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On Super Duplicate Graphs With Respect to Complementation

T. N. Janakiraman[1], S. Muthammai[2] and M. Bhanumathi[2]

[1]*Department of Mathematics .National Institute of Technology, Triuchirapalli, 620015,TamilNadu, India*

E-Mail: janaki@nitt.edu, tnjraman2000@yahoo.com

[2]*Government Arts College for Women, Pudukkottai.622001, TamilNadu, India.*

E-Mail: bhanu_ksp@yahoo.com, muthammai_s@yahoo.com

Abstract: *For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. Let V′(G) = {v′ : v∈V(G)} be a copy of V(G). The Super duplicate graph with respect to complementation D_c*(G) of G is the graph whose vertex set is V(G)∪V′(G) and edge set is E(G)∪E(D($\bar{G}$)) where D($\bar{G}$) is the Duplicate graph of the complement $\bar{G}$ of G. In this paper, some basic properties of D_c*(G) are studied. Also a criterion for D_c*(G) to be Eulerian and a sufficient condition for Hamiltonicity are obtained. In addition, the parameters girth, connectivity, covering number, independence number, chromatic number determined for this graph. Finally, eccentricity properties of D_c*(G) are discussed.*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. *Eccentricity* of a vertex u∈V(G) is defined as $e_G(u)$ = max {$d_G(u, v)$: v∈V(G)}, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of vertex v in G as e(v) and d(u, v) to denote the distance between two vertices u, v in G respectively. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted r(G) and diam(G) respectively. When diam(G) = r(G), G is called a *self-centered* graph with radius r, equivalently G is r-self-centered. $N_i(v)$ = {u∈V(G) : d(u, v) = i} is called the i^{th} neighborhood of v. Let u∈$N_i(v)$. A vertex w∈$N_j(v)$, for i < j is said to be the j^{th} *successor* of u, if d(u, w) = j - i and for j < i, the vertex w is said to be j^{th} predecessor, if d(u, w) = i – j. An edge set e = uv is a *dominating edge* in a graph G, if every vertex of G is adjacent to at least one of u and v. A *dominating trail* T of a graph G is a closed trail in G (which may be just a single vertex) such that every edge of G not in T is incident with T.

Let V′(G) = {v′ : v∈V(G)} be a copy of V(G). Then the *Duplicate graph* D(G) of G is the graph whose vertex set is V(G)∪V′(G) and edge set is {u′v and uv′ : uv∈E(G)}. This graph was first studied by Sampathkumar [3]. The *Super duplicate graph with respect to complementation* D_c*(G) of G is defined as the graph whose vertex set is as that of D(G)

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and edge set is $E(D(\overline{G})) \cup E(G)$, where D(G) and $\overline{G}$ are respectively, the Duplicate graph and the complement of G.

The concept of super duplicate graph with respect to complementation of a given graph defines Boolean function of a graph based on the adjacency of the vertices of the given graph. The important application of facility location on networks is based on various types of graphical centrality, all of which are defined using distance. The concept of distance and related concept of eccentricity in graph play a dominated role in the study of structural properties such as fault tolerant and extremal nature of links and nodes of the underlying graphs of the corresponding communication networks.

In this paper, some basic properties of $D_c^*(G)$ are studied. Also a criterion for $D_c^*(G)$ to be Eulerian and a sufficient condition for Hamiltonicity are obtained. In addition, the parameters girth, connectivity, covering number, independence number, chromatic number determined for this graph. Finally, eccentricity properties of $D_c^*(G)$ are discussed. The definitions and details not furnished in this paper may found in [2].

2. Prior Results

In this section, we list results with indicated references, which will be used in the subsequent main results.

Theorem 2.1[2]: For any nontrivial connected graph G, $\alpha_0 + \beta_0 = p = \alpha_1 + \beta_1$, where p is the number of vertices in G.

Theorem 2.2 [1]: If G is a planar (p, q) graph ($p \geq 3$), then $q \leq 3p - 6$.

Corollary 2.3 [1]: If G is a planar graph, then $\delta(G) \leq 5$.

