International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 2, No. 4, October –December 2011, pp.162- 197

Development of New Fifth-Order Fifth-Stage Runge Kutta Method based On Heronian Mean

R.Ponalagusamy[1], P.J.A. Alphonse[2] and M.Chandru[3]

[1]*Department of Mathematics, National Institute of Technology, Triuchirapalli, India*

E-Mail: rpalagu@nitt.edu

[2, 3]*Department of Computer Applications, National Institute of Technology, Triuchirapalli, India*

E-Mail: alphonse@nitt.edu, leochandru@gmail.com

Abstract: *The aim of this paper is focused on developing a new fifth order fifth-stage Runge-Kutta method based on Heronian Mean to compute numerical solutions of Initial Value Problems (IVPs) in ODEs. The stability polynomial and the stability region of the new formula are obtained. A modest effort has been taken to examine the suitability, adoptability and accuracy of the method. Based on the numerical results, it is observed that the presently developed new method is superior compared to some existing methods including fifth order Runge-Kutta methods based on Harmonic Mean and Contra Harmonic Mean.*

Key words: *Runge-Kutta, Heronian Mean, Fifth order, Stability Region, Stability Polynomial.*

1 Introduction

It is well known that most of the Initial Value Problems(IVPs) are solved by Runge-Kutta methods which in turn being applied to compute numerical solutions for variety of problems that are modeled as the differential equations and their systems(Alexander and Coyle[1], Evans[2] , Hung[3] and Shampine and Gorden[4]). Runge-Kutta algorithms are used to solve differential equations efficiently that are equivalent to approximate the exact solutions by matching 'n' terms of the Taylor series expansion. Shampine and Watts [5], [6] and [7] have developed mathematical codes for the Runge-Kutta fourth order method. Ponalagusamy and Senthilkumar [8, 9] have successfully demonstrated the applicability of Runge-Kutta methods to investigate the robot arm problem and the problem of multilayer rater CNN simulation.

In order to compute highly accurate numerical solutions, Ponalagusamy and Senthilkumar [10] have introduced a new embedded Runge-Kutta RK (4, 4) method which is actually two different RK methods but of the same order p = 4. This embedded method has been developed using Runge-Kutta methods based on arithmetic mean (RKAM) and Heronian mean (RKHeM). Ponalagusamy and Senthilkumar [11] introduced embedded fourth order Runge-Kutta root mean square (RKARMS) to

Received: 22 December, 2010; Revised: 25 February, 2011; Accepted: 9 April, 2011

invesgate raster CNN simulation. Evans and Yaakub [12] introduced embedded fourth order contraharmonic mean and Yaacob and Sanugi [13] adopted embedded fourth order harmonic mean.

Butcher [14] has developed Runge-Kutta formula of fifth order. A new fifth order Runge-Kutta RK (5, 5) method with error control has been introduced by Evans and Yaakub [15]. They computed numerical results and compared with other well known methods RKF (4, 5) and RK (4, 5) Merson. Razali et al.[16] have applied the fifth order Runge-Kutta method to investigate the problem of Lorenz system. Evans and Yaakub [17] computed approximate solutions of several types of differential equations using fifth order Runge-Kutta method based on contraharmonic mean. It is of importance to pin point out here that the errors involved in the numerical solutions of ordinary differential equations computed by using fifth order Runge-Kutta methods based on harmonic mean, contraharmonic mean ([17], [18]) are found to be high in comparison with that of heronian mean. In view of this, a modest effort has been made to develop a new fifth order Runge-Kutta method based on heronian mean which out performs well as compared to the exiting fifth order Runge-Kutta methods based on various means ([17], [18]).

