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Structural properties of k-Distance closed domination critical graphs for k = 5 and 6

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Abstract: *In a graph G = (V, E), a set $S \subset V(G)$ is a distance closed set of G if for each vertex $u \in S$ and for each $w \in V - S$, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$. Also, S is said to be a distance closed dominating set of G if (i) <S> is distance closed and (ii) S is a dominating set. The critical concept in graphs plays an important role in the study of structural properties of graphs and hence it will be useful to study any communication model. The critical concept of distance closed domination which deals with those graphs that are critical in the sense that their distance closed domination number drops when any missing edge is added. The structural properties of k-distance closed domination critical graphs for k ≤ 4 are studied in [13]. In this paper, we analyze the structural properties of k- distance closed domination critical graphs for k = 5 and 6.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self-centered graph, neighborhood, induced sub graph, distance closed dominating set, distance closed domination critical graph.*

1. Introduction

Graphs discussed in this paper are connected and simple graphs only. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $deg_G(v)$ and minimum degree and maximum degree of G are indicated by $\delta(G)$ and $\Delta(G)$ respectively. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max \{d_G(u, v): \forall u \in V(G)\}$ and the eccentric set $E_G(v) = \{u \in V(G): d(v, u) = e_G(v)\}$. If there is no confusion, we simply use the notation deg (v), d(u, v), e(v) and E(v) to denote degree, distance, eccentricity and eccentric set respectively for a connected graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted by r(G) and d(G) respectively. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r self-centered* graph. Such graphs are 2-connected graphs. A vertex u is said to be an eccentric vertex of v in a graph G, if d(u, v) = e(v) in

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that graph. For $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood* of v.

One important aspect of the concept of distance and eccentricity is the existence of polynomial time algorithm to analyze them. The concept of distance and related properties are studied in [3], [6] and [15] and the structural properties of radius and diameter critical graphs are studied in [7] and [16] respectively. The new concepts such as ideal sets, distance preserving sub graphs, eccentricity preserving sub graphs, super eccentric graph of a graph, pseudo geodetic graphs are introduced and structural properties of those graphs are studied in [11].

The concept of domination in graphs originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters. A set $D \subseteq V(G)$ is called a dominating set of G if every vertex in V(G) – D is adjacent to some vertex in D and D is said to be a minimal dominating set if D – {v} is not a dominating set for any $v \in D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on dominating sets. A dominating set D is called a *connected* dominating set if the induced sub graph <D> is connected. The list of survey of domination and connected domination papers are in [4], [5], [8], [9], [14] and [17]. Janakiraman and Alphonse [2] introduced and studied the concept of weak convex dominating sets, which mixes the concept of dominating set and distance preserving set. Using these, structural properties of various dominating parameters are studied.

Graphs which are critical with respect to a given property frequently play an important role in the investigation of that property. Not only are such graphs of considerable interest in their own right, but also a knowledge of their structure often aids in the development of general theory. In particular, when investigating any finite structure, a great number of results are proven by induction. Consequently, it is desirable to learn as much as possible about those graphs that are critical with respect to a given property so as to aid and abet such investigation. A graph G is said to be domination critical if for every edge $e \notin E(G)$, $\gamma(G + e) < \gamma(G)$. If G is a domination critical graph with $\gamma(G) = k$, we will say G is k-domination critical or just k-critical. The 1-critical graphs are K_n, for $n \geq 1$. The concept of domination critical graphs and their structural properties are studied in [1], [10] and [18].

In this paper, we studied the distance closed domination critical graphs through which the structural properties of those graphs are studied. Since this concept deals the reduction in the cardinality of distance closed dominating set for any addition of one new link in the original structure, it will be useful to study the communication model, which reduces it dominating parameters by simple addition of a link, which doesn't exist in the system. Hence this critical concept can be directly applied to the construction of a fault tolerant communication model.

2. Prior Results

The concept of ideal set is defined and studied in the doctoral thesis of Janakiraman [11] and the concept of ideal sets in graph theory is due to the related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to the multiplicative closure property with the elements of that ring. Similarly, the ideal set in a graph is defined with respect to the distance property between the ideal set and the vertices of the graph. Thus, the ideal set of a graph G is defined as follows:

Let I be a vertex subset of G. Then I is said to be an *ideal set* of G if

(i) For each vertex $u \in I$ and for each $w \in V - I$, there exists at least one vertex $v \in I$ such that $d_{<I>}(u, v) = d_G(u, w)$.

(ii) I is the minimal set satisfying (i).