3. Main Results

The following elementary properties of a super duplicate graph with respect to complementation are immediate.

Let G be a (p, q) graph. Then

(a) Duplicate graph $D(\overline{G})$ of $\overline{G}$ is a spanning sub graph of $D_c^*(G)$ and $D_c^*(G)$ is a (2p, p(p - 1) - q) graph.

(b) $V(D_c^*(G))$ can be partitioned into two sets $V(G)$ and $V'(G)$ such that the sub graph of $D_c^*(G)$ induced by the vertices in $V(G)$ is G and that of $D_c^*(G)$ induced by the vertices in $V'(G)$ is totally disconnected.

(c) For any vertex $v \in V(G)$, there are two vertices v and v′ in $D_c^*(G)$, such that $deg_{D_c^*(G)}(v) = p - 1$ and $deg_{D_c^*(G)}(v') = p - 1 - deg_G(v)$ and hence $\Delta(D_c^*(G)) = p - 1$ and $\delta(D_c^*(G)) = p - 1 - \Delta(G)$.

(d) For $p \geq 3$, $D_c^*(G)$ contains pendant vertices if and only if there exists a vertex v in $V(G)$ such that $deg_G(v) = p-2$.

(e) For any graph G with $\delta(G) \geq 1$, $D_c^*(G)$ is biregular if and only if G is regular and $D_c^*(G)$ is regular if and only if G is totally disconnected.

(f) $D_c^*(G)$ is disconnected if and only if either $r(G) = 1$ or G is disconnected with exactly two complete components.

In the following, the solution for $D_c^*(G)$, which is bipartite is obtained.

Theorem 3.1: $D_c^*(G)$ is bipartite if and only if G is complete bipartite.

Proof: Suppose G is a complete bipartite graph with bipartition $[V_1, V_2]$. Let $V(D_c^*(G)) = V(G) \cup V'(G)$, where $V(G) = V_1 \cup V_2$ and $V'(G) = V_1' \cup V_2'$. Then $[W_1, W_2]$, where $W_i = V_i \cup V_i'$ $(i = 1, 2)$ is a bipartition of $D_c^*(G)$ and hence $D_c^*(G)$ is bipartite. Conversely, assume $D_c^*(G)$ is bipartite. Since G is an induced sub graph of $D_c^*(G)$, G is bipartite and let $[V_1, V_2]$ be the bipartition of G. If G is not complete bipartite, then there exists a vertex say, $v_1 \in V_1$, not adjacent to at least one vertex say, $v_2 \in V_2$. Then the vertex v_2' in $D_c^*(G)$ is adjacent to v_1 and v_3, where $v_3 \in V_2$ and $(v, v_3) \in E(G)$, thereby forming a triangle in $D_c^*(G)$. Hence, G is a complete bipartite graph.

Theorem 3.2: $D_c^*(G)$ is not a tree, for any graph G.

Proof: Since G is an induced sub graph of $D_c^*(G)$, $D_c^*(G)$ is not a tree, if G is not a tree. Assume G is a tree. If G contains $K_2 \cup K_1$ or $K_{1,3}$ as an induced sub graph, then $D_c^*(G)$ contains C_3 or C_4 as an induced sub graph respectively. Therefore, $G \cong K_2$ or P_3. But $D_c^*(K_2)$ and $D_c^*(P_3)$ are disconnected. Hence, $D_c^*(G)$ is not a tree.

A connected graph G is said to be ***geodetic***, if a unique shortest path joins any two of its vertices. In the following, the geodetic graphs G for which $D_c^*(G)$ are also geodetic are characterized.

Theorem 3.3: For any geodetic graph G with $r(G) \geq 2$, $D_c^*(G)$ is geodetic if and only if G is a path on four vertices.

Proof: Let G be geodetic graph with $r(G) \geq 2$. Therefore G is connected. Assume $D_c^*(G)$ is geodetic. If G contains $K_2 \cup 2K_1$ or $P_3 \cup K_1$ as an induced sub graph, then $D_c^*(G)$ contains $K_4 - e$ as an induced sub graph and hence not geodetic. Similarly, if G contains $K_{1,3}$ (or C_5) as an induced sub graph, then $D_c^*(G)$ contains C_4 (or C_6) as an induced sub graph and hence not geodetic. Consequently, G is a path on four vertices. Converse is obvious.

