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Domination and Neighborhood Numbers in Super Duplicate Graph With Respect to Complementation

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Abstract: *For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. Let V′(G) = {v′ : v∈V(G)} be a copy of V(G). The Super duplicate graph withrespect to complementation* D_c**(G) *of G is the graph whose vertex set is V(G)∪V′(G) and edge set is E(G) ∪E(D($\bar{G}$)) where D($\bar{G}$) is the duplicate graph of the complement* $\bar{G}$ *of G. In this paper, bounds for the domination number of* D_c**(G) *are obtained. Also, relationships between the domination number of* D_c**(G) *and other domination parameters are discussed.*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. *Eccentricity* of a vertex u∈V(G) is defined as $e_G(u) = \max \{d_G(u, v): v \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of vertex v in G as e(v) and d(u, v) to denote the distance between two vertices u, v in G respectively. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called a *self-centered* graph with radius r, equivalently G is r-self-centered. A vertex v∈V(G) is a *central vertex* if $e_G(v) = r(G)$ and the center C(G) is the set of all the central vertices. Thus, the center consists of all vertices having minimum eccentricity.

The concept of domination in graphs was introduced by Ore [6]. A set D ⊆ V(G) is said to be a *dominating set* of G, if every vertex in V(G)−D is adjacent to some vertex in D. D is said to be a minimal dominating set if D−{u} is not a dominating set for any u∈D. The *domination number* γ(G) of G is the minimum cardinality of a dominating set. A dominating set D is called an *independent dominating set*, if the induced sub graph <D>

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is independent [1]. D is called a *total dominating set*, if every vertex in V(G) is adjacent to some vertex in D. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set. This concept was introduced in Cockayne *et al* [2]. In the following, some bounds for the domination number $\gamma(D_c^*(G))$ of $D_c^*(G)$ are determined. D is a *global dominating set*, if it is a dominating set of both G and its complement $\overline{G}$. The global domination number γ_g of G is defined similarly [8]. A total dominating set T of G is a *total global dominating set* (t.g.d. set), if T is also a total dominating set of $\overline{G}$. The total global dominating number $\gamma_{tg}(G)$ of G is the minimum cardinality of a t.g.d. set [5]. $\gamma_t(G)$ is defined for G with $\delta(G) \geq 1$ and $\gamma_{tg}(G)$ is only defined for G with $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$, where $\delta(G)$ is the minimum degree of G. A γ-set is a minimum dominating set. Similarly, a γ_g-set, γ_t-set and γ_{tg}-set are defined.

For $v \in V(G)$, the *neighborhood* N(v) of v is the set of all vertices adjacent to v in G. $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v. A subset S of V(G) is a *neighborhood set* (n-set) of G, if $G = \cup_{v\in S} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by N[v]. The *neighborhood number* $n_0(G)$ of G is the minimum cardinality of an n-set of G [9].

For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. Let $V'(G) = \{v' : v \in V(G)\}$ be a copy of V(G). Then the *Duplicate graph* D(G) of G is the graph whose vertex set is $V(G) \cup V'(G)$ and edge set is $\{u'v \text{ and } uv' : uv \in E(G)\}$. This graph was first studied by Sampathkumar [7]. The *Super duplicate graph with respect to complementation* $D_c^*(G)$ of G is the graph whose vertex set is as that of D(G) and edge set is $E(D(\overline{G})) \cup E(G)$, where D(G) and $\overline{G}$ are respectively, the duplicate graph and the complement of G. In this paper, bounds for the domination number of $D_c^*(G)$ are obtained. Also, relationships between the domination number of $D_c^*(G)$ and other domination parameters are discussed.

2.Prior Results

In this section, we list results with indicated references, which will be used in the subsequent main results.

Observation 2.1[4]:

(b). $V(D_c^*(G))$ can be partitioned into two sets $V(G)$ and $V'(G)$ such that the sub graph of $D_c^*(G)$ induced by the vertices in $V(G)$ is G and that of $D_c^*(G)$ induced by the vertices in $V'(G)$ is totally disconnected.