Also, a graph G is said to be a 0-ideal graph if it has no non-trivial ideal set other than G. The ideal set without the minimality condition is taken as a distance closed set in the present work. Thus, the distance closed dominating set of a graph G is defined as follows:

A subset $S \subseteq V(G)$ is said to be a *distance closed dominating* (*D.C.D*) *set*, if

(i) <S> is distance closed;

(ii) S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the *distance closed domination number* of G and is denoted by γ_{dcl}.

Clearly from the definition, $1 \le \gamma_{dcl} \le p$ and a graph with $\gamma_{dcl} = p$ is called a 0-distance closed dominating graph. If S is a D.C.D set of a graph G, then <S> need not be connected and also the complement V – S need not be a D.C.D set of G. The definition and the extensive study of the above said distance closed domination in graphs are studied in [12].

A graph G is said to be a *distance closed domination critical* if for every edge $e \notin E(G)$, $\gamma_{dcl}(G + e) < \gamma_{dcl}(G)$. If G is a D.C.D critical graph with $\gamma_{dcl}(G) = k$, then G is said to be k-D.C.D critical. There is no 2-D.C.D critical graph. The structural properties of k-distance closed domination critical graphs for k = 3 and 4 are studied in [13] and following are some of the results given in [13].

Theorem 2.1 [13]: A graph G is 3-D.C.D critical if and only if

(i) G is connected.

(ii) G has $\gamma_{dcl}(G) = 3$.

(iii) G has exactly one vertex with eccentricity equal to 1.

(iv) For every pair of non adjacent vertices at least one of them is of degree p – 2.

Theorem 2.2 [13]: Any 3-D.C.D critical graph has diameter equal to two.

Theorem 2.3 [13]: Any 3-D.C.D critical graph with $\delta \geq 2$ is a block.

Theorem 2.4 [13]: A graph G is 4-D.C.D critical if and only if

(i) G is connected.

(ii) G has $\gamma_{dcl}(G) = 4$.

(iii) For any two non adjacent vertices at least one of them is of degree p – 2.

Theorem 2.5 [13]: Any 4-D.C.D critical graph is self centered of diameter 2.

Theorem 2.6 [13]: Any 4-D.C.D critical graph is a block.

3. Main Results

Continuing the above, we studied the structural properties of k-distance closed domination critical graphs for k = 5 and 6. There are many structures possible for a k-distance closed domination critical graph for $k \geq 5$. Here, we analyzed the type (I) and type (II) structures of 5-D.C.D and 6-D.C.D critical graphs respectively.

3.1 5-D.C.D critical graphs:

If G is a 5-D.C.D critical graph, then addition of an edge will reduce the D.C.D number of G by exactly 1. Also, if D is a minimum D.C.D set of a 5-D.C.D critical graph, then <D> need not be a path (it may contain cycle also). Thus, there are many structures possible for a 5-D.C.D critical graph depends on the structure of the induced sub graph of the minimum D.C.D set. For example, the structural design of 5-D.C.D critical graphs, in which the induced sub graph of every minimum D.C.D set is a path (type (I) structure), is given below in Figure 3.1 and hereafter this type of structure is represented by 5-D.C.D critical graphs with type (I) structure or simply 5-D.C.D type (I) critical graphs.

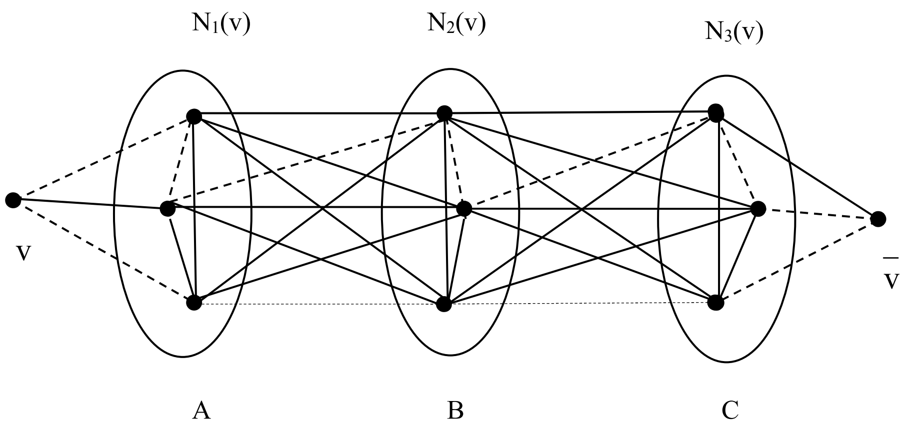

Figure 3.1 - ***Structural design of 5-D.C.D type (I) critical graphs***

The following theorems are based on the above 5-D.C.D type (I) critical graphs.