2 A New Fifth Order Heronian Mean Runge-Kutta Formula

The standard fifth order Heronian Mean (HeM) Runge-Kutta formula for solving initial value problems of the form $y' = f(x, y)$ may be written as follows:

$$y_{n+1} = y_n + \frac{h}{14}\left[k_1 + 2\left(k_2 + k_3\right) + 2\left(k_3 + k_4\right) + k_5 + \sqrt{\left|k_1 k_2\right|} + \sqrt{\left|k_2 k_3\right|} + \sqrt{\left|k_3 k_4\right|} + \sqrt{\left|k_4 k_5\right|} \right] \tag{2.1}$$

where,

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + a_1 h, y_n + a_1 h k_1)$$

$$k_3 = f(x_n + (a_2 + a_3)h, y_n + a_2 h k_1 + a_3 h k_2)$$

$$k_4 = f(x_n + (a_4 + a_5 + a_6)h, y_n + a_4 h k_1 + a_5 h k_2 + a_6 h k_3)$$

$$k_5 = f(x_n + (a_7 + a_8 + a_9 + a_{10})h, y_n + a_7 h k_1 + a_8 h k_2 + a_9 h k_3 + a_{10} h k_4) \tag{2.2}$$

and the parameters $a_1, a_2, \ldots, a_{10}$ are to be determined. It is to be noticed that for simplicity of the algebra the function f is considered as a function of y only, without loss of generality. Taylor series expansion of an exact solution $y(x_n + h)$ up to sixth order is given by

$$
\begin{aligned}
y(x_n + h) = {} & y_n + hf + \frac{1}{2}h^2 f f_y + \frac{1}{6}h^3 (f f_y^2 + f^2 f_{yy}) + \frac{1}{24}h^4 (f^3 f_{yyy} + 4f^2 f_y y_{yy} \\
& + f f_y^3) + \frac{1}{120}h^5 (f f_y^4 + 11 f^2 f_y^2 f_{yy} + 4 f^3 f_y^2 f_{yy} + 7 f^3 f_y f_{yyy} \\
& + f^4 f_{yyyy}) + \frac{1}{720}h^6 (f^5 f_{yyyyy} + 11 f^4 f_y f_{yyyy} + 15 f^4 f_{yy} f_{yyy} \\
& + 32 f^3 f_y^2 f_{yyy} + 34 f^3 f_y f_{yy}^2 + 26 f^2 f_y^3 f_{yy} + f f_y^5) + O(h^7)
\end{aligned} \tag{2.3}
$$

Expanding k1, k_2, k_3, k_4 and k_5 in Taylor series about x_n and substituting in equation (1) and comparing the coefficients of the same in equation (2.3), we can obtain 11 equations with 10 parameters. The equations are given below.

$h^2 f f_y$:

$$
\frac{3a_1}{14} + \frac{3a_{10}}{28} + \frac{5a_2}{14} + \frac{5a_3}{14} + \frac{3a_4}{14} + \frac{3a_5}{14} + \frac{3a_6}{14} + \frac{3a_7}{28} + \frac{3a_8}{28} + \frac{3a_9}{28} = \frac{1}{2} \tag{2.4 –i}
$$

$h^3 f f_y^2$:

$$
\begin{aligned}
& -\frac{3a_1^2}{56} + \frac{a_{10}^2}{112} + \frac{a_1 a_2}{56} - \frac{a_2^2}{56} + \frac{3a_1 a_3}{8} - \frac{a_2 a_3}{28} - \frac{a_3^2}{56} + \frac{a_{10} a_4}{8} + \frac{a_2 a_4}{56} + \frac{a_3 a_4}{56} - \frac{a_4^2}{56} \\
& + \frac{3a_1 a_5}{14} + \frac{a_{10} a_5}{8} + \frac{a_2 a_5}{56} + \frac{a_3 a_5}{56} - \frac{a_4 a_5}{28} - \frac{a_5^2}{56} + \frac{a_{10} a_6}{8} + \frac{13 a_2 a_6}{56} + \frac{13 a_3 a_6}{56} \\
& - \frac{a_4 a_6}{28} - \frac{a_5 a_6}{28} - \frac{a_6^2}{56} - \frac{a_{10} a_7}{56} + \frac{a_4 a_6}{56} + \frac{a_5 a_6}{56} + \frac{a_6 a_8}{56} - \frac{a_7 a_8}{56} - \frac{a_8^2}{112} - \frac{a_{10} a_9}{56} \\
& + \frac{3a_2 a_9}{28} + \frac{3a_3 a_9}{28} + \frac{a_4 a_9}{56} + \frac{a_5 a_9}{56} + \frac{a_6 a_9}{56} - \frac{a_7 a_9}{56} - \frac{a_8 a_9}{56} + \frac{a_9^2}{112} = \frac{1}{6}
\end{aligned} \tag{2.4 –ii}
$$