(c). For any vertex $v \in V(G)$, there are two vertices v and v' in $D_c^*(G)$, such that $deg_{Dc^*(G)}(v) = p - 1$ and $deg_{Dc^*(G)}(v') = p - 1 - deg_G(v)$ and hence $\Delta(D_c^*(G)) = p - 1$ and $\delta(D_c^*(G)) = p - 1 - \Delta(G)$.

(f). $D_c^*(G)$ is disconnected if and only if either $r(G) = 1$ or G is disconnected with exactly two complete components not adjacent to u.

Theorem 2.1[5]: Let G be a graph with $diam(G) \geq 5$. Then $T \subseteq V$ is a total dominating set of G if and only if T is a total global dominating set.

Theorem 2.2[5]: Let G be a graph such that neither G nor $\overline{G}$ have an isolated vertex and D be a γ_g-set of G with $\delta(\langle D \rangle) \geq 1$. If $diam(G) \geq 5$, then $\gamma_t(G) = \gamma_{tg}(G)$ and $\gamma_g(G) = \gamma_{tg}(G)$.

Theorem 2.3 [5]: Let diam(G) = k.

(i). If k = 4, then $\gamma_{tg}(G) \leq \gamma_t(G) + 1$.

(ii). If k = 3, then $\gamma_{tg}(G) \leq \gamma_t(G) + 2$.

Theorem 2.4 [5]: Let D be a γ_g-set of G such that <D> has no isolated vertices and diam(G)=k.

(i). If k = 4, then $\gamma_{tg}(G) \leq \gamma_g(G) + 1$.

(ii). If k = 3, then $\gamma_{tg}(G) \leq \gamma_g(G) + 2$.

Theorem 2.5 [9]: For a graph G of order p, $n_0(G) = 1$ if and only if G has a vertex of degree p−1.

2. Main Results

In the following, we give bounds for the domination number of $D_c^*(G)$.

Theorem 3.1: For any graph G with $\delta(G) = 1$, $\gamma(D_c^*(G)) \leq 3$.

Proof: Let $v \in V(G)$ be such that $deg_G(v) = 1$ and u be adjacent to v in G. Then $D = \{v, v', u'\} \subseteq V(D_c^*(G))$ is a dominating set of $D_c^*(G)$ and hence $\gamma(D_c^*(G)) \leq 3$.

This bound is sharp, since $\gamma(D_c^*(G)) = 3$, if either G is a path on at least 5 vertices or G is a star on at least 4 vertices.

Theorem 3.2: If G is any connected graph with diameter at least three, then $\gamma(D_c^*(G)) \leq 3$.

Proof: Since diameter of G is 3, there exists at least one vertex v in V(G) such that e(v) = d, where d is the diameter of G. Let u be an eccentric point of v.

Then $D = \{v, u, u'\} \subseteq V(D_c^*(G))$ is a dominating set of $D_c^*(G)$ and hence $\gamma(D_c^*(G)) \leq 3$.

This bound is sharp, for $G \cong C_n$ with $n \geq 6$. That is, $\gamma(D_c^*(G)) = 3$.

Observation:

3.1: $2 \leq \gamma(D_c^*(G)) \leq p$.

3.2: If $\delta(G) = 0$, then $\gamma(D_c^*(G)) = 2$.

3.3: If there exists a γ-set of G having two vertices at distance at least 3, then $\gamma(D_c^*(G))=2$.

3.4: If $\delta(G) = 1$ and $r(G) = 1$, then $\gamma(D_c^*(G)) = 3$.

3.5: Let G be a self-centered graph with radius 2 and triangle-free. If G has a dominating edge, then $\gamma(D_c^*(G)) = 2$.

Theorem 3.3: Let G be a graph with $r(G) = 1$ and $\delta(G) \geq 2$. If there exists two nonadjacent vertices u, v in G with $N(u) \cap N(v) = \{w \in V(G) : e(w) = 1\}$, then $\gamma(D_c^*(G)) \leq k + 2$, where k is the number of central vertices in G.