Theorem 3.1.1: If G is a 5-D.C.D type (I) critical graph, then G has a unique pair of peripheral nodes.

Proof: Let G be a 5-D.C.D type (I) critical graph and let D be a minimum D.C.D set of G. Then <D> is the diametral path on 5 vertices and it must contain exactly the two peripheral nodes, say v and $\bar{v}$. Suppose if G has a third peripheral node w, then it must be adjacent to either v or $\bar{v}$ (Otherwise addition of an edge between v and w (or) $\bar{v}$ and w will not affect the D.C.D number of G). Now, addition of an edge between w to any of the central vertex of G will not reduce the D.C.D number of G. Hence, G has a unique pair of peripheral nodes and every D.C.D set of G must contain this pair of peripheral nodes.

Lemma 3.1.1: If G is a 5-D.C.D type (I) critical graph, then diameter of G is equal to 4.

Proof: If D is a minimum D.C.D set of a 5-D.C.D type (I) critical graph G, then D must contain the unique pair of peripheral nodes of G and the induced sub graph of D, <D> is the diametral path on 5 vertices. Since, it is the distance preserving sub graph of G having diameter 4, G must have diameter equal to 4.

Corollary 3.1.1: If G is a 5-D.C.D type (I) critical graph, then radius of G is equal to 2.

Lemma 3.1.2: If v is a vertex of a 5-D.C.D type (I) critical graph with e(v) = 2, then d(v) = p – 3.

Proof: Let G be a 5-D.C.D type (I) critical graph and let v be vertex with e(v) = 2. Then, v must be adjacent to all the vertices u of eccentricity 2 and 3, otherwise addition of an edge between u and v will not affect the D.C.D number of G. Hence, the only possibility is that v is non adjacent to the unique pair of peripheral nodes of G and hence d(v) = p – 3.

Corollary 3.1.2: If G is a 5-D.C.D type (I) critical graph, then $\delta(G) \geq 1$ and $\Delta(G) = p - 3$.

Note 3.1.1: C_5 is the smallest 5-D.C.D type (I) critical graph having the above bound.

Theorem 3.1.2: Let G be a 5-D.C.D type (I) critical graph. Then for every pair of non adjacent vertices of G, there exists a minimum D.C.D set that contains both.

Proof: Let G be a 5-D.C.D type (I) critical graph and let D = {x, u, v, w, y} be a minimum D.C.D set of G. Then, <D> is a path and x, y are the unique pair of peripheral nodes of G. Therefore, any vertex $z \in V - D$ implies that e(z) = 2 or e(z) = 3.

Claim: ***There exists a minimum D.C.D set different from D, containing the vertex z and the vertices non adjacent to z***

Case (i): ***e*(*z*) = 2**

If e(z) = 2, then d(z) = p – 3 and is adjacent to all the vertices of D except the two peripheral nodes x and y, i.e., z is adjacent to all the vertices of V(G) – {x, y}. Also, there exists a set {x, u, z, w, y}, which is also a minimum D.C.D set of G that contains both the pair of non adjacent vertices (z, x) and (z, y).

Case (ii): ***e*(*z*) = 3**

If e(z) = 3, then z must be adjacent to v and x (as z is of eccentricity 3 it must be adjacent to any one of the peripheral node, say x). Also, there exists a minimum D.C.D set {x, z, v, w, y} that contains the pair of non adjacent vertices (z, w) and (z, y).

Hence, the theorem.

Corollary 3.1.3: If G is a 5-D.C.D type (I) critical graph, then every vertex must be in at least one minimum D.C.D set of G.

Theorem 3.1.3: If G is a 5-D.C.D type (I) critical graph, then for every peripheral vertex v of G, $<N_i(v)>$ is a clique for i = 1 to 4 and also $< N_i(v) \cup N_{i+1}(v) >$ is a clique for i = 1 to 3.

Proof: Let G be a 5-D.C.D type (I) critical graph and let D be a minimum D.C.D set of G. Then G has a unique pair of peripheral nodes (say $v, \bar{v}$) and D must contain these peripheral nodes. Also, the eccentric node of each vertex in G is either v or $\bar{v}$. Therefore, each $N_i(v)$ for i = 1 to 4 has vertices with same eccentricity and D must contain exactly one vertex from each $N_i(v)$ for i = 1 to 4, as <D> is a path.

