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Excellent – Domination Dot Stable Graphs

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Abstract: *A graph G is said to be domination dot stable (DDS) if γ (G•uv) = γ (G), ∀ u, v∈ V(G), u adjacent to v. In this paper we introduce a DDS graph. We have obtained a necessary and sufficient condition for a graph to DDS. We have initiated a study on DDS graph. We have discussed the properties of graph that are excellent and DDS.*

Key words : *domination dot stable, just excellent, very excellent.*

1. Introduction

A set of vertices D in a graph G is a dominating set if every vertex of G – D is adjacent to some vertex of D. If D has the smallest possible cardinality of any dominating set of G, then D is called a minimum dominating set — abbreviated MDS. The cardinality of any MDS for G is called the domination number of G and is denoted by γ (G). For graph theoretic terminologies refer to [4].

The private neighborhood of v ∈ D is denoted by PN [v, D], is defined by PN [v, D] = N (v) – N (D – { v }). A vertex v is said to be a, down vertex if γ (G – u) < γ (G), level vertex if γ (G – u) = γ (G), up vertex if γ (G – u) > γ (G). A vertex v is said to be selfish in the γ- set D, if v is needed only to dominate itself.

A vertex v is said to be good it there is a γ - set of G containing v. If there is no γ - set of G containing v, then v is said to be a bad vertex. A graph G is said to be excellent if every vertex of G is good. In [5] Yamuna. M had defined a graph G to be

1. Just excellent if it to each u ∈ V, there is a unique γ - set of G containing u.
2. Very excellent if there is a γ - set D of G such that, to each vertex u ∈ V – D there exists a vertex v ∈ D such that (D – v) ∪ { u } is a γ - set of G. A γ - set D of G satisfying this property is called a very excellent γ - set of G. It has been proved that R_1. If G ≠ $\overline{K_n}$ is Just Excellent, then | PN (u, D) | ≥ 2 ∀u ∈ D where D is any γ - set of G

For a pair of adjacent vertices u, v of G, we denote by G•uv the graph obtained by identifying u and v. Let uv denote the identified vertex. In [1] Tamara Burton and David.

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P. Sumner defined a graph to be domination dot critical if γ ($G \bullet uv$) < γ (G) $\forall$ u, v. u adjacent to v denoted by u $\perp$ v. They have been proved that

R_2. Let a, b $\in$ V (G) for a graph G. Then γ ($G \bullet ab$) $<$ γ (G) if and only if either there exists an MDS S of G such that a, b $\in$ S or atleast one of a or b is critical in G.

In this paper we introduce a new kind of graphs called domination dot stable graphs and initiate a study on them.

2. Domination Dot Stable Graphs

A graph G is said to be dominating dot stable (DDS) if γ ($G \bullet uv$) = γ (G) $\forall$ u, v $\in$ V (G), u $\perp$ v.

Eg : P_n, path with n vertices is DDS iff γ (P_{n-1}) = γ (P_n). C_n, cycle with n vertices is DDS iff γ (C_{n-1}) = γ (C_n). The complete graph K_n, Star graph S_n are DDS graphs.

Theorem2.1. A graph G is DDS iff every γ - set of G is an independent dominating set.

Proof. Let G be a DDS graph and D be a γ - set of G. Let u, v $\in$ D such that u $\perp$ v. Then γ ($G \bullet uv$) $<$ γ (G). [By (R_2)]. Hence D is an independent dominating set.

Conversely, let us assume that every γ - set of G is independent. If G is not DDS then $\exists$ at least one pair of adjacent vertices u, v such that γ ($G \bullet uv$) $<$ γ (G). Since γ - set of G is independent u, v does not belongs to a common MDS. [By (R_2)] either u or v is critical. Let us assume that u is critical. Let H = G – { u }. Then γ (H) = γ (G) – 1. Let D' be a γ - set for H.

Case 1 If N (u) $\notin D'$

Let v $\in$ N (u). Since N (u) $\notin D'$ $\exists$ one x such that x $\in D'$ where x $\perp$ v, $D'' = D' \cup$ { v } is a γ - set for G, where x, v $\in D''$ such that x $\perp$ v which is contradiction as D'' is not an independent dominating set.

Case 2 If there exists at least one vertex in N (u) $\in D'$.

Let v $\in$ N (u) such that v $\in D'$. Then $D' \cup$ { u } is a γ - set for G where v $\perp$ u, which is contradiction as every γ - set of G is an independent dominating set.

ie., u is not critical. Also each γ - set of G is independent. Hence G is DDS.