$h^3 f^2 f_{yy}$:

$$\frac{3a_1^2}{28}+\frac{3a_{10}^2}{56}+\frac{5a_2^2}{28}+\frac{5a_2a_3}{14}+\frac{5a_3^2}{28}+\frac{3a_4^2}{28}+\frac{3a_4a_5}{14}+\frac{3a_5^2}{28}+\frac{3a_4a_6}{14}+\frac{3a_5a_6}{14}+\frac{3a_6^2}{28} \quad (2.4\text{ –iii})$$
$$+\frac{3a_{10}a_7}{28}+\frac{3a_7^2}{56}+\frac{3a_{10}a_8}{28}+\frac{3a_7a_8}{28}+\frac{3a_8^2}{56}+\frac{3a_{10}a_9}{28}+\frac{3a_7a_9}{28}+\frac{3a_8a_9}{28}+\frac{3a_9^2}{56}=\frac{1}{56}$$

$h^4 f^3 f_{yyy}$:

$$\frac{a_1^3}{28}+\frac{a_{10}^3}{56}+\frac{5a_2^3}{28}+\frac{5a_2^2a_3}{28}+\frac{5a_2a_3^2}{28}+\frac{5a_3^3}{84}+\frac{a_4^3}{28}+\frac{3a_4^2a_5}{28}+\frac{3a_4a_5^2}{28}+\frac{a_5^3}{28}+\frac{3a_4^2a_6}{28}$$
$$+\frac{3a_4a_5a_6}{14}+\frac{3a_5^2a_6}{28}+\frac{3a_4a_6^2}{28}+\frac{3a_5a_6^2}{28}+\frac{a_6^3}{28}+\frac{3a_{10}a_7^2}{56}+\frac{a_7^3}{56}+\frac{3a_{10}^2a_8}{56} \quad (2.4\text{ –iv})$$
$$+\frac{3a_{10}a_7a_8}{28}+\frac{3a_7^2a_8}{56}+\frac{3a_{10}a_8^2}{56}+\frac{3a_7a_8^2}{56}+\frac{a_8^3}{56}+\frac{3a_{10}^2a_9}{56}+\frac{3a_{10}a_7a_9}{28}+\frac{3a_7^2a_9}{56}$$
$$+\frac{3a_{10}a_8a_9}{56}+\frac{3a_7a_8a_9}{28}+\frac{3a_8^2a_9}{56}+\frac{3a_{10}a_9^2}{56}+\frac{3a_7a_9^2}{56}+\frac{3a_8a_9^2}{56}+\frac{a_9^3}{56}=\frac{1}{24}$$

$h^4 f^2 f_y f_{yy}$:

$$\frac{-a_1^3}{56}-\frac{a_{10}^3}{112}+\frac{a_1^2a_2}{112}+\frac{a_1a_2^2}{112}-\frac{a_2^3}{56}+\frac{3a_1^2a_3}{16}+\frac{3a_1a_2a_3}{8}-\frac{3a_2^2a_3}{56}+\frac{41a_1a_3^2}{112}-\frac{3a_2a_3^2}{56}$$
$$-\frac{a_3^3}{56}+\frac{13a_{10}^2a_4}{112}+\frac{a_2^2a_4}{112}+\frac{a_2a_3a_4}{56}+\frac{a_3^2a_4}{112}+\frac{a_{10}a_4^2}{16}+\frac{a_2a_4^2}{112}+\frac{a_3a_4^2}{112}-\frac{a_4^3}{56}+\frac{3a_1^2a_5}{28}$$