Suppose that, if x and y are any two non adjacent vertices of $N_i(v)$ for i = 1 to 4, then e(x) = e(y) and also it means that x and y are in different D.C.D sets of G, a contradiction. Hence every pair of vertices in $N_i(v)$, for i = 1 to 4 are adjacent and hence each $N_i(v)$, for i = 1 to 4 is a clique. Also, every vertex with eccentricity equal to 2 must be adjacent to all the (p – 3) vertices (except v and $\bar{v}$) of G and a vertex with eccentricity equal to 3 must be adjacent to all the vertices with eccentricity equal to 2. Hence for a peripheral vertex v of G, $<N_i(v) \cup N_{i+1}(v)>$ is a clique, for i = 1 to 3.

Proposition 3.1.1: If G is a 5-D.C.D type (I) critical graph, then G can have at most 2 pendant vertices.

Proof: Let G be a 5-D.C.D type (I) critical graph. Then every vertex u in G belongs to at least one minimum D.C.D set D of G. Also, <D> is the diametral path and it contains exactly the two peripheral vertices $\{v, \bar{v}\}$ of G. Therefore, d(u) ≥ 2, for every vertex u in

$V(G) - \{v, \overline{v}\}$. Hence, the vertices $\{v, \overline{v}\}$ can be the pendant vertices of G and hence G can have at most 2 pendant vertices.

Theorem 3.1.4: If G is a 5-D.C.D type (I) critical graph, then G can have at most 3 cut vertices.

Proof: If u is a cut vertex of a 5-D.C.D type (I) critical graph G, then u must be in every D.C.D set $\{v, u_1, u_2, u_3, \overline{v}\}$ of G, where $u_1 \in A$, $u_2 \in B$ and $u_3 \in C$ and also u cannot be a peripheral node of G. Hence, u can be any one of the vertex in the set $\{u_1, u_2, u_3\}$ and hence G can have at most 3 cut vertices.

Theorem 3.1.5: If u is a cut vertex of a 5-D.C.D type (I) critical graph, then

(i) G – u can have at most two components and

(ii) One of the components is a clique.

Proof: Let u be a cut vertex of a 5-D.C.D type (I) critical graph G. Then, u must be in every D.C.D set $\{v, u_1, u_2, u_3, \overline{v}\}$ of G, where $u_1 \in A$, $u_2 \in B$ and $u_3 \in C$. Also, u can be any one of the vertex in $\{u_1, u_2, u_3\}$.

Case (i): $u = u_1$

If $u = u_1$, then $e(u) = 3$ and u is the only vertex in A. Also, G – u has two components C_1 and C_2 such that $C_1 = \{v\}$ and $C_2 = B \cup C \cup \{\overline{v}\}$. Hence, in this case G – u has two components and one of them is a clique.

Case (ii): $u = u_2$

If $u = u_2$, then $e(u) = 2$ and u is the only vertex in B. Also, G – u has two components C_1 and C_2 such that $C_1 = \{v\} \cup A$ and $C_2 = C \cup \{\overline{v}\}$. Hence, in this case G – u has two components and both of them are cliques.

Case (iii): $u = u_3$

If $u = u_3$, then $e(u) = 3$ and u is the only vertex in C. Also, G – u has two components C_1 and C_2 such that $C_1 = \{v\} \cup A \cup B$ and $C_2 = \{\overline{v}\}$. Hence, in this case G – u has two components and one of them is a clique.

Hence from all the three cases, we have the theorem.

Proposition 3.1.2: If G is a 5-D.C.D type (I) critical graph G, then we have the following:

(i) If u is a unique central vertex (cut vertex with $e(u) = 2$) of G, then $|N_2(u)| = 2$ and $\langle N_2(u) \rangle$ is independent.

(ii) If u is cut vertex of G with $e(u) = 3$, then $|N_3(u)| = 1$.

Proof: Let G be a 5-D.C.D type (I) critical graph.

(i) If u is a unique central vertex of G, then $d(u) = p - 3$ and u is non adjacent to only the unique pair of peripheral nodes $(v, \overline{v})$ of G. Hence, $|N_2(u)| = 2$ and $\langle N_2(u) \rangle$ is independent (as v and $\overline{v}$ are non adjacent).

(ii) If u is a cut vertex with e(u) = 3, then u must be in either A or C. If $u \in A$ then $N_3(u) = \{\bar{v}\}$ and if $u \in C$ then $N_3(u) = \{v\}$. Hence in both the cases, $|N_3(u)| = 1$.