Observation: Let G be DDS graph and let D = { u_1, u_2, …, u_k } be a γ - set for G. Let x, y $\in$ V (G), u $\perp$ v.

Case 1 $x \in D, y \notin D$.

Let $x = u_i$. Then $D' = D - \{ u_i \} \cup \{ u_i y\}$ is γ - set for $G \bullet xy$.

Case 2 $x, y \notin D$.

Then D itself is a γ - set for $G \bullet xy$.

By the above cases we observe that if G is a DDS graph then $\forall$ $x, y \in V(G)$ either D or D' is a γ - set for $G \bullet xy$. This is true $\forall$ $x, y \in V(G)$, $u \perp v$.

Theorem 2.2. A DDS graph cannot have a down vertex.

Proof. Let G be DDS. Let D be a γ - set of G. Let $u, v \in V(G)$ such that $u \perp v$ and u is critical. Then $\gamma(G \bullet uv) < \gamma(G)$. Since by converse part of theorem [2.1], if u is critical then $\gamma(G)$ is not an independent. ie., G has no down vertex.

Theorem 2.3. Let G be a DDS graph. Any vertex $v \in V(G)$ is not selfish.

Proof. Let $v \in V(G)$ such that v is selfish and let D be a γ - set for G. Let $u \in N(v)$. $\exists$ one $w \in D$ such that $w \perp u$. $D - \{ u \}$ is a γ - set for $G \bullet uv$, since in $G \bullet uv$, uv will be dominated by w ie., $\gamma(G \bullet uv) = \gamma(G) - 1$, which is contradiction as G is DDS. Hence v is not a selfish vertex. ie., G has no selfish vertex.

Theorem 2.4. If G is DDS graph, u is a support vertex such that,

1. $PN[u, D] = 1$, then u is a level vertex $\forall$ γ - set D containing u.
2. $PN[u, D] \geq 2, \forall$ possible γ - set D such that $u \in D$, then u is an up vertex.

Proof

1. Let G be a DDS graph. Let D be a dominating set for G. Let u be a support vertex such that $PN[u, D] = 1$, say $PN[u, D] = v$, where v is a pendant vertex. Consider $G - u$. Since $PN[u, D] = v$ in G, $N(u) - v$ is 2 – dominated. Hence $(D - u) \cup \{ v \}$ is a γ - set for $G - u$. ie., $\gamma(G - u) = \gamma(G)$. Hence u is a level vertex.
2. Let G a DDS graph and let D be a γ - set for G. Let u be a support vertex in G such that $PN[u, D] \geq 2$ $\forall$ γ - set of G. Let x be a pendant vertex adjacent to u. We know that a DDS graph does not have a down vertex. If possible let u be a level vertex. Let D' be a γ - set for $G - u - x$. Then $D = D' \cup \{ u \}$ is a γ - set for G such that $PN[u, D] = 1$, which is contradiction. Hence u is an up vertex.

Theorem 2.5. If G is DDS, then a pendant vertex is always a level vertex.

Proof. Let G be a DDS graph. Let v be a pendant vertex and e = (uv) $\in$ E (G). Then G$\bullet$ uv = G – v as G is DDS graph. γ (G) = γ (G$\bullet$ uv) = γ (G – uv). Hence u is a level vertex. ie., In DDS graph a pendant vertex is always a level vertex.

Theorem 2.6. Let G be a graph such that γ (G) = γ_2 (G). Then G is not DDS.

Proof. Let G be a graph such that γ (G) = γ_2 (G). Let D be a γ - set of G. We know that for any u $\in$ V – D $\exists$ vertices x, y $\in$ D such that x, y dominates u. Now D' = (D – x) $\cup$ { u } and D'' = (D – y) $\cup$ { u } are γ - sets of G such that y, u $\in$ D' where y $\perp$ u and x, u $\in$ D'', where x $\perp$ u which are not independent γ - sets. Hence G is not DDS.

Theorem 2.7. Let G be DDS graph,

1. If PN [u, D] = 1 say PN [u, D] = v then G$\bullet$ uv is not DDS.
2. Let u $\in$ V (G) and D be a γ - set for G. Let x, y $\in$ D such that x $\perp$ u and y $\perp$ u. Then (G$\bullet$ ux) and (G$\bullet$ uy) are not DDS.
3. Let x, y $\in$ V (G) such that PN [u, D] = 1 say PN [u, D] = x and PN [v, D] = 1 say PN [v, D] = y and x $\perp$ y for some γ - set D of G, then G$\bullet$ xy is not DDS.
4. Let PN [u, D] = 2 say PN [u, D] = { x, y } such that x $\perp$ y. Let z be a 2 – dominated vertex such that z $\perp$ u, v, y. Where u, v $\in$ D then G$\bullet$ zy is not DDS.