$$
\begin{aligned}
&+\frac{13a_{10}^2a_5}{112}+\frac{a_2^2a_5}{112}+\frac{a_2a_3a_5}{56}+\frac{a_3^2a_5}{112}+\frac{3a_1a_4a_5}{14}+\frac{a_{10}a_4a_5}{8}+\frac{a_2a_4a_5}{56}+\frac{a_3a_4a_5}{56}\\
&-\frac{3a_4^2a_5}{56}+\frac{3a_1a_5^2}{14}+\frac{a_{10}a_5^2}{16}+\frac{a_3a_5^2}{112}-\frac{3a_4a_5^2}{56}-\frac{a_5^3}{56}+\frac{13a_{10}^2a_6}{112}+\frac{13a_2^2a_6}{112}+\frac{13a_2a_3a_6}{56}\\
&+\frac{a_{10}a_4a_6}{8}+\frac{13a_2a_4a_6}{56}+\frac{13a_3a_4a_6}{56}-\frac{3a_4^2a_6}{56}+\frac{3a_1a_5a_6}{14}+\frac{a_{10}a_5a_6}{8}+\frac{13a_2a_5a_6}{56}\\
&+\frac{13a_3a_5a_6}{56}-\frac{3a_4a_5a_6}{28}-\frac{3a_5^2a_6}{56}+\frac{a_{10}a_6^2}{16}+\frac{25a_2a_6^2}{112}+\frac{25a_3a_6^2}{112}-\frac{3a_4a_6^2}{56}-\frac{3a_5a_6^2}{56}\\
&-\frac{a_6^3}{56}-\frac{3a_{10}^2a_7}{112}+\frac{a_{10}a_4a_7}{8}+\frac{a_4^2a_7}{112}+\frac{a_{10}a_5a_7}{8}+\frac{a_4a_5a_7}{56}+\frac{a_5^2a_7}{112}+\frac{a_{10}a_6a_7}{8}\\
&+\frac{a_4a_6a_7}{56}+\frac{a_5a_6a_7}{56}+\frac{a_6^2a_7}{112}-\frac{3a_{10}a_7^2}{112}-\frac{a_4a_7^2}{112}+\frac{a_5a_7^2}{112}+\frac{a_6a_7^2}{112}-\frac{a_7^3}{112}+\frac{3a_1^2a_8}{56}\\
&+\frac{3a_1a_{10}a_8}{28}-\frac{3a_{10}^2a_8}{112}+\frac{a_{10}a_4a_8}{8}+\frac{a_4^2a_8}{112}+\frac{a_{10}a_5a_8}{8}+\frac{a_4a_5a_8}{56}+\frac{a_5^2a_8}{112}\\
&+\frac{a_{10}a_6a_8}{8}+\frac{a_4a_6a_8}{56}+\frac{a_5a_6a_8}{56}+\frac{a_6^2a_8}{112}+\frac{3a_1a_7a_8}{28}-\frac{3a_{10}a_7a_8}{56}+\frac{a_4a_7a_8}{56}\\
&+\frac{a_5a_7a_8}{56}+\frac{a_6a_7a_8}{56}-\frac{3a_7^2a_8}{112}+\frac{3a_1a_8^2}{28}-\frac{3a_{10}a_8^2}{112}+\frac{a_4a_8^2}{112}+\frac{a_5a_8^2}{112}+\frac{a_6a_8^2}{112}-\frac{3a_7a_8^2}{112}\\
&-\frac{a_8^3}{112}-\frac{3a_{10}^2a_9}{112}+\frac{3a_{10}a_2a_9}{28}+\frac{3a_2^2a_9}{56}+\frac{3a_{10}a_3a_9}{28}+\frac{3a_2a_3a_9}{28}+\frac{3a_3^2a_9}{56}+\frac{a_{10}a_4a_9}{8}\\
&+\frac{a_4^2a_9}{112}+\frac{a_{10}a_5a_9}{8}+\frac{a_4a_5a_9}{56}+\frac{a_5^2a_9}{112}+\frac{a_{10}a_6a_9}{8}+\frac{a_4a_6a_9}{56}+\frac{a_5a_6a_9}{56}+\frac{a_6^2a_9}{112}\\
&-\frac{3a_{10}a_7a_9}{56}+\frac{3a_2a_7a_9}{28}-\frac{3a_7a_8a_9}{56}-\frac{3a_8^2a_9}{112}-\frac{3a_{10}a_9^2}{112}+\frac{3a_2a_9^2}{28}+\frac{3a_3a_9^2}{28}+\frac{a_4a_9^2}{112}+\frac{a_5a_9^2}{112}\\
&+\frac{a_6a_9^2}{112}-\frac{3a_7a_9^2}{112}-\frac{3a_8a_9^2}{112}-\frac{a_9^3}{112}=\frac{1}{6}
\end{aligned}
\tag{2.4 –v}
$$