Theorem 3.4: For any graph G having at least four vertices, girth of $D_c^*(G)$ is either 3 or 4.

Proof: Obviously girth of $D_c^*(G)$ is 3, if G contains either a triangle or $K_2 \cup K_1$ as an induced sub graph. If not, then either G is a star or G contains C_4 as an induced sub graph. In both the cases $D_c^*(G)$ contains C_4, which is the smallest cycle and hence girth of $D_c^*(G)$ is 4.

In the following, a necessary and sufficient condition that a cut-vertex of G is also a cut-vertex of $D_c^*(G)$ is given. For simplicity, neighborhood of a vertex v in $D_c^*(G)$ is denoted by $N_{c^*}(v)$.

Theorem 3.5: Let G be a connected graph with $r(G) \geq 2$. Then any cut-vertex v of G is also a cut-vertex of $D_c^*(G)$ if and only if

(i). $\omega(G - v) = 2$, where $\omega(G)$ is the number of components of G.

(ii). Each of the components of $G - v$ is complete

(iii). v is adjacent to all the vertices of one of the components of $G - v$.

Proof: Let v be a cut-vertex of G satisfying (i), (ii) and (iii). Let G_1 and G_2 be the two complete components of $G - v$ such that v is adjacent to all the vertices in G_1 (say). Then for all $v_1 \in V(G_1)$, $N_{c^*}(v_1') = V(G_2)$ and for all $v_2 \in V(G_2)$, $N_{c^*}(v_2') = V(G_1)$ and hence v' in $D_c^*(G)$ is adjacent to vertices in G_2 only. Since G is an induced sub graph of $D_c^*(G)$, $D_c^*(G) - v$ is disconnected and hence v is a cut-vertex of $D_c^*(G)$. Conversely, assume v is a cut-vertex of both G and $D_c^*(G)$.

(i). If $G - v$ contains at least three components say, G_1, G_2, …, G_n ($n \geq 3$), then for all $u_i \in V(G_i)$, $(1 \leq i \leq n)$, $N_{c^*}(u_i')$ contains vertices in $V(G_i)$, for at least one i and hence $G - v$ is connected. This is a contradiction. Hence, n=2.

(ii). Let G_1 and G_2 be the two components of $G-v$ with G_2 be not complete. Then there exists at least one vertex say, $u \in V(G_2)$ such that $N_{c^*}(u')$ contains $V(G_1)$ and at least one vertex in G_2 and hence $G-v$ is connected.

By (i) and (ii), $G-v$ contains exactly two components each of which is complete.

(iii). If v is not adjacent to all the vertices in say, G_1, then $v' \in V(D_c^*(G))$ is adjacent to vertices in both G_1 and G_2 and hence $G-v$ is connected, which is a contradiction.

Remark 3.1: The cut vertices $v_1, v_2, \ldots, v_k$ $(k \geq 1)$ of G are also cut vertices of $D_c^*(G)$ if and only if $G-\{v_1, v_2, .., v_k\}$ contains exactly two components, each of which is complete and v_i' s (i = 1, 2,...,k) are adjacent to all the vertices in one of the components.

Remark 3.2: $\kappa(D_c^*(G)) = 1$, if there exists a vertex of degree p-2 in G and $\kappa(D_c^*(G) \leq p - 1 - \Delta(G)$. This is sharp, because for $G \cong C_5$ or C_6, the equality holds.

Note 3.1: Similar results hold for graphs having a cut-edge.

In the following, a criterion for $D_c^*(G)$ being Eulerian is established.

Theorem 3.6: Let G be any (p, q) graph such that $D_c^*(G)$ is connected. Then $D_c^*(G)$ is Eulerian if and only if p is odd and each vertex in G is of even degree.