Proof. Let G be DDS graph,

1. If PN [u, D] = v, then D – { u } $\cup$ { uv } is γ - set for G$\bullet$ uv ie., uv is a selfish vertex for G$\bullet$ uv. Hence by theorem [2.3] G$\bullet$ uv is not DDS.
2. Let u $\in$ V (G) and D be a γ - set for G. Let x, y $\in$ D such that x $\perp$ u and y $\perp$ u and D' = D – { u } $\cup$ { ux } is a γ - set for G$\bullet$ ux , where xu $\perp$ y and xu, y $\in$ D'. Similarly D'' = D – { u } $\cup$ { uy } is a γ - set for G$\bullet$ uv such that uy $\perp$ y, uy, y $\in$ D''. Hence G$\bullet$ ux and G$\bullet$ uy are not DDS.
3. Let x, y $\in$ V (G), x $\perp$ y and let D be a γ - set for G such that PN [u, D] = { x } and PN [v, D] = { y }. As G is DDS, γ (G) = γ (G$\bullet$ xy). D' is a γ - set for G$\bullet$ xy. PN [v, D'] = ϕ and PN [u, D'] = ϕ. Hence G$\bullet$ xy is not DDS.
4. Let PN [u, D] =2 say PN [u, D] = x, y such that x $\perp$ y. Let z be 2 – dominated vertex such that z $\perp$ u, v, y where u, v $\in$ D. In G$\bullet$ zy, D' = D – { u } $\cup$ { zy } is a γ - set for G$\bullet$ zy, where yz $\perp$ u, x, v. Here v, zy $\in$ D' such that v $\perp$ zy. Hence G $\bullet$ zy is not DDS.

Remark. If PN [u, D] = 1 for a DDS graph G, then G$\bullet$ uv has at least one selfish vertex.

2.1. Excellent and DDS Graphs

In this section, we discuss properties of graphs which are DDS and excellent.

Theorem 2.1.1. Let G be DDS and excellent, then G• uv is excellent, $\forall$ u, v $\in$ V (G), u $\perp$ v.

Proof. Let G be DDS. So, γ (G• uv) = γ (G) $\forall$ u, v $\in$ V (G), u $\perp$ v. Since G is excellent let D_1 be the γ - set for G containing u and D_2 be a γ - set for G containing v. In G• uv all the vertices that are dominated by u and v is now dominated by uv. Hence D_1 – { u } $\cup$ { uv } and D_2 – { v }) $\cup$ { uv } will be a γ - sets for G• uv ie., all γ - sets of G including u or v will now be γ - sets for G• uv including uv instead of u or v. The remaining γ - set for G will be a γ - set for G• uv also. Every vertex of G• uv is included in some γ - set for G• uv. Hence G• uv is an excellent graph.

Remark. If G is not DDS ie., then if G is excellent, G• uv may or may not be excellent.

Theorem 2.1.2. If G is non – excellent and DDS graph then number of bad vertices in G• uv is at least one less than the number of bad vertices in G.

Proof. Let G be DDS and non – excellent graph. Let D = { u_1, u_2, …, u_k } be a γ - set for G. Let v be a bad vertex. Since v is a bad $\exists$ one u_i $\perp$ v. Then {u_1, u_2, …, u_{i-1}, u_{i+1}, …, u_k } $\cup$ { u_iv } is γ - set for G• u_iv. By observation u_iv is a good vertex ie., G• u_iv has at least one bad vertex less than the number of bad vertices in G.

Remark. If G is a DDS and non – excellent graph with exactly one bad vertex u, then we can generate N (u) excellent graphs.

Theorem 2.1.3. Let G be DDS and non – excellent graph. Let u be a bad vertex in G. Then uv is never a selfish vertex in G• uv $\forall$ v $\in$ V (G) such that u $\perp$ v.

Proof. Let G be a DDS and non – excellent graph. Let u be a bad vertex. Let v $\in$ V (G) such that u $\perp$ v.

