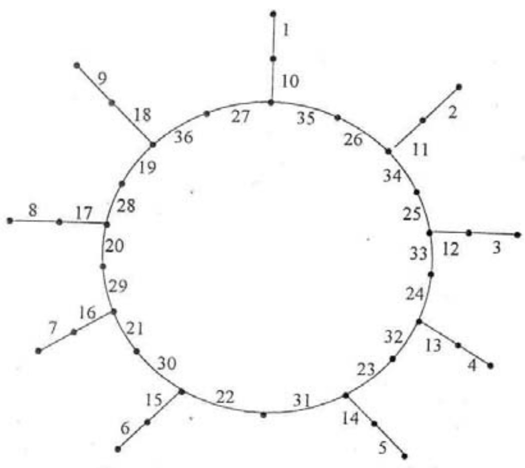

Figure 2.4: An antimagic labeling for $S_1(C_9^+)$.

Theorem 2.5: Let $n \geq 5$ be a prime and $k \geq 2$ be a positive integers such that $k < \frac{n}{2}$. The generalized Peterson graph P(n, k) is antimagic , for all prime n.

Proof:

Let G = P (n, k) be the generalized Peterson graph.

Let $V(G) = \{v_i, u_i \ / \ 0 \leq i \leq n\text{-}1\}$ and let $E(G) = \{u_i\, v_i\, ;\, v_iv_{i+1},\, u_iu_{i+k} \ / \ 0 \leq i \leq n\text{-}1\}$. (for suffixes, the addition i+k is under addition modulo n). As n and K are prime to each other, gcd(n, k) = 1 and k is a generator for the group Z_n . Hence each $i = m_i k$, for some unique integer m_i ($0 \leq m_i \leq n\text{-}1$) in the group Z_n . It is clear that if $i \neq j$, then $m_i \neq m_j$ ($0 \leq i, j \leq n\text{-}1$).

Define $f: E(G) \longrightarrow \{1,2,\ldots,3n\}$ as follows :

$f(v_iv_{i+1}) = i+1$ for $0 \leq i \leq n\text{-}2$; $f(v_{n\text{-}1}v_0) = n$; $f(v_iu_i) = 2n\text{-}i$ for $0 \leq i \leq n\text{-}1$

$f(u_iu_{i+k}) = 3n\text{-}m_i$ for $0 \leq i \leq n\text{-}1$

Clearly, $i+1 < n < 2n\text{-}j < 3n\text{-}s$ for all $0 \leq i \leq n\text{-}2$; $0 \leq j,s \leq n\text{-}1$ and hence the map f is bijective .

The induced map f^* is obtained as follows:

$f^*(v_i) = 2n + i + 1$ for $1 \leq i \leq n\text{-}1$; $f^*(v_0) = 3n+1$; $f^*(u_0) = 7n+1$ and $f^*(u_i) = 8n\text{-}2m_i - i + 1$ for $1 \leq i \leq n\text{-}1$.

As $f^*(v_i) < 4n < f^*(u_j)$ for all $0 \leq i,j \leq n\text{-}1$, $f^*(v_i) \neq f^*(v_j)$.

Now if $1 \leq i \leq n\text{-}1$, $f^*(u_i) = f^*(u_j)$

$$\Longrightarrow 8n - 2m_i - i + 1 = 8n - 2m_j - j + 1$$

$$\Longrightarrow 2m_i + i = 2m_i + j$$

$$\Longrightarrow 2(m_i - m_j) + (i-j) = 0$$

$$\Longrightarrow 2(m_i - m_j) + (m_i - m_j)\, k = 0 \pmod n$$

$$\Longrightarrow (k+2)\,(m_i - m_j) = 0 \pmod n$$

$$\Longrightarrow \text{either } m_i - m_j = 0 \text{ or n divides } k+2,$$

(as n is prime)

$$\Longrightarrow m_i = m_j \quad \text{as } k+2 < n$$

$$\Longrightarrow i = j$$

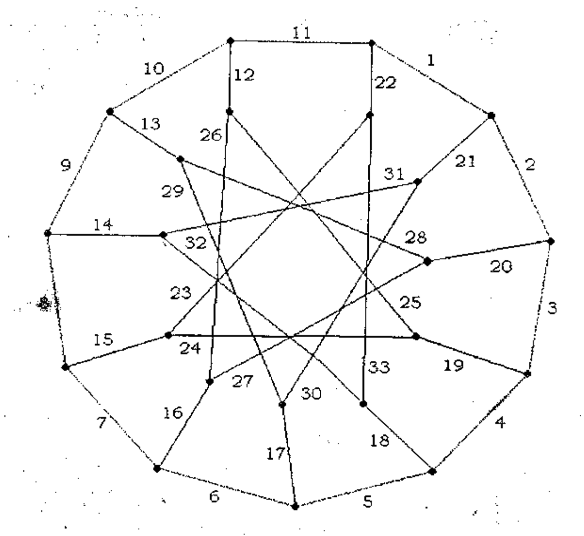

Figure 2.5 : An antimagic labeling for P (11,4).