Proof: The degree of the vertices v and v' in $D_c^*(G)$ are p - 1 and p – 1 - $deg_G(v)$ respectively. Assume $D_c^*(G)$ is Eulerian. Then it is clear that p is odd and each vertex in G is of even degree. Conversely, assume the given conditions. Then each vertex in $D_c^*(G)$ is of even degree and hence, $D_c^*(G)$ is Eulerian.

Theorem 3.7: Let G be a connected graph with $r(G) \geq 2$. If $\overline{G}$ contains an odd Hamiltonian cycle, then $D_c^*(G)$ is Hamiltonian.

Proof: Since G is connected and $r(G) \geq 2$, $D_c^*(G)$ is connected. Let $v_1 v_2 \ldots v_n v_{2n+1} v_1$ $(n \geq 2)$ be an odd Hamiltonian cycle in $\overline{G}$. Then $v_1 v_2' v_3 \ldots v_{2n-1} v_{2n}' v_{2n+1} v_1' v_2 v_3' \ldots v_{2n-1}' v_{2n} v_{2n+1}' v_1$ is a Hamiltonian cycle in $D_c^*(G)$ and hence $D_c^*(G)$ is Hamiltonian.

Remark 3.3: 1. Let G be a disconnected graph such that at least one of its components is not complete. Then $D_c^*(G)$ is Hamiltonian, if the complement of each of the component of G contains an odd Hamiltonian cycle.

2. If $\Delta(G) \geq p - 2$, then $D_c^*(G)$ is not Hamiltonian.

Theorem 3.8: Let G be any connected graph with $\delta(G) \geq 2$ and $\Delta(G) \leq p-3$. Then each edge of $D_c{}^*(G)$ lies on a triangle if and only if

(i). For each edge (u, v) in G not lying in any triangle, there exists at least one vertex in G not adjacent to both u and v; and

(ii). For any two vertices v_1 and v_2 at distance 2 in G, $N_G(v_1) \cap N_{\bar{G}}(v_2) \neq \Phi$ and $N_{\bar{G}}(v_1) \cap N_G(v_2) \neq \Phi$.

Proof: Assume each edge of $D_c{}^*(G)$ lies on a triangle in G. Edges lying on a triangle in G, also lie on a triangle in $D_c{}^*(G)$, since G is an induced sub graph of $D_c{}^*(G)$. Let e = (u, v) be an edge in G not lying in any triangle in G, where $u, v \in V(G)$. By the assumption, e = (u, v) lies on a triangle in $D_c{}^*(G)$. Hence, there exists a vertex $w \in V(G)$ such that w' in $D_c{}^*(G)$ is adjacent to both u and v. That is, there exists a vertex $w \in V(G)$ not adjacent to both u, v in G, which proves (i). Let v_1 and v_2 be two nonadjacent vertices in G. If $d_G(v_1, v_2) = 2$, then the edges v_1v_2', $v_1'v_2$ in $D_c{}^*(G)$ lie on a triangle in $D_c{}^*(G)$ by the assumption. Hence, there must exist vertices x, y in G such that $x \in N_G(v_1) \cap N_{\bar{G}}(v_2)$ and $y \in N_{\bar{G}}(v_1) \cap N_G(v_2)$, which proves (ii). If $d_G(v_1,v_2) \geq 3$, then the edges v_1v_2', $v_1'v_2$ in $D_c{}^*(G)$ lie on a triangle in $D_c{}^*(G)$. Conversely, if the conditions (i) and (ii) are true, then each edge of $D_c{}^*(G)$ lies on a triangle.

Remark 3.4: If each edge of $D_c{}^*(G)$ lies on a triangle, then $D_c{}^*(G)$ has a dominating trail and hence $L(D_c{}^*(G))$ is Hamiltonian.

Theorem 3.9: Let G be any (p, q) graph. Then $D_c{}^*(G)$ is non-planar, if one of the following holds.

(i). G contains $K_{1,5}$ as an induced sub graph.

(ii). $p > \Delta(G) + 6$ and (iii). $q \geq p - 1$ and $p > 7$.