Proof: Assume G is a graph with $r(G) = 1$ and $\delta(G) \geq 2$. Let C(G) be the center of G and $|C(G)| = k$ and let $u, v \in V(G)$ be such that $N(u) \cap N(v) = \{w \in V(G) : e(w) =1\}$. Then $D = \{u, v\} \cup \{w' : w \in C(G)\}$ is a dominating set of $D_c^*(G)$ and hence, $\gamma(D_c^*(G)) \leq k + 2$.

This bound is attained, if G is a star.

The next theorem relates $\gamma(D_c^*(G))$ with the domination number of $\overline{G}$, if $r(G) = 1$ and $\delta(G) \geq 2$

Theorem 3.4: If $r(G) = 1$ and $\delta(G) \geq 2$, then $\gamma(D_c^*(G)) \leq 2\gamma(\overline{G})$.

Proof: Let D be a γ-set of $\overline{G}$. If $r(\langle D \rangle_G) \geq 2$, then D is also a dominating set of $D_c^*(G)$. In this case, $\gamma(D_c^*(G)) \leq \gamma(\overline{G})$. If $r(\langle D \rangle_G)$ is equal to one, then $D' = D \cup D''$ is a dominating set of $D_c^*(G)$, where $D'' = \{v' : e(v) = 1 \text{ in } \langle D \rangle_G\}$. If $\langle D \rangle_G$ is complete, then D' contains $2\gamma(\overline{G})$ vertices and hence $\gamma(D_c^*(G)) \leq 2\gamma(\overline{G})$.

Now, we give an upper bound of $\gamma(D_c^*(G))$ in terms of the minimum degree of G.

Theorem 3.5: For any graph G, $\gamma(D_c^*(G)) \leq \delta(G) + 2$, where $\delta(G)$ is the minimum degree of G.

Proof: Let $v \in V(G)$ be such that $deg_G(v) = \delta(G)$ and let $v_1, v_2,..,v_{\delta(G)}$ be the vertices in G adjacent to v. Then $D = \{v, v', v_1', v_2', \ldots v_{\delta(G)}'\} \subseteq V(D_c^*(G))$ is a dominating set of $D_c^*(G)$ and hence $\gamma(D_c^*(G)) \leq \delta(G) + 2$.

The complete graph attains this upper bound.

The next theorem relates $\gamma(D_c^*(G))$ with global domination number γ_g of G.

Theorem 3.6: For any graph G, $\gamma(D_c^*(G)) \leq \gamma_g(G)$ if and only if there exists a γ_g-set D of G with $r(\langle D \rangle_G) \geq 2$.

Proof: Let D be a γ_g-set of G with $r(\langle D \rangle_G) \geq 2$. Since D is a dominating set of $\overline{G}$ and $r(\langle D \rangle_G) \geq 2$, D dominates v' in $V(D_c^*(G))$ for all $v \in V(G)$. Also since G is an induced sub graph of $D_c^*(G)$, D is also a dominating set of $D_c^*(G)$ and hence $\gamma(D_c^*(G)) \leq \gamma_g(G)$. Conversely, assume any γ_g-set D of G is also a dominating set of $D_c^*(G)$ and $r(\langle D \rangle_G) = 1$. Let u be a vertex in $\langle D \rangle_G$ with $e(u) = 1$ in $\langle D \rangle_G$. Then D does not dominate u' in $V(D_c^*(G))$. This is a contradiction. Hence, $r(\langle D \rangle) \geq 2$.

Remark 3.1: If $r(\langle D \rangle_G) = 1$, then $D' = D \cup \{v' : e_{\langle D \rangle}(v) = 1\}$ is a dominating set of $D_c^*(G)$.

Corollary 3.6.1: Any total global dominating set of G is also a dominating set of $D_c^*(G)$ and hence $\gamma(D_c^*(G)) \leq \gamma_{tg}(G)$, where $\gamma_{tg}(G)$ is the total global domination number of G and this bound is sharp. Sharpness is realized by the cycle of length 4.