Case 1 v is a bad vertex. Consider G• uv. If uv is a selfish vertex then $\exists$ a γ - set D for G• uv such that N (uv) is 2 – dominated. Then D – { uv } $\cup$ { u } or D – { uv } $\cup$ { v } is a γ - set for G. ie., $\exists$ γ - sets for G containing u and v which is not possible, since u and v are bad vertices.

Case 2 v is a good vertex. $D' = D - \{ uv \} \cup \{ u \}$ is a γ - set for G, where D is a γ - set for G• uv where uv is a selfish vertex. Thus $\exists$ a γ - set for G containing u which is not possible.

Hence uv is never a selfish vertex in G• uv.

Theorem 2.1.4. Let G be a DDS and just excellent such that $\gamma (G) \geq 2$. Then G• uv is not just excellent $\forall$ u, v $\in$ V (G), $u \perp v$.

Proof. Let G be DDS and JE graph. Let u, v $\in$ V (G), $u \perp v$. Since G is excellent let D_1 be a γ - set containing u and D_2 be a γ - sets containing v. Since G is DDS, $D_1 \neq D_2$. By theorem [2.1], we know that G• uv is an excellent graph. Also $(D_1 - \{ u \}) \cup \{ uv \}$ and $(D_2 - \{ v \}) \cup \{ uv \}$ are two distinct γ - sets for G containing the vertex uv. Hence G• uv is not just excellent $\forall$ u, v $\in$ V (G), $u \perp v$.

Theorem 2.1.5. A JE graph ($\neq K_n$) is not DDC.

Proof. It is enough to prove the result for graphs with $\gamma(G) \neq 1$ (ie., G is not K_n, $n \geq 2$). Let G be JE and DDC.

Claim 1. A JE graph does not have a selfish vertex.

Let G be just excellent. Suppose $\exists$ a selfish vertex say u. Let D be a γ - set for G including u. Then PN [u, D] = ϕ. Also $D_1 = D - \{ u \} \cup \{ v \}$, where v $\in$ N (u) is also a γ - set for G, ie., D, D_1 are two distinct γ - sets for G containing D – u which is contradiction as G is just excellent. Hence G has no selfish vertex.

Claim 2. A JE graph has no down vertex.

Suppose a JE graph G has a down vertex u , then $\gamma (G - u) = \gamma (G) - 1$. Let D' be a γ - set for G – u. Then $D' \cup \{ u \}$ is a γ - set for G where u is selfish , which is not possible by Claim 1ie., a JE graph does not have a down vertex.

G is DDC and has no down vertex. [By R_2] $\forall$ u, v $\in$ V (G), $u \perp v$ $\exists$ a γ - set containing u and v ie., if $v_1, v_2, \ldots, v_n \in N (u)$, $\exists$ γ - sets containing $uv_1, uv_2, \ldots, uv_n$. Since G is JE $\exists$ a unique γ - set containing u which implies N [u] belongs to a common MDS set say D. In D, PN [u, D] = ϕ, which is not possible by R_1. ie.,$\exists$ no common MDS set containing uv $\forall$ u, v $\in$ V (G). ie., a JE graph is not DDC.

Theorem 2.1.6. Let G be a very excellent graph. Let D be γ - set of G. Each one of the following conditions implies that G is not DDS graph.

1. If v $\in$ D is selfish,
2. If v $\in$ D is not used for vertex exchange,

3. If w $\in$ V – D is 2 – dominated,

Proof

1. If v is selfish then G is not DDS by theorem [2.3].
2. If v $\in$ D is not used for vertex exchange. Let u $\in$ N (v). Now as G is very excellent $\exists$ one w $\in$ D such that D – { w } $\cup$ { u } is a γ - set. Hence $\exists$ one γ - set for G where u and v are adjacent. Hence G is not DDS.
3. Suppose x is 2 – dominated say x is dominated by u and v. $\exists$ one y $\in$ D such that D – { y } $\cup$ { x } is a γ - set for G. But G is not DDS as u $\perp$ x $\perp$ v (y may be one of u or v). Hence G is not DDS.

Remark

If G is DDS and VE, then G

a. Cannot have a selfish vertex.

b. $\exists$ no vertex that cannot be used for vertex exchange.

c. $\exists$ no vertex that is 2 – dominated.

d. Any vertex can be interchanged only with the vertex dominating it ie., PN [u, D] = N (u) and N (u) is complete for every γ - set of G.

Conclusion

From theorem [2.1.6] and its remark we conclude that, K_n is the only DDS and VE connected graph $\forall$ n $\geq$ 2.

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