Proof: If (i) holds, then $D_c{}^*(G)$ contains $K_{3,3}$ as a sub graph and if (ii) holds, then $\delta(D_c{}^*(G)) \geq 6$. Assume $q \geq p-1$ and $p > 7$. Since $D_c{}^*(G)$ contains 2p vertices and p(p-1)-q edges, $q' > 3p' - 6$, where $p' = 2p$ and $q' = p(p-1) - q$. Thus, $D_c{}^*(G)$ is non-planar.

Observation:

3.1: Since G is an induced sub graph of $D_c{}^*(G)$, $D_c{}^*(G)$ is non-planar if G is non-planar.

3.2: $D_c{}^*(C_n)$ $(n \geq 7)$ has a sub graph homoeomorphic to $K_{3,3}$ and hence non-planar.

3.3: $D_c{}^*(C_n)$ $(3 \leq n \leq 6)$ and $D_c{}^*(K_m)$ $(m < 5)$ are planar graphs.

In the following, covering, independence and chromatic numbers for $D_c{}^*(G)$ are determined.

Theorem 3.10: Let G be any graph such that $D_c^*(G)$ is connected. Then

(i). $\alpha_0(D_c^*(G)) = p = \beta_0(D_c^*(G))$; and

(ii). $\alpha_1(D_c^*(G) = 2\alpha_1(\overline{G})$ and $\beta_1(D_c^*(G)) = 2\beta_1(\overline{G})$, where p is the number of vertices in G.

Proof: Let V be the vertex set of G and V' be the set of new points introduced in the construction of $D_c^*(G)$. Since V' is an independent set with p points, $\beta_0(D_c^*(G)) \geq p$. Since $D_c^*(G)$ is connected, each vertex in V is adjacent to at least one vertex in V' and hence any independent set in $D_c^*(G)$ can have at most p points. Thus, $\beta_0(D_c^*(G)) = p$. Since $D_c^*(G)$ has 2p points and $\alpha_0(D_c^*(G)) + \beta_0(D_c^*(G)) = 2p$, $\alpha_0(D_c^*(G)) = p$. It remains to prove that $\beta_1(D_c^*(G)) = 2\beta_1(\overline{G})$. It is to be observed that corresponding to each edge uv of $\overline{G}$, there are two independent edges uv' and $u'v$ in $D_c^*(G)$. Thus, each edge of $\overline{G}$ gives rise to two independent edges in $D_c^*(G)$. So $\beta_1(\overline{G})$ independent edges of G give $2\beta_1(\overline{G})$ independent edges in $D_c^*(G)$ and this is the maximum number of independent edges in $D_c^*(G)$. Hence, $\beta_1(D_c^*(G)) = 2\beta_1(\overline{G})$. From the equation $\beta_1(D_c^*(G)) + \alpha_1(D_c^*(G)) = 2p = 2\beta_1(\overline{G}) + 2\alpha_1(\overline{G})$, it follows that $\alpha_1(D_c^*(G)) = 2\alpha_1(\overline{G})$.

Theorem 3.11: For any graph G, $\chi(D_c^*(G)) = \chi(G)$ or $\chi(G) + 1$.

Proof: Since G is an induced sub graph of $D_c^*(G)$, $\chi(D_c^*(G)) \geq \chi(G)$. Let $\chi(G) = k$. Then V(G) can be partitioned into k sets $V_1, V_2, \ldots, V_k$ such that no two vertices in V_i (i = 1, 2, …, k) are adjacent. Hence, the vertices in $V(G) \cap V(D_c^*(G))$ can be colored by k colors. Let v be any vertex in G. If v is adjacent to all the vertices in V_j for at least one j, then color the vertex v' in $D_c^*(G)$ by the color j and hence $D_c^*(G)$ is k-colorable. Thus, $\chi(D_c^*(G)) \leq \chi(G)$. If there exists no V_j (j = 1, 2, ... , k) such that $v \in V(G)$ is adjacent to all the vertices in V_j, then color the vertex v' in $D_c^*(G)$ by a new color say, k + 1. Since the sub graph of $D_c^*(G)$ induced by $V'(G)$ is independent, $D_c^*(G)$ is (k + 1)-colorable. Hence, $\chi(D_c^*(G)) = \chi(G) + 1$.