$h^4 f f_y^3$:

$$-\frac{3a_{10}a_7^2}{224}-\frac{a_4a_7^2}{224}-\frac{a_5a_7^2}{224}-\frac{a_6a_7^2}{224}+\frac{a_7^3}{224}-\frac{a_1a_{10}a_8}{56}+\frac{3a_{10}^2a_8}{224}+\frac{a_1a_4a_8}{56}$$

$$-\frac{3a_{10}a_{14}a_8}{112}-\frac{a_4^2a_8}{224}+\frac{a_1a_5a_8}{28}-\frac{a_3a_6a_8}{56}-\frac{a_4a_6a_8}{56}-\frac{a_5a_6a_8}{112}-\frac{a_6^2a_8}{224}-\frac{a_1a_7a_8}{56}$$

$$+\frac{3a_{10}a_7a_8}{112}-\frac{a_4a_7a_8}{112}-\frac{a_5a_7a_8}{112}-\frac{a_6a_7a_8}{112}+\frac{3a_7^2a_8}{228}-\frac{a_1a_8^2}{56}+\frac{3a_{10}a_8^2}{224}-\frac{a_4a_8^2}{224}$$

$$-\frac{a_5a_8^2}{224}-\frac{a_6a_8^2}{224}+\frac{3a_7a_8^2}{224}+\frac{a_8^3}{224}+\frac{a_{10}^2a_9}{224}-\frac{a_{10}a_2a_9}{56}+\frac{3a_1a_3a_9}{28}-\frac{a_{10}a_3a_9}{56}$$

$$-\frac{3a_{10}a_4a_9}{112}+\frac{a_2a_4a_9}{56}+\frac{a_3a_4a_9}{56}-\frac{a_4^2a_9}{224}+\frac{a_1a_5a_9}{56}-\frac{3a_{10}a_5a_9}{112}+\frac{a_2a_5a_9}{56}$$

$$+\frac{a_3a_5a_9}{56}-\frac{a_4a_5a_9}{112}-\frac{a_5^2a_9}{224}-\frac{3a_{10}a_6a_9}{112}+\frac{a_2a_6a_9}{28}+\frac{a_3a_6a_9}{28}-\frac{a_4a_6a_9}{112}$$

$$-\frac{a_5a_6a_9}{112}-\frac{a_6^2a_9}{224}\frac{3a_{10}a_7a_9}{112}-\frac{a_2a_7a_9}{56}-\frac{a_3a_7a_9}{56}-\frac{a_4a_7a_9}{112}-\frac{a_5a_7a_9}{112}-\frac{a_6a_7a_9}{112}$$

$$+\frac{3a_7^2a_9}{224}-\frac{a_1a_8a_9}{56}+\frac{3a_{10}a_8a_9}{112}-\frac{a_2a_8a_9}{56}-\frac{a_3a_8a_9}{56}-\frac{a_4a_8a_9}{112}-\frac{a_5a_8a_9}{112}$$

$$-\frac{a_6a_8a_9}{112}+\frac{3a_7a_8a_9}{112}+\frac{3a_8^2a_9}{224}+\frac{3a_{10}a_9^2}{224}-\frac{a_2a_9^2}{56}-\frac{a_3a_9^2}{56}-\frac{a_4a_9^2}{224}-\frac{a_6a_9^2}{224}$$

$$+\frac{3a_7a_9^2}{224}+\frac{3a_8a_9^2}{224}+\frac{a_9^3}{224}=\frac{1}{24} \tag{2.4 –vi}$$