Proposition 3.1.3: Any 5-D.C.D type (I) critical graph G, which is also a block, is Hamiltonian.

Proof: For any vertex v in G, $\langle N_i(v) \rangle$, i = 1 to 4 is a clique and also $|A|$, $|B|$ and $|C| \geq 2$ (as G is a block). Hence, we can have a cycle that covers all the vertices of G (refer Figure 3.1) and hence G is Hamiltonian.

Proposition 3.1.4: Every 5-D.C.D type (I) critical graph with a unique central vertex has an eccentricity preserving spanning tree.

Proof: Let G be a 5-D.C.D type (I) critical graph and let $D = \{v, u_1, u_2, u_3, \bar{v}\}$, where $u_1 \in A$, $u_2 \in B$ and $u_3 \in C$ be a minimum D.C.D set of G. Then, $\langle D \rangle$ is a path in which u_2 is the unique central vertex and $(v, \bar{v})$ is the unique pair of peripheral nodes of G. Also, $d(u_2) = p - 3$ and u_2 is adjacent to all the vertices of V(G) – D. Thus, the induced sub graph of D together with the edges from u_2 to the vertices of V(G) – D will form an eccentricity preserving spanning tree of G.

Theorem 3.1.6: Any 5-D.C.D type (I) critical graph is diameter edge (addition) critical.

Proof: Let G be a 5-D.C.D type (I) critical graph. Then G is of diameter 4 and radius 2. In G, for every pair of non adjacent vertices (u, v), there exists a minimum D.C.D set D that contains both (u, v) and also $\langle D \rangle$ is the diametral path of G. Hence, addition of an edge between any pair of non adjacent vertices in G will reduce the diameter of G to 3 or 2 and hence G is diameter edge (addition) critical.

Proposition 3.1.5: There exists no graph G for which both G and $\bar{G}$ are 5-D.C.D type (I) critical.

Proof: Without loss of generality, assume that G is a 5-D.C.D type (I) critical graph. Then, the diameter of G is 4 and $\bar{G}$ contains a dominating edge. Hence, $\gamma_{dcl}(\bar{G}) = 4$ and hence $\bar{G}$ cannot be 5-D.C.D type (I) critical.

3.2 6-D.C.D critical graphs:

If G is a 6-D.C.D critical graph, then addition of an edge will reduce the D.C.D number of G by at most 2. Also, if D is a minimum D.C.D set of a 6-D.C.D critical graph, then $\langle D \rangle$ cannot be a path (as addition of an edge between the two peripheral nodes of that path will not reduce the D.C.D number of G). Thus, $\langle D \rangle$ may contain a cycle of length less than or equal to 6. The structural design of 6-D.C.D critical graphs, in which the induced sub graph of every minimum D.C.D set is a cycle C_6 (type (II) structure), is given below and hereafter this type of structure is represented by 6-D.C.D critical graphs with type (II) structure or simply 6-D.C.D type (II) critical graphs.

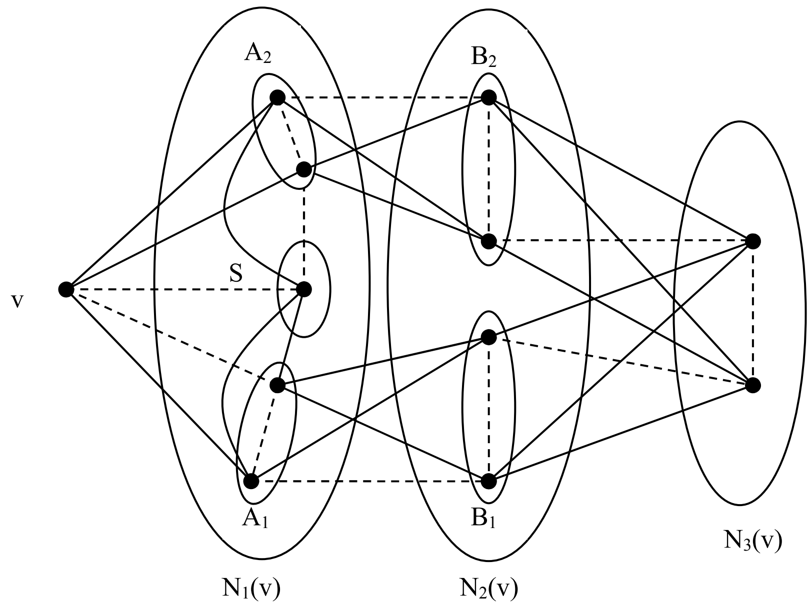

Figure 3.2 - ***Structural design of 6-D.C.D type (II) critical graphs***

The following theorems are based on the above structure.