Remark : The antimagic labeling f defined in the theorem 2.5 also is an antimagic labeling for P (n,k) , Where $k < \frac{n}{2}$ and both k and k+2 are prime to n. (The integer n need not be a prime).

Thorem 2.6: Every gear graph is antimagic .

Proof:

Let G be the gear graph obtained from a Wheel W_n by subdividing each edge on the cycle C_n once. Let $V(G) = \{v, v_i, u_i \mid 1 \leq i \leq n\}.(n \geq 3)$ and

let $E(G) = \{vu_i, v_i u_i \mid 1 \leq i \leq n\} \cup \{u_i v_{i+1}, u_n v_1 \mid 1 \leq i \leq n-1\}$.

Define $f: E(G) \longrightarrow \{1,2,\ldots,3n\}$ as follows :

$f(v_i u_i) = i$ for $1 \leq i \leq n$

$f(u_i v_{i+1}) = n + i$ for $1 \leq i \leq n-1$

$f(u_n v_1) = 2n$

$f(v v_i) = 3n - i + 1$ for $1 \leq i \leq n$

The induced map f^* on V(G) is obtained as

$f^*(u_i) = n+2i$ for $1 \leq i \leq n$

$f^*(v_i) = 4n+i$ for $2 \leq i \leq n$

$f^*(v_1) = 5n+1$

$$f^*(v) = \frac{n(5n+1)}{2}$$

As $n+2i < 4n+j < 5n+1 < \frac{n(5n+1)}{2}$ for all $1 \le i \le n$; $2 \le j \le n$ (as $n \ge 3$),

The map f^* is injective and hence f is an antimagic labeling for G.

Figure 2.6 illustrates an antimagic labeling for the gear graph G_{12}

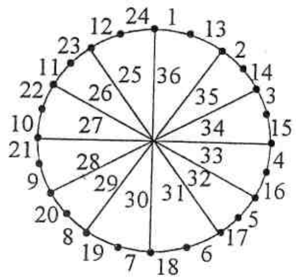

Figure 2.6 : An antimagic labeling for the gear graph G_{12}.

The helm H_n the graph obtained from the Wheel W_n, ($n \ge 3$), by attaching a pendant edge at vertex of the n cycle of W_n .(H_n can also be obtained from C^+ by joining all the vertices of C with degree 3 to a new vertex v).

Thorem 2.7: Helm H_n is antimagic , for all $n \ge 3$.

Proof:

Let $G = H_n$ and let $n \ge 4$. Let $V(G) = \{v, v_i, u_i \mid 1 \le i \le n\}$ and

let $E(G) = \{v v_i, v_i u_i, v_n v_1 / 1 \le i \le n\} \cup \{v_i v_{i+1} \mid 1 \le i \le n-1\}$

G-v is the graph C_n^+ Let f be the antimagic labeling for C_n^+ as defined in the proof of the Theorem 2.3. Now we extend that map f to E(G) , by defining $f(vv_i) = 3n-i+2$, $2 \le i \le n$ and $f(vv_1) = 2n+1$

The induced map f^* on V(G) is given by

$$\left.\begin{array}{l} f^{'}(u_i) = i \\ f^{'}(u_{n-i+1}) = 5n+3+2i \end{array}\right\} \text{for } 1 \le i \le n$$

$$f^*(v) = \frac{n(5n+1)}{2}$$

As $n \ge 4$, $\frac{n(5n+1)}{2} \ge 10n+2 > 5n+3+2i$ for $1 \le i \le n$.

Hence f^* is injective and H_n , ($n \ge 4$) , is antimagic .

The helm H_3 is also antimagic , an antimagic labeling for H_3 is exhibited in the Figure 2.7 (a).

A flower F_n is the graph obtained from the helm H_n by joining each pendent vertex of H_n to the centrel vertex of H_n .

Thorem 2.8 : For $n \geq 3$, flower F_n is antimagic .

Proof :

Let $G = F_n$, $V(G) = \{v, v_i, u_i \,/\, 1 \leq i \leq n\}$ and let $E(G) = \{vv_i, vu_i, v_iu_i \,/\, 1 \leq i \leq n\} \cup \{v_nv_1; v_iv_{i+1} \,/\, 1 \leq i \leq n-1\}$

Define f: $E(G) \longrightarrow \{1,2,\ldots,4n\}$ as follows :

$f(u_i v_i) = i$ for all $1 \leq i \leq n$; $f(v_iv_{i+1}) = 2n-i+1$, for all $1 \leq i \leq n-1$; $f(v_1v_n) = n+1$

$f(vv_i) = 3n - i+2$ for $2 \leq i \leq n$; $f(vv_1) = 2n+1$; $f(vu_i) = 3n+i$ for all $1 \leq i \leq n$.