Example 3.1: $\chi(D_c^*(C_n)) = \chi(C_n) + 1$, if $n \geq 6$ or n = 3; and

$= \chi(C_n)$, if n = 4, 5.

In the following, the eccentricity properties of $D_c^*(G)$ are discussed. First, all self-centered graphs G with radius 2 for which $D_c^*(G)$ is self-centered with radius 3 are characterized.

For simplicity, $e_{Dc^*}(v)$ and $d_{Dc^*}(u, v)$ are used to denote the eccentricity of a vertex v and the distance between u and v in $D_c^*(G)$ respectively.

Theorem 3.12: Let G be self-centered with radius 2. Then $D_c^*(G)$ is self-centered with radius 3 if and only if G has no dominating edge.

Proof: Assume G is self-centered with radius 2 and has no dominating edge. For each vertex v in $D_c^*(G)$, there is a vertex v' in $D_c^*(G)$. Let $v_i, v_j \in V(G)$. Since G is an induced sub graph of $D_c^*(G)$, $d_{Dc^*}(v_i, v_j) \leq 2$.

(i). Since $e(v_i) = 2$, there exists a vertex $v_i^{(2)} \in N_2(v_i)$ with $d(v_i, v_i^{(2)}) = 2$ in G. Then $d_{Dc^*}(v_i, v_i') = 3$, since $v_i\, v_i^{(1)}\, v_i^{(2)}\, v_i'$ is geodesic in $D_c^*(G)$, where $v_i^{(1)} \in N_1(v_i)$.

(ii). If $(v_i, v_j) \notin E(G)$, then $d_{Dc^*}(v_i, v_j') = 1$. Let $(v_i, v_j) \in E(G)$. If there exists a vertex $v_i^{(1)} \in N_1(v_i)$ not adjacent to v_j, then $v_i\, v_i^{(1)}\, v_j'$ is geodesic in $D_c^*(G)$ and hence $d_{Dc^*}(v_i, v_j') = 2$. If not, there exists a vertex $v_i^{(2)} \in N_2(v_i)$ not adjacent to v_j, since G has no dominating edge, then $v_i\, v_i^{(1)}\, v_i^{(2)}\, v_j'$ is geodesic and hence $d_{Dc^*}(v_i, v_j') = 3$. Thus, it follows that $e_{Dc^*}(v_i) = 3$.

(iii)(a). Let $(v_i, v_j) \in E(G)$. Since G has no dominating edge, there exists a vertex $v_k \in V(G)$ not adjacent to both v_i and v_j and hence $d_{Dc^*}(v_i', v_j') = 2$.

(b). If $(v_i, v_j) \notin E(G)$, then $v_j \in N_2(v_i)$. If there exists a vertex v_k in $N_2(v_i)$ such that $d(v_j, v_k) = 2$, then $v_i'\, v_k\, v_j'$ is geodesic in $D_c^*(G)$ and hence $d_{Dc^*}(v_i', v_j') = 2$. If not, $\langle N_2(v_i)\rangle_G$ has radius one and there exists at least one vertex $v_l \in N_1(v_i)$, having eccentric point v_j. Let v_m be a vertex in $N_2(v_i)$ adjacent to both v_l and v_j, then $v_i'\, v_m\, v_l\, v_j'$ is geodesic in $D_c^*(G)$ and hence $d_{Dc^*}(v_i', v_j') = 3$. Thus, $e_{Dc^*}(v_i') = 3$. Hence, each vertex in $D_c^*(G)$ has eccentricity 3 and $D_c^*(G)$ is self-centered with radius 3.

Conversely, assume G and $D_c^*(G)$ are self-centered with radii 2 and 3 respectively. If G has a dominating edge, then there exists at least one pair of vertices v_i, v_j in G with $d(v_i, v_j) = 2$ and $d_{Dc^*}(v_i', v_j') = 4$. This is a contradiction. Thus, G has no dominating edge.