Theorem 3.2.1: Let G be a 6-D.C.D type (II) critical graph. If u and v are any two non adjacent vertices of G, then there exists a minimum D.C.D set that contains both u and v.

Proof: Let G be a 6-D.C.D type (II) critical graph and let D = {x, u, v, y, v^1, u^1} where u, $u^1 \in N_1(x)$; v, $v^1 \in N_2(x)$ and $y \in N_3(x)$ be a minimum D.C.D set of G. Then, <D> is a cycle and a vertex $z \in V - D$, which is non adjacent to x implies that either $z \in N_2(x)$ or $z \in N_3(x)$.

Claim: ***There exists a minimum D.C.D set different from D, that contains both x and z***

Case (i): $z \in N_2(x)$

If $z \in N_2(x)$, then z must be adjacent to y and it is adjacent to exactly one vertex of {u, u^1}, say u^1. Now, the set of vertices {x, u, v, y, z, u^1} forms a minimum D.C.D set of G in which x and z are non adjacent.

Case (ii): $z \in N_3(x)$

If $z \in N_3(x)$, then z must be adjacent to y and it is adjacent to both the vertices of {v, v^1}. Now, the set of vertices {x, u, v, z, v^1, u^1} forms a minimum D.C.D set of G in which x and z are non adjacent.

Therefore from cases (i) and (ii), we have the result.

Corollary 3.2.1: If G is a 6-D.C.D type (II) critical graph, then every vertex must be in at least one minimum D.C.D set of G.

Lemma 3.2.1: Any 6-D.C.D type (II) critical graph is self centered of diameter 3.

Proof: If G is a 6-D.C.D type (II) critical graph, then the induced sub graph of any minimum D.C.D set of G is a cycle C_6 and every vertex of G must lies in at least one C_6. Hence, G must be self centered of diameter 3.

Corollary 3.2.2: If G is a 6-D.C.D type (II) critical graph, then G is a block.

Proof: As any 6-D.C.D type (II) critical graph is self centered of diameter 3, G must be a block.

Proposition 3.2.1: If G is a 6-D.C.D type (II) critical graph, then $\delta(G) \geq 2$ and $\Delta(G) \leq p - 4$.

Proof: As G is 3-self centered, $\delta(G) \geq 2$. Also, every vertex v of G must lies in at least one C_6. Hence, v must be non adjacent at least three vertices of G and hence $\Delta(G) \leq p - 4$.

Note 3.2.1: C_6 is the smallest 6-D.C.D type (II) critical graph having the above bound.

Theorem 3.2.2: If G is a 6-D.C.D type (II) critical graph, then for every vertex $v \in V(G)$,

(i) $<N_3(v)>$ is complete.

(ii) Every vertex in $N_2(v)$ is adjacent to all the vertices of $N_3(v)$.

Proof: Let G be a 6-D.C.D type (II) critical graph and let v be a vertex in G.

(i) Suppose that, if x and y are any two non adjacent vertices of $N_3(v)$, then addition of an edge between x and y will not affect the D.C.D number of G (as they are eccentric nodes of v), a contradiction to G is 6-D.C.D critical. Hence, every pair of vertices x and y in $<N_3(v)>$ are adjacent and hence $<N_3(v)>$ is complete.

(ii) Suppose, if a vertex u in $N_2(v)$ is not adjacent to a vertex w in $N_3(v)$, then u and w must lie on a C_7 in which addition of an edge uw will not reduce the D.C.D number of G, a contradiction to G is 6-D.C.D critical. Hence, the vertices u and w are adjacent and hence every vertex in $N_2(v)$ is adjacent to all the vertices of $N_3(v)$.

Theorem 3.2.3: Let G be a 6-D.C.D type (II) critical graph and let $v \in V(G)$. If a vertex u in $N_1(v)$ has no successors in $N_2(v)$ then we have the following:

(i) u is adjacent to all the vertices of $N_1(v)$ and

(ii) $E(u) = N_3(v)$.

Proof Let G be a 6-D.C.D type (II) critical graph and for a vertex $v \in V(G)$, let u be a vertex in $N_1(v)$ which has no successors in $N_2(v)$.

(i) Suppose that, if u is not adjacent to a vertex w in $N_1(v)$, then the vertices u, v and w lie on a C_7 (as u has no successors and $N_1(v)$ has at least one pair of non adjacent vertices), in which addition of an edge uw will not reduce the D.C.D number of G, a contradiction to G is 6-D.C.D critical. Hence, the vertices u and w are adjacent and hence every vertex in $N_2(v)$ is adjacent to all the vertices of $N_3(v)$.