(The map f is an extension of the antimagic labeling for H_m, defined in the proof of the Theorem 2.7)

The induced map f^* on V(G) is given by :

$f^*(u_i) = 3n+2i$ for $1 \leq i \leq n$; $f^*(v_i) = 7n-2i+5$ for $2 \leq i \leq n-1$; $f^*(v_1) = 5n+3$, $f^*(v) = (6n+1)$

As $3n + 2i < 5n + 3 < 7n - 2j + 5 < (6n + 1)$ n, for all $n \geq 3$

$1 \leq i \leq n$ and $2 \leq j \leq n-1$, the map f^* is injective. So the flower F_n ($n \geq 3$), is antimagic

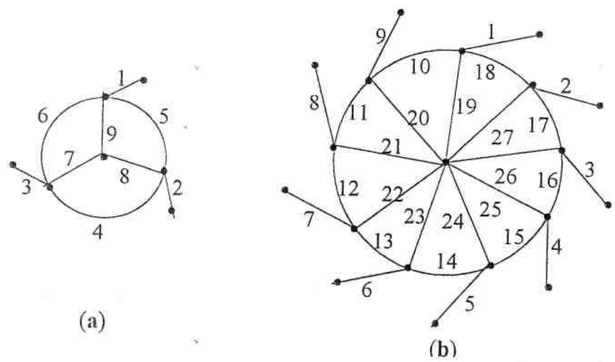

Figure 2.7 (a) An antimagic labeling for H_3 and (b) An antimagic labeling for H_9

The shell graph of order n, $n \geq 4$, denoted by H(n,n-3) is obtained from the cycle C_n of order n by adding (n-3) chords incident with a common vertex.

Theorem 2.9 : For every $n \geq 4$, the shell graph H(n, n-3) is antimagic.

Proof :

Let G = H(n,n-3). We assume that $n \geq 6$, (An antimagic labelings for H(4,1) and H(5,2) are shown in figure 2.8)

Let $V(G) = \{ v_i \mid 1 \leq i \leq n \}$ and $E(G) = \{ v_1v_i \mid 2 \leq i \leq n \} \cup \{ v_{i+1}v_i \mid 2 \leq i \leq n-1 \}$

Define $f : E(G) \longrightarrow \{1,2,\ldots,2n-3\}$ as follows :

$f(v_1 v_i) = n - i$ for $3 \leq i \leq n-1$; $f(v_1 v_{i+1}) = n - 3 + i$ for $1 \leq i \leq n-1$and $f(v_n v_1) = 2n - 3$.

The induced map f^* on V(G) is given by

$f^*(v_1) = \dfrac{n^2+n-4}{2}$; $f^*(v_2) = 2n - 3$; $f^*(v_i) = 3n + i - 7$ for $3 \leq i \leq n-1$ and $f^*(v_n) = 4n - 7$

For some i , $3 \le i \le n-1$, $f^*(v_i) = f^*(v_n) \Longrightarrow 3n + i - 7 = 4n - 7 \Longrightarrow i = n$, which is a contradiction.

We have $f^*(v_i) \ne f^*(v_n)$ for all $2 \le i \le n-1$. If $n \ge 6$, $\frac{n^2 - 5n + 10}{2} \ge n$, it follows that $f^*(v_i)$ $\ne f^*(v_1)$

for all i, $2 \le i \le n-1$. Also as $n \ge 6$, $\frac{n^2 + n - 4}{2} \ne 4n - 7$.

Thus if $n \ge 6$, $f^*(v_1),\ldots f^*(v_n)$ are all distinct and hence f is an antimagic labeling for H(n, n-3) , $\forall$ $n \ge 6$.

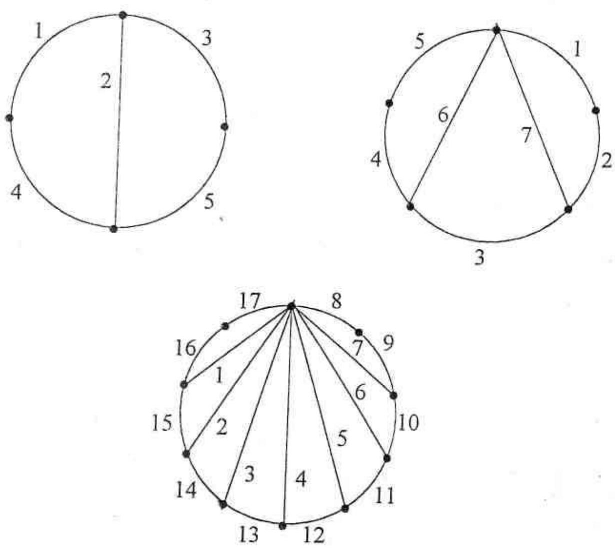

Figure 2.8 Antimagic labeling for H(4,1), H(5,2),and H(10,7).