Theorem 3.7: For a graph G, any independent dominating set D of G is also an independent dominating set of $D_c^*(G)$ if and only if D is a dominating set of $\overline{G}$.

Proof: Assume D is an independent dominating set of G and is a dominating set of $\overline{G}$. Since $V(D_c^*(G)) = V(G) \cup V'(G)$, where $V'(G)$ is a copy of $V(G)$, it is enough to prove D dominates all the vertices in $V'(G)$. Since D is a dominating set of $\overline{G}$, each vertex in $V(G)-D$ is not adjacent to at least one vertex in D. Hence, D dominates v' in $V(D_c^*(G))$ for all $v \in V(G)-D$. Also since D is an independent set in G, D dominates v' in $V(D_c^*(G))$ for all $v \in D$. Thus, D dominates $V'(G)$ and is a dominating set of $D_c^*(G)$. Conversely, if D is not a dominating set of $\overline{G}$, then there exists a vertex v in $V(G)-D$ adjacent to all the vertices in D. Then the corresponding vertex v' in $V(D_c^*(G))$ is not adjacent to any of the vertices in D. This is a contradiction. Thus, D is a dominating set of $\overline{G}$.

Note 3.1: Theorem 3.7., implies that $\gamma_i(D_c^*(G)) \leq \gamma_i(G)$ if and only if $\gamma(\overline{G}) \leq \gamma_i(G)$ and the equality holds, if G is a path on four vertices.

Kulli and Janakiram[5] defined the total global domination number γ_{tg} of a graph and obtained several results. Based on these, the following theorems are stated without proof. Here, the graphs G for which both $\delta(D_c^*(G)) \geq 1$ and $\delta(\overline{D_c^*(G)}) \geq 1$ are considered.

Theorem 3.8: Let G be a graph such that neither G nor $\overline{G}$ has an isolated vertex and D be a γ_g-set of G with $\delta(\langle D \rangle) \geq 1$. If $diam(G) \geq 5$, then $\gamma(D_c^*(G)) \leq \gamma_t(G)$ and $\gamma(D_c^*(G)) \leq \gamma_g(G)$, where $\gamma_t(G)$ is the total domination number of G.

Theorem 3.9:

(i). If $diam(G) = 4$, then $\gamma(D_c^*(G)) \leq \gamma_t(G) + 1$

(ii). If $diam(G) = 3$, then $\gamma(D_c^*(G)) \leq \gamma_t(G) + 2$.

Theorem 3.10: Let D be a γ_g-set of G such that <D> has isolated vertices.

(i). If $diam(G) = 4$, then $\gamma(D_c^*(G)) \leq \gamma_g(G) + 1$.

(ii). If $diam(G) = 3$, then $\gamma(D_c^*(G)) \leq \gamma_g(G) + 2$.

Example 3.1:

1.For any complete bi-partite graph $K_{m,n}$ with $2 \leq m \leq n$, $\gamma(D_c^*(K_{m,n})) = 4$.

2.For any cycle C_n with $n \geq 3$ vertices, $\gamma(D_c^*(C_n)) = 4$, if $n = 3, 4$;
$= 2$, if $n = 6$; and
$= 3$, if $n \geq 5, n \neq 6$.

3. For any path P_n with $n \geq 3$ vertices, $\gamma(D_c^*(P_n))$ = 3, if n = 3, $n \geq 7$; and
= 2, if n = 4, 5, 6.

In the following, the bounds for neighborhood number $n_0(D_c^*(G))$ of $D_c^*(G)$ are determined.

Theorem 3.11: For any graph G having p vertices, $2 \leq n_0(D_c^*(G)) \leq p$.

Proof: Since $D_c^*(G)$ has no vertex of degree 2p-1, $n_0(D_c^*(G)) \geq 2$. Further, V(G) is an n-set for $D_c^*(G)$ and hence $n_0(D_c^*(G)) \leq p$.