Corollary 3.12.1: If G is self-centered with radius 2 and has a dominating edge, then $D_c^*(G)$ is bi-eccentric with radius 3.

Next, the radius and diameter of $D_c^*(G)$ are determined, where G is bi-eccentric with radius 2.

Theorem 3.13: If G is a bi-eccentric graph with radius 2 and $\delta(G) \geq 2$, then $D_c^*(G)$ has radius 3 and diameter 3 or 4.

Proof: Let G be any bi-eccentric graph with radius 2 and $\delta(G) \geq 2$. Let $v_i, v_j \in V(G)$. Since G is an induced sub graph of $D_c^*(G)$, $d_{Dc^*}(v_i, v_j) \leq 3$. By Theorem 3.1.(i)., $d_{Dc^*}(v_i, v_i') = 3$.

(i)(a). Let $(v_i, v_j) \in E(G)$. If $e(v_i) = 3$, then $v_i\, v_i^{(3)}\, v_j'$ is geodesic in $D_c^*(G)$, where $v_i^{(3)} \in N_3(v_i)$ in G. If $e(v_i) = 2$, since $\delta(G) \geq 2$, there exists either $v_i^{(1)} \in N_1(v_i)$ or $v_i^{(2)} \in N_2(v_i)$, not adjacent to v_j. Then either $v_i\, v_i^{(1)}\, v_j'$ or $v_i\, v_i^{(1)}\, v_i^{(2)}\, v_j'$ is geodesic in $D_c^*(G)$, where $v_i^{(1)} \neq v_j$ and $(v_i^{(1)}, v_i^{(2)}) \in E(G)$. Thus, $d_{Dc^*}(v_i, v_i') = 2$ or 3.

(b). Let $(v_i, v_j) \notin E(G)$, then $d_{Dc^*}(v_i, v_j') = 1$.

(ii)(a). Let $(v_i, v_j) \in E(G)$. Then either there exists a vertex w in $N_2(v_i)$ such that $d(v_j, w) \geq 2$ or all the vertices of $N_2(v_i)$ are adjacent to v_j. In the first case, $v_i'\, w\, v_j'$ is geodesic and hence, $d_{Dc^*}(v_i', v_j') = 2$. In the second case, G has a dominating edge. If w_1 is an eccentric point of v_j in $N_1(v_i)$ and is a predecessor of w, then $v_i'\, w\, w_1\, v_j'$ is geodesic and hence $d_{Dc^*}(v_i', v_j') = 3$.

(b). Let $(v_i, v_j) \notin E(G)$. Then $d(v_i, v_j) = 2$ or 3. If $e(v_i) = e(v_j) = 3$, then either v_j is an eccentric point of v_i or there exists a vertex v_k in G not adjacent to both v_i and v_j. In the first case, $d_{Dc^*}(v_i', v_j') = 3$, since $v_i'\, v_i^{(2)}\, v_i^{(1)}\, v_j'$ is geodesic in $D_c^*(G)$, where $(v_i^{(2)}, v_i^{(1)}) \in E(G)$ and in the second case, $d_{Dc^*}(v_i', v_j') = 2$. Let $e(v_i) = 2$, $e(v_j) = 3$. If there exist vertices $v_i^{(1)} \in N_1(v_i)$, $v_i^{(2)} \in N_2(v_i)$ with $(v_i^{(1)}, v_i^{(2)}) \in E(G)$ and $(v_i^{(1)}, v_j) \notin E(G)$, then $v_i'\, v_i^{(2)}\, v_i^{(1)}\, v_j'$ is geodesic in $D_c^*(G)$ and hence, $d_{Dc^*}(v_i', v_j') = 4$. If $e(v_i) = e(v_j) = 2$, then there exists a vertex v_k not adjacent to both v_i and v_j and hence $d_{Dc^*}(v_i', v_j') = 2$. Thus, it follows that $e_{Dc^*}(v_i) = 3$ and $e_{Dc^*}(v_i') = 3$ or 4. Hence, $D_c^*(G)$ has radius 3 and diameter 3 or 4.