$BT(n_1,n_2)$ is the tree obtained by joining a new vertex w to one pendant vertex of each stars k_{1,n_1} **and** k_{1,n_2} . $BT(n_1,n_2)$is called a banana tree.

Theorem 2.10 : Banana tree $BT(n_1,n_2)$ admits an antimagic labeling for all $2 \le n_1 \le n_2$.

Proof :

Let $V(G) = \{ v,u,w,u_i,v_j \mid 1 \le i \le n_1; 1 \le i \le n_2 \}$ and

$E(G) = \{ u\,u_i \mid 1 \le i \le n \} \cup \{ v\,v_j \mid 1 \le j \le n_2 \} \cup \{ wu_{n_1}, wu_{n_2} \}$

Define $f : E(G) \longrightarrow \{ 1,2,\ldots..n_1 + n_2 + 2 \}$ as follows :

$f(u\,u_i) = i$ for $1 \le i \le n_1 - 1$;

$f(v\,v_j) = n_1 + j - 1$ for$1 \le i \le n_2$

$$f(uu_{n_1}) = \begin{cases} n_1 + n_2 \text{ if } n_2 \neq \dfrac{n_1^2 - 3n_1 - a}{2} \\ n_1 + n_2 + 1 \text{ if } n_2 \neq \dfrac{n_1^2 - 3n_1}{2} \\ n_1 + n_2 + 2 \text{ if } n_2 \neq \dfrac{n_1^2 - 3n_1 - 4}{2}, \quad \text{or } \dfrac{n_1^2 - 3n_1 - 6}{2} \end{cases}$$

$$f(wu_{n_1}) = \begin{cases} n_1 + n_2 + 1 \text{ if } n_2 \neq \dfrac{n_1^2 - 3n_1 - 6}{2} \\ n_1 + n_2 \quad \text{if } n_2 \neq \dfrac{n_1^2 - 3n_1 - 4}{2}, \quad \text{or } \dfrac{n_1^2 - 3n_1 - 6}{2} \end{cases}$$

$$f(wu_{n_2}) = \begin{cases} n_1 + n_2 + 1 \text{ if } n_2 \neq \dfrac{n_1^2 - 3n_1}{2} \\ n_1 + n_2 \quad \text{if } n_2 \neq \dfrac{n_1^2 - 3n_1}{2} \end{cases}$$

The induced map f^* on V(G) is given by

$f^*(u_i) = i$ for $1 \le i \le n_1$

$$f^*(u_{n_1}) = \begin{cases} 2n_1 + 2n_2 + 2 \text{ if } n_2 \neq \dfrac{n_1^2 - 3n_1}{2} \\ 2n_1 + 2n_2 + 3 \text{ if } n_2 \neq \dfrac{n_1^2 - 3n_1}{2} \end{cases}$$

$f^*(v_j) = n_1+j-1$ for $1 \le j \le n_2 - 1$

$$f^*(v_{n_2}) = \begin{cases} 2n_1 + 2n_2 \quad \text{if } n_2 \neq \dfrac{n_1^2 - 3n_1}{2} \\ 2n_1 + 2n_2 - 1 \text{ if } n_2 \neq \dfrac{n_1^2 - 3n_1}{2} \end{cases}$$

$$f^*(v_{n_2})=\begin{cases}2n_1+2n_2+3 \text{ if } n_2\neq\frac{n_1^2-3n_1-a}{2} \quad a=0,4 \text{or} 6\\ 2n_1+2n_2 \text{ if } n_2\neq\frac{n_1^2-3n_1}{2}\\ 2n_1+2n_2+1 \text{ if } n_2\neq\frac{n_1^2-3n-4}{2} \text{ or } \frac{n_1^2-3n-6}{2}+1\end{cases}$$

$$f^*(u)=\begin{cases}\frac{n_1(n_1+1)}{2}+n_2 \text{ if } n_2\neq\frac{n_1^2-3n_1-a}{2} \quad a=0,4 \text{or} 6\\ \frac{n_1(n_1+1)}{2}+n_2+1 \text{if } n_2\neq\frac{n_1^2-3n}{2}\\ \frac{n_1(n_1+1)}{2}+n_2+2 \text{ if } n_2\neq\frac{n_1^2-3n_1-4}{2} \text{ or } \frac{n_1^2-3n_1-6}{2}\end{cases}$$

$$f^*(v)=\frac{n_2(2n_1+n_2-1)}{2}$$

Clearly (i) $f^*(u_i) < f^*(v_j) < f^*(v_{n_2}) < f^*(u_{n_1})$ for all $1 \le i \le n_1 -1$ and $1 \le j \le n_2 -1$. So $f^*(u_i)$, $f^*(v_j)$ $1 \le i \le n_1$ and $1 \le j \le n_2$, are all distinct. (ii) $f^*(u)$, $f^*(v)$, $f^*(w)$,$f(u_{n_1})$, $f^*(v_{n_2}) \notin \{ f^*(u_i), f^*(v_j) \mid 1 \le i \le n_1-1, 1 \le j \le n_2-1\}$.