The lower bound is attained, if $G \cong K_m \cup K_n$, for m, $n \geq 2$ and the upper bound is attained, if $G \cong C_4$.

Next, a necessary and sufficient condition for an n-set of G to be an n-set of $D_c^*(G)$ is found.

Theorem 3.12: Let G be any graph with $r(G) \geq 2$. Then $n_0(D_c^*(G)) \leq n_0(G)$ if and only if there exists an n-set D of G with $|D| = n_0(G)$ satisfying,

1. For each pair of non–adjacent vertices u, v in V(G)-D, $N_G(u) \cap N_{\bar{G}}(v) \subseteq D$ and $N_{\bar{G}}(u) \cap N_G(v) \subseteq D$; and

(2). For each pair of nonadjacent vertices u, v in G with $u \in D$, $v \in V(G) - D$, $N_{\bar{G}}(u) \cap N_G(v) \subseteq D$.

Proof: Assume conditions (i) and (ii). Let D be an n-set of G with $|D| = n_0(G)$.

(a). Since G is an induced sub graph of $D_c^*(G)$, each edge xy in $D_c^*(G)$ with x, $y \in V(G)$ belongs to $\cup_{w \in D}(E<N[w]>)$.

(b). Let $xy \notin E(G)$. Then xy', $x'y \in E(D_c^*(G))$.

(i). If both x, $y \in D$, then xy', $x'y \in \cup_{w \in D}(E<N[w]>)$.

(ii). If both x, $y \in V(G) - D$, then by condition (1) there exists a vertex w in D such that $wx \in E(G)$ and $wy \notin E(G)$ and hence both wx, $wy' \in E(D_c^*(G))$ and $xy' \in E(<N[w]>)$. Similarly, $x'y$ belongs to $E(<N[w]>)$.

(iii). If $x \in V(G) - D$ and $y \in D$, then by condition (2) there exists a vertex z in D such that $zx \in E(G)$ and $zy \notin E(G)$ and as in (ii), both $x'y$ and xy' belongs to $\cup_{w \in D}(E<N[w]>)$. Hence, $D_c^*(G) = \cup_{w \in D}(E<N[w]>)$ and D is an n-set for $D_c^*(G)$. Conversely, assume $n_0(Dc^*(G)) \leq n_0(G)$ and D is an n-set for both G and $D_c^*(G)$. If condition (1) is not true, corresponding to two non-adjacent vertices x, y in $V(G) - D$, the edge $x'y$ in $D_c^*(G)$ does not belong to $E(<N[w]>)$ for every w in D. This is a contradiction. If condition (2) is not true, corresponding to the non-adjacent vertices x, y in G, where $x \in D$, $y \in V(G) - D$, the edge $xy' \notin E(<N[w]>)$ for every w in D. This is a contradiction to the fact that D is an n-set for $D_c^*(G)$.

Corollary 3.12.1: Let v be a vertex of minimum degree in G and $N(v) = \{v_1, v_2,..,v_{\delta(G)}\}$. Then $D = N[v] \cup \{v', v_1', \ldots, v_{\delta(G)}'\}$ is an n-set for $D_c^*(G)$ if and only if for each pair of non-adjacent vertices u, v in V(G)-D, $N_G(u) \cap N_{\bar{G}}(v) \subseteq D$ and $N_{\bar{G}}(u) \cap N_G(v) \subseteq D$.

Corollary 3.12.2: Let v be a vertex of minimum degree in G such that $<V(G) - N[v]> \cong K_n$, where $n \geq 3$. If there exists a vertex u in V(G) with $N(u) \cap N[v] = \Phi$, then $N[v] \cup \{u\}$ is an n-set for $D_c^*(G)$.

Example 3.2:

1. $n_0(D_c^*(C_n)) = \{n/2\}$, if $n \geq 4$.
2. $n_0(D_c^*(P_n)) = \{n/2\}$, if $n \geq 4$.
3. $n_0(D_c^*(K_{m,n})) = m + n$, if $m, n \geq 2$.

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