Example 3.2: If $G \cong K_2 + K_1 + K_1 + K_2$, then G is bi-eccentric with radius 2 and $D_c^*(G)$ has radius 3 and diameter 4.

Corollary 3.13.1: Let G be bi-eccentric with radius 2 and $\delta(G) \geq 2$. Then $D_c^*(G)$ is self-centered with radius 3 if and only if one of the following holds.

(i). For every pair of non-adjacent vertices u, v in G there exists a vertex in G, not adjacent to both u and v; and

(ii). For every pair of vertices u, v in G, with $e(u) = 2$ and $e(v) = 3$, v is an eccentric point of u.

Observation 3.4: Let G be bi-eccentric with radius 2 and $\delta(G) = 1$.

(a). If G has a dominating edge, then $D_c*(G)$ has radius 3 and diameter 4 or 5. For example, $D_c*(P_4)$ has radius 3 and diameter 5, where P_4 is a path on four vertices and $D_c*(B)$ is bi-eccentric with radius 3, where B is the Bull graph.

(b). If G has no dominating edge and for every pair of non-adjacent vertices u, v in G there exists a vertex in G adjacent to both u and v, then $D_c*(G)$ is self-centered with radius 3. Otherwise, $D_c*(G)$ is bi-eccentric with radius 3.

Observation 3.5: let G be any graph radius 2 and diameter 4. Then $D_c*(G)$ is self-centered with radius 3 if and only if one of the following holds.

(i). For every pair of non-adjacent vertices u, v in G, there exists a vertex in G, not adjacent to both u and v; and

(ii). For every pair of vertices u, v in G with e(u) = 2 and e(v) = 3, v is an eccentric point of u.

Theorem 3.14: If G is any connected graph with $r(G) \geq 3$, then $D_c*(G)$ is self-centered with radius 3.

Proof: Assume G is a connected graph with $r(G) \geq 3$. Let $v_i, v_j \in V(G)$.

(i). If $d(v_i, v_j) \leq 3$, then, since G is an induced sub graph of $D_c*(G)$, $d_{Dc^*(G)}(v_i, v_j) \leq 3$. If $d(v_i, v_j) \geq 4$, then $d_{Dc^*}(v_i, v_j) = 2$.

(ii). Since $e_G(vi) \geq 3$, $d_{Dc^*}(v_i, v_i') = 3$.

(iii). $d_{Dc^*}(v_i, v_j') = 1$, if $(v_i, v_j) \notin E(G)$; and

$= 2$, if $(v_i, v_j) \in E(G)$.

(iv)(a). If $(v_i, v_j) \in E(G)$, then $d_{Dc^*}(v_i', v_j') = 2$, since $r(G) \geq 3$.

(b). Let $(v_i, v_j) \notin E(G)$. If $d_G(v_i, v_j) = 2$, then $d_{Dc^*}(v_i', v_j') = 2$ and if $d_G(v_i, v_j) \geq 3$, then $d_{Dc^*}(v_i', v_j') = 2$ or 3.

From (i), (ii), (iii) and (iv), it follows that $e_{Dc^*}(v_i) = e_{Dc^*}(v_i') = 3$ for all $v_i \in V(G)$. Hence, $D_c*(G)$ is self-centered with radius 3.

Corollary 3.14.1: Let G be a disconnected graph. Then $D_c*(G)$ is self-centered with radius 3 if and only if one of the following is true.

(i). G contains at least three components.

(ii). If G contains exactly two components, then radius of at least one of the components is at least 2.

Corollary 3.14.2: Let G be a disconnected graph with exactly two components, at least one of them is not complete. Then $D_c*(G)$ is bi-eccentric with radius 3 if and only if both the components have radius one.

Note 3.2: The complement $\overline{D_c^*(G)}$ of $D_c^*(G)$ is self-centered with radius 2, if one of the following holds.

(i). G is self-centered with radius 2;

(ii). $\delta(G) \geq 2$ and G is bi-eccentric with radius 2; and

(iii). G is connected and $r(G) \geq 3$.

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