It is enough to show that $f^*(u)$, $f^*(v)$, $f^*(w)$,$f^*(u_{n_1})$,$f^*(v_{n_2})$ are all distinct.

Let $\lambda = 2n_1 + 2n_2$.

$$\text{Thus } f^*(u_{n_1})=\begin{cases}\lambda+2 \text{if } n_2\neq\frac{n_1^2-3n_1}{2}\\ \lambda+3 \text{ if } n_2\neq\frac{n_1^2-3n_1}{2}\end{cases}$$

$$f^*(v_{n_2})=\begin{cases}\lambda \quad \text{if } n_2\neq\frac{n_1^2-3n_1}{2}\\ \lambda-1 \text{ if } n_2=\frac{n_1^2-3n_1}{2}\end{cases}$$

$$f^*(w)=\begin{cases}\lambda+3 & \text{if } n_2 \neq \dfrac{n_1^2-3n_1-a}{2} \quad a=0,4 \text{ or } 6\\ \lambda+2 & \text{if } n_2=\dfrac{n_1^2-3n_1}{2}\\ \lambda+1 & \text{if } n_2=\dfrac{n_1^2-3n-4}{2} \text{ or } \dfrac{n_1^2-3n-6}{2}\end{cases}$$

$$f^*(v) = \frac{n_2(2n_1+n_2-1)}{2} = \frac{2n_1n_2+n_2^2-n_2}{2} \geq \frac{10n_1+5n\ \ -n_2}{2} \text{ if } n_2 \geq 5$$

$= 5n_1 + 2n_2$

$= \lambda + 3n_1 + \geq \lambda + 6$

If $n_2 = 4$, $f^*(v) = \dfrac{8n_1+4n_2-4}{2} = \lambda + 2n_1 - 2 \geq \lambda + 4$ as $(n_1,n) \neq (2,4)$

If $n_2 = 3$ $f^*(v) = 3n_1+3 = (2n_1+6) + (n_1-3) = \lambda - 1$ as $(n_1,n_2) \neq (3,3)$.

$$f^*(v)=\begin{cases}\geq \lambda+6 & \text{if } n_2 \geq 5\\ \geq \lambda+4 & \text{if } n_2=4\\ =\lambda-1 & \text{if } (n_1,n_2)=(2,3)\end{cases}$$

As $n_1 \leq n_2$, $n_2 = \dfrac{n_1^2-3n_1-a}{2}$,a = 0 or 4 or 6 $\Rightarrow n_1 \geq 5$.

Hence in all the cases, $f^*(u_{n1})$, $f^*(u_{n2})$, $f^*(w)$ and $f^*(v)$ are all distinct

$f^*(u) \leq \dfrac{n_1(n_1+1)}{2} + n_2 + 2 = \dfrac{n_1n_1+n_1+2n_2+4}{2} \leq \dfrac{n_2(2n_1+n_2-1)}{2} = f^*(v)$

As $n_1+4 \leq 2n_2$ for all $(n_1,n_2) \neq (3,3)$, we have $n_1+3n_2+4 \leq 5n_2 \leq n_2(n_1+n_2)$

$n_1^2 + n_1+3n_2+4 \leq n_2(2n_1+n_2)$

$n_1^2 + n_1+2n_2+4 \leq n_2(2n_1+n_2-1)]$

if $n_2 \geq 5$, then $5n_2 \leq n_2(n_1+n_2)$ and if $n_2 < 5$, then

$n_2 \neq \dfrac{n_1^2-3n_1-a}{2}$, a = 0,4 or 6 or and $f^*(u) \leq \dfrac{n_1(n_1+1)}{2} + n_2 + 2$.

Thus, in all the cases $f^*(u) < f^*(v)$. Now it is enough to show that $f^*(u) \neq f^*(u_{n_1})$, $f(u_{n_2})$, $f^*(w)$.