(ii) By result (i), the vertex u is adjacent to all the vertices of $N_1(v)$. Thus the distance between u to any vertex x in $N_2(v)$ must be equal to 2. That is d(u, x) = 2, for all $x \in N_2(v)$. Also every vertex in $N_2(v)$ is adjacent to all the vertices of $N_3(v)$ and $\langle N_3(v) \rangle$ is a clique in G. Hence, d(u, y) = 3 for every vertex $y \in N_3(v)$ and hence the eccentric set of u is $N_3(v)$.

Theorem 3.2.4: Let v be a vertex of a 6-D.C.D type (II) critical graph and let S be the set of vertices in $N_1(v)$ having no successors. Then we have the following:

(i) $\langle N_1(v) - S \rangle$ and $\langle N_2(v) \rangle$ has exactly two components A_1, A_2 and B_1, B_2 respectively such that each $\langle A_i \rangle$, $\langle B_i \rangle$, i = 1 to 2 is a clique;

(ii) $\langle A_1 \cup B_1 \rangle$and $\langle A_2 \cup B_2 \rangle$ are also cliques.

Proof: Let v be a vertex of a 6-D.C.D type (II) critical graph and let S be the set of vertices in $N_1(v)$ having no successors.

(i) Now, let u and w are any two non adjacent vertices of $N_1(v) - S$. Then, every vertex in $N_1(v) - S \cup \{u, w\}$ is adjacent to exactly one vertex of {u, w}. Hence, the set of vertices in $N_1(v) - S$ has two components A_1 and A_2 where A_1 is the set of vertices in $N_1(v) - S$ which are adjacent to u and A_2 is the set of vertices in $N_1(v) - S$ which are adjacent to w. Since u and w are non adjacent, their corresponding successors are also pair wise non adjacent. Hence $\langle N_2(v)$ has exactly two components B_1 and B_2, where B_1 is the set of vertices in $N_2(v)$ which are the successors of A_1 and B_2 is the set of vertices in $N_2(v)$ which are the successors of A_2.

The vertices of B_1 and B_2 are adjacent to all the vertices of $N_3(v)$ and $\langle N_3(v) \rangle$ is a clique. Also, the eccentric set of each vertex in A_1 is the set B_2 and vice versa. Hence $\langle A_1 \rangle$ and $\langle B_2 \rangle$ are cliques. Similarly the eccentric set of each vertex in A_2 is the set B_1 and vice versa. Hence $\langle A_2 \rangle$ and $\langle B_1 \rangle$ are cliques.

(ii) As every vertex u in B_1 has $N_2(u) = B_2$ and $N_3(u) = A_2$, every vertex in B_2 is adjacent to all the vertices of A_2 and also every vertex w in A_1 has $N_2(u) = A_2$ and $N_3(u) = B_2$, every vertex in A_2 is adjacent to all the vertices of B_2. Hence, $\langle A_2 \cup B_2 \rangle$ is a clique. Similarly we can prove $\langle A_1 \cup B_1 \rangle$ is also a clique.

Therefore, we have the results (i) and (ii).

Theorem 3.2.5: Let v be any vertex of a 6-D.C.D type (II) critical graph and let u and w be any two non adjacent vertices of $N_1(v)$. Then, every vertex in $N_1(v)$ – {u, w} which is having successors in $N_2(v)$ is adjacent to exactly one vertex of {u, w}.

Proof: Let v be any vertex of a 6-D.C.D type (II) critical graph and let u and w be any two non adjacent vertices of $N_1(v)$. If x is a vertex of $N_1(v)$, which has a successor in $N_2(v)$, then we need to prove the following.

Claim 1: ***x must be adjacent to at least one of vertex of {u, w}***

Suppose that, if x is not adjacent to both u and w then addition of an edge between (u, x) or (w, x) will not affect the D.C.D number of G, as {u, x} is the eccentric set of w^1 and {w, x} is the eccentric set of u^1, where u^1 and w^1 are the successors of u and w respectively in $N_2(v)$. Hence, x must be adjacent to at least one vertex of {u, w}.

Claim 2: ***x cannot be adjacent to both the vertices of {u, w}***

If x is adjacent to both u and w, then x is a common vertex of two cycles each of length 5, namely {u, x, x^1, y, u^1} and {x, w, w^1, y, x^1} where u^1, w^1 and x^1 are the successors of u w and x respectively in $N_2(v)$. Hence, the eccentricity of x will be 2, a contradiction to G is 3-self centered and hence x cannot be adjacent to both the vertices of {u, w}.