Now $\dfrac{n_1(n_1+1)}{2} + 2 = 2n1 + 2n_2 + k = \lambda + k$ $(k = 2, 3, 0)$ $\Leftrightarrow n_1^2 + n_1+2n_2 = 4n_1+ 4n_2+2k$

$$\Leftrightarrow n_2 = \frac{n_1^2 - 3n_1 - 2k}{2}$$

$$\frac{n_1(n_1+1)}{2} + n_2 = \lambda + k \ (k = 2, 3, 0) \ \text{ iff } n_2 = \frac{n_1^2 - 3n_1 - 2k}{2}$$

$$f^*(u) = \begin{cases} = \lambda + 1 \ \text{ if } n_2 = \dfrac{n_1^2 - 3n_1}{2} \\ = \lambda + 4 \ \text{ if } n_2 = \dfrac{n_1^2 - 3n_1 - 4}{2} \\ = \lambda + 5 \ \text{ if } n_2 = \dfrac{n_1^2 - 3n_1 - 6}{2} \\ = \lambda, \lambda + 2, \lambda + 3 \text{ if } n_2 = \dfrac{n_1^2 - 3n_1 - a}{2} \quad a = 0, 4 \text{ or } 6 \end{cases}$$

So $f^*(u) \neq f^*(u_{n_1}), f^*(u_{n_2}), f^*(w)$. in all the cases.

Thus, if $(n_1,n_2) \neq (2,2),(2,4),(3,3)$, f^* is an injective map and so f is an antimagic labeling for $BT(n_1,n_2)$, $\forall$ $(n_1,n_2) \neq (2,2), (3,3), (2,4)$.

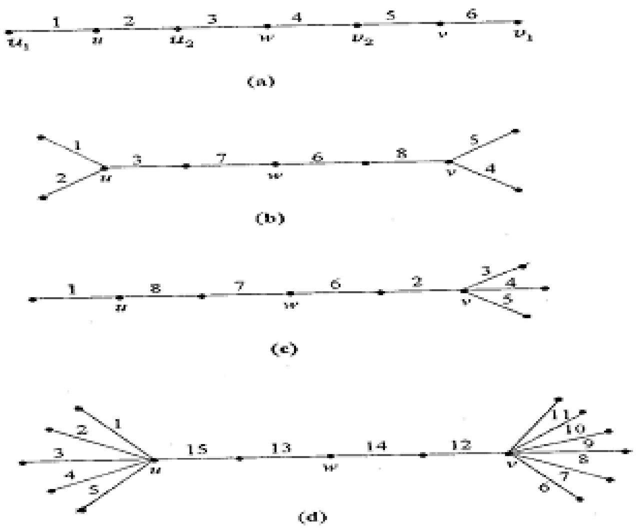

Figure: 2. 9 Antimagic Labeling for (a) BT (2,2), (b) BT(3,3) , (c) BT(2,4) and (d) BT(6,7).

So we assume that $(n_1,n_2) \neq (2,2),(2,4),(3,3)$.

Thorem 2.11: Friendship graph $C_3^{(t)}$ is antimagic for all $t \geq 2$.

Proof:

Let $G = C_3^{(t)}$, $t \geq 2$, and let $V(G) = \{ v,u_i,v_i / 1\leq i \leq t\}$ and $E(G) = \{u_iv_i, vu_i, vv_i / i=1,\ldots,t\}$.

The vertex v is common to the t-triangles $\{v, u_i, v_i\}, 1\leq i \leq t$.

Define $f : E(G) \longrightarrow \{1,2,\ldots,3t\}$ as follows:

For all $i=1,2,\ldots,t$, $f(vv_i) = 3i$; $f(vu_i) = 3i-2$; $f(u_iv_i) = 3i-1$.

Then the induced map f^* an V(G) is obtained as follows :

For all $i=1,2,\ldots,t$, $f^*(u_i) = 6i-3$; $f^*(v_i) = 6i-1$; and $f^*(v) = t(3t+1)$.

As $t \geq 2$, $t(3t+1) \geq 7t$, it follows that f^* is injective and the map f defines an antimagic labeling for $C_3^{(t)}$

Theorem 2.12: Fan graph F_n admits an antimagic labeling, for all positive integers $n \geq 2$.

Proof:

Let $v_1v_2.....v_n$ be the path P_n and the fan graph F_n be obtained from P_n by introducing a new vertex and joining it with every vertex of P_n.

So $V(G) = \{v,v_i / 1\leq i \leq n\}$ and $E(G) = \{vv_i , v_j v_{j+1} / 1 \leq i \leq n ; 1 \leq j \leq n-1\}$.

Define $f: E(G) \longrightarrow \{1,2,\ldots,2n-1\}$ as follows :

$f(v_iv_{i+1}) = i$ for $1 \leq i \leq n-1$, and

$f(vv_i) = 2n-i$ for $1 \leq i \leq n$.

Then the induced map f^* is given by

$f^*(v_i) = 2n+i-1$ for i == 1,2,…,n-1

$f^*(u_n) = 2n-1$

$$f^*(v) = \frac{n(3n-1)}{2}.$$

As $n \geq 2$, $f^*(v) = \frac{n(3n-1)}{2} \geq 3n-1 > f^*(v_i)$, for all $i = 1,2,\ldots,n-1$.

So f induces an antimagic labeling for $F_n (n \geq 2)$.

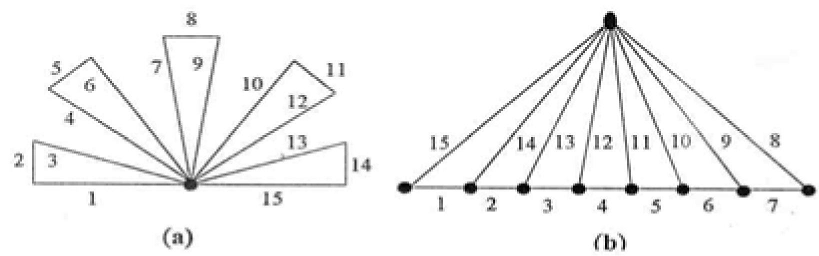

Figure 2.10 antimagic labeling for (a) $C^{(5)}_3$ and (b) F_8

A graph G is said to be a Lantern if it has two adjacent vertices u and v such that all the other vertices of G are adjacent to both u and v and G has no other edges . In fact G = $K_2 + \overline{K_n}$, for some (n ≥ 2).

Theorem 2.13: Lantern G = $K_2 + \overline{K_n}$, (n ≥ 2) , is antimagic .

Proof: Let G = $K_2 + \overline{K_n}$, (n ≥ 2).

Let V(G) = { u, v, u_i / 1≤ i ≤ n} and E(G) = {uv, uu_i, vu_i / i = 1≤ i ≤ n }.

Define f : E(G) ⟶{1,2,…,2n+1 } as follows:

$$f(uu_i) = 2i-1;\ f(vu_i) = 2i;\ \text{for } 1 \le i \le n \text{ and } f(uv) = 2n+1.$$

Then the mapping f^* on V(G) is obtained as follows :

$f^*(u_i) = 4i-1$ for 1≤ i ≤n ; $f^*(u) = n^2$ and $f^*(u) = n^2+ n$.

Clearly f^* is injective if n =2 or 3. If n ≥ 4, then $n^2+ n > n^2 > 4n-1$ and hence f^* is injective in all the cases. Thus, f defines an antimagic for $K_2 + \overline{K_n}$, for n ≥ 2.

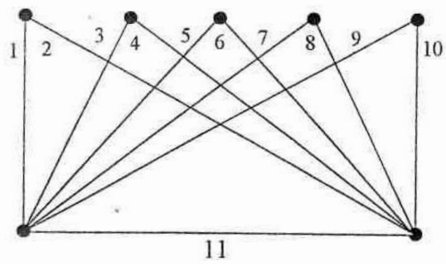

Figure 2.11 : an antimagic labeling for $K_2 + \overline{K_5}$

A triangular snake is obtained from a path P_n, $u_1u_2 \ldots u_n$,(n ≥3), by introducing new vertices $v_1, v_2, \ldots, v_{n-1}$ and joining v_i ,(1≤ i ≤n-1), with the vertices u_i and u_{i+1}.

Theorem 2.14: Every triangular snake is antimagic.

Proof:

Let G be the triangular snake obtained from the path P_n.

Let V(G) = { u_i, u_j / 1≤ i ≤ n; 1≤ j ≤ n -1} and E(G) = {$u_iu_{i+1}, u_iv_i, u_{i+1}u_i$ / i = 1≤ i ≤ n-1 }.

Define f : E(G) ⟶{1,2,…,3n-3 } as follows:

$$\left.\begin{array}{l} f(u_i v_i) = i \\ f(u_{i+1}, v_i) = n + i - 1 \end{array}\right\} \text{for } i = 1,2,\ldots,n-1$$

$$f(u_1u_2) = \begin{cases} 2n & \text{if n is odd} \\ 2n-1 & \text{if n is even} \end{cases}$$

$$f(u_2u_3)=\begin{cases}2n-1 & \text{if } n \text{ is odd}\\ 2n & \text{if } n \text{ is even}\end{cases}$$

$f(u_iu_{i+1}) = 2n+(i-2)$ for $3 \le i \le n-1$.

The induced map f^* is obtained as $f^*(v_i) = n+i$ for $1 \le i \le n-1$.

$$f^*(u_1)=\begin{cases}2n & \text{if } n \text{ is even}\\ 2n+1 & \text{if } n \text{ is odd}\end{cases}$$

$f^*(u_2) = 5n+1$

$$f^*(u_3)=\begin{cases}5n+5 & \text{if } n \text{ is even}\\ 5n+4 & \text{if } n \text{ is odd}\end{cases}$$

$f^*(u_i) = 5n+4i-7$ for $4 \le i \le n-1$.