Therefore, every vertex in $N_1(v)$ – {u, w} which is having successors in $N_2(v)$ is adjacent to exactly one vertex of {u, w}.

Proposition 3.2.2: Let G be a 6-D.C.D type (II) critical graph and let $v \in V(G)$. For a pair of non adjacent vertices u and w of $N_1(v)$ define the sets X and Y as follows:

$X = \{x \in N_2(v) \mid x \text{ is a successor of } u\}$ and

$Y = \{y \in N_2(v) \mid y \text{ is a successor of } w\}$.

Then every pair of vertices (x, y) where $x \in X$ and $y \in Y$ are non adjacent.

Proof: Let G be a 6-D.C.D type (II) critical graph and let $v \in V(G)$. Suppose that, if a vertex $x \in X$ and $y \in Y$ are non adjacent, then the set of vertices { v, u, x, y, w} forms a C_5 in G, a contradiction to G is 3-self centered. Hence, every pair of vertices (x, y) where $x \in X$ and $y \in Y$ are non adjacent.

Theorem 3.2.6: Any 6-D.C.D type (II) critical graph is radius edge (addition) critical.

Proof: Let G be a 6-D.C.D type (II) critical graph. Then G is self centered of diameter 3. If x and y are any two non adjacent vertices of G, then there exists a D.C.D set D that contains both x and y. Also addition of the edge xy in G will reduce the eccentricity of x and y to 2 and at least one pair of vertices in V(G) – {x, y} must have the same eccentricity 3 in (G + xy). Hence, for every pair of non adjacent vertices x and y, (G + xy) is of diameter 3 and radius 2 and hence G is radius edge (addition) critical.

Theorem 3.2.7: Any 6-D.C.D type (II) critical graph G is Hamiltonian.

Proof: If G is a 6-D.C.D type (II) critical graph and if v is a vertex of G, then each component of $<N_i(v)>$, for i = 1 to 3 is a clique. Hence, we can have a cycle

($v \to S \to A_2 \to B_2 \to N_3(v) \to B_1 \to A_1 \to v$) that covers all the vertices of G (refer Figure 3.2) and hence G is Hamiltonian.

Proposition 3.2.3: Let G be a 6-D.C.D type (II) critical graph. If v is a vertex with degree δ and if it has k eccentric nodes, then $q = 1 + \delta(G) + \{\binom{m}{2} + \binom{n}{2} + \binom{k}{2}\} + rk$, where $m = |A_1 \cup B_1|$, $n = |A_2 \cup B_2|$ and $r = |B_1 \cup B_2| = |N_2(v)|$.

Proof: Let G be a 6-D.C.D type (II) critical graph and let v be a vertex with degree δ. Also, $|E(v)| = k$. Then we have

(i) $\langle N_1(v) - S\rangle$, where S is the set of vertices in $N_1(v)$ having no successors and $\langle N_2(v)\rangle$ has exactly two components $\{A_1, A_2\}$ and $\{B_1, B_2\}$ respectively such that each $\langle A_i\rangle$, $\langle B_i\rangle$, i = 1 to 2 is a clique.

(ii) $\langle A_1 \cup B_1\rangle$ and $\langle A_2 \cup B_2\rangle$ are also cliques.

(iii) Every vertex in $N_2(v)$ is adjacent to all the k vertices of $N_3(v)$.

(iii) $\langle N_3(v)\rangle$ is a clique.

Therefore,

$$q = 1 + \delta(G) + \binom{|A_1 \cup B_1|}{2} + \binom{|A_2 \cup B_2|}{2} + (|B_1 \cup B_2|)\,k + \binom{k}{2}$$

$$= 1 + \delta(G) + \binom{|A_1 \cup B_1|}{2} + \binom{|A_2 \cup B_2|}{2} + (|N_2(v)|)\,k + \binom{k}{2}$$

$$q = 1 + \delta(G) + \{\binom{m}{2} + \binom{n}{2} + \binom{k}{2}\} + rk,$$

where $m = |A_1 \cup B_1|$, $n = |A_2 \cup B_2|$ and $r = |N_2(v)|$.

Open Problems

1. Find the structures other than the type (I) and type (II) of k-distance closed domination critical graphs for k ≥ 5.
2. Analyze the structural properties of distance closed domination critical graphs with respect to edge deletion.

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