$f^*(u_n) = 5n-5$.

Clearly, f^* is injective and f is an antimagic labeling for G.

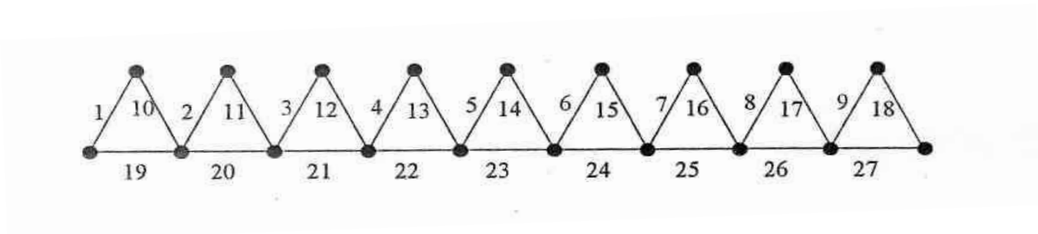

Figure 2.12 : An antimagic labeling for a triangular snake ,(n is even).

Theorem2.15: Let $\{u_i, v_i\, w_iu_i\,;\ 1\le i \le n\}$ be a collection of n disjoint triangles . Let G be the graph obtained by joining w_i to u_{i+1}, $1\le i \le n-1$ and joining u_i to u_{i+1} and v_{i+1}, $1\le i \le n-1$. Then the graph G is antimagic .

Proof: The order and size of G are 3n and 6n-3 respectively.

Assume that $n \ge 3$. Define f: $E(G) \longrightarrow \{1,2,\ldots,6n-3\}$ as follows:

$$f(u_iv_i)=\begin{cases}i & \text{if } i \ne \frac{n+1}{3}, \frac{n+4}{3}\\ i+1 & \text{if } i=\frac{n+1}{3}\\ i-1 & \text{if } i=\frac{n+4}{3}\end{cases}$$

$$\left.\begin{aligned} f(v_i w_i) &= n+i \\ f(u_i, w_i) &= 2n+i \end{aligned}\right\} \text{ for } i = 1,2,\dots,n$$

$$\left.\begin{aligned} f(u_i v_{i+1}) &= 3n+i \\ f(w_i u_{i+1}) &= 4n+i-1 \\ f(u_i u_{i+1}) &= 5n+i-2 \end{aligned}\right\} \text{ for } 1 \le i \le n-1$$

Then the induced map f^* an V(G) is obtained as:

$f^*(v_1) = n+2$; $f^*(v_n) = 7n-1$

$$f^*(v_i) = \begin{cases} 4n+3i-1 & \text{for } 2 \le i \le n-1 \text{ but } i \ne \frac{n+1}{3}, \frac{n+4}{3} \\ 4n+3i & \text{if } i = \frac{n+1}{3} \\ 4n+3i-2 & \text{if } i = \frac{n+4}{3} \end{cases}$$

$f^*(u_1) = 10n+2$; $f^*(u_n) = 15n-5$.

$$f^*(v_i) = \begin{cases} 19n+6i-7 & \text{for } 2 \le i \le n-1 \text{ but } i \ne \frac{n+1}{3}, \frac{n+4}{3} \\ 19n+6i-6 & \text{if } i = \frac{n+1}{3} \\ 19n+6i-8 & \text{if } i = \frac{n+4}{3} \end{cases}$$

$$f^*(w_i) = \begin{cases} 7n+3i-1 & \text{for } 1 \le i \le n-1 \\ 5n & \text{if } i = n \end{cases}$$

The map f is an antimagic labeling for G.

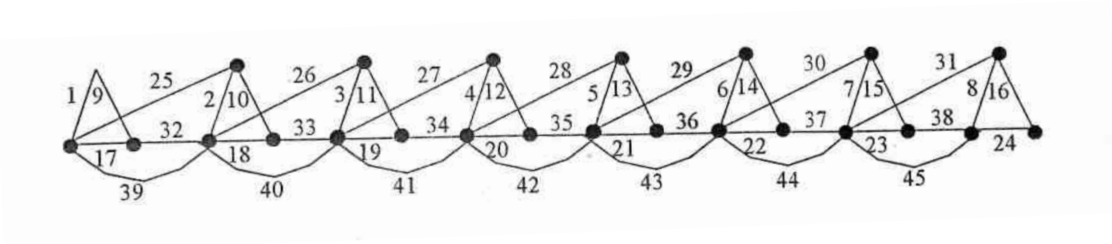

Figure 2.13: An antimagic labelling for G (Theorem 2.15) when n = 8.

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