$h^5 f f_y^4$:

$$-\frac{5a_4^3a_5}{224}-\frac{a_1^2a_5^2}{56}+\frac{3}{112}a_1a_{10}a_5^2-\frac{15a_{10}^2a_5^2}{896}-\frac{1}{112}a_1a_2a_5^2+\frac{a_2^2a_5^2}{896}-\frac{3}{224}a_1a_3a_5^2$$

$$+\frac{1}{448}a_2a_3a_5^2+\frac{a_3^2a_5^2}{896}+\frac{3}{56}a_1a_4a_5^2-\frac{3}{448}a_{10}a_4a_5^2+\frac{3}{448}a_2a_4a_5^2+\frac{3}{448}a_3a_4a_5^2$$

$$-\frac{15a_4^2a_5^2}{448}+\frac{3a_1a_5^3}{112}-\frac{a_{10}a_5^3}{448}+\frac{a_2a_5^3}{448}+\frac{a_3a_5^3}{448}-\frac{5a_4a_5^3}{224}-\frac{5a_5^4}{896}+\frac{a_{10}^3a_6}{64}$$

$$-\frac{5}{224}a_{10}^2a_2a_6-\frac{a_2^3a_6}{448}+\frac{1}{8}a_1a_{10}a_3a_6-\frac{5}{224}a_{10}^2a_3a_6+\frac{3}{112}a_1a_2a_3a_6$$

$$-\frac{3}{448}a_2^2a_3a_6+\frac{3}{112}a_1a_3^2a_6-\frac{3}{448}a_2a_3^2a_6-\frac{a_3^3a_6}{448}-\frac{15}{448}a_{10}^2a_4a_6$$

$$+\frac{3}{112}a_{10}a_2a_4a_6-\frac{3}{448}a_2^2a_4a_6-\frac{5}{112}a_1a_3a_4a_6+\frac{3}{112}a_{10}a_3a_4a_6$$

$$-\frac{3}{224}a_2a_3a_4a_6-\frac{3}{448}a_3^2a_4a_6-\frac{3}{448}a_{10}a_4^2a_6+\frac{15}{448}a_2a_4^2a_6+\frac{15}{448}a_3a_4^2a_6$$

$$-\frac{5a_4^3a_6}{224}+\frac{3}{112}a_1a_{10}a_5a_6-\frac{15}{448}a_{10}^2a_5a_6-\frac{5}{112}a_1a_2a_5a_6+\frac{3}{112}a_{10}a_2a_5a_6$$

$$-\frac{3}{448}a_2^2a_5a_6-\frac{5}{56}a_1a_3a_5a_6+\frac{3}{112}a_{10}a_3a_5a_6-\frac{3}{224}a_2a_3a_5a_6-\frac{3}{448}a_3^2a_5a_6$$

$$+\frac{3}{56}a_1a_4a_5a_6-\frac{3}{224}a_{10}a_4a_5a_6+\frac{15}{224}a_2a_4a_5a_6+\frac{15}{224}a_3a_4a_5a_6$$

$$-\frac{15}{224}a_4^2a_5a_6+\frac{3}{56}a_1a_5^2a_6-\frac{3}{448}a_{10}a_5^2a_6+\frac{15}{448}a_2a_5^2a_6+\frac{15}{448}a_3a_5^2a_6$$

$$-\frac{15}{224}a_4a_5^2a_6-\frac{5a_5^3a_6}{224}-\frac{15a_{10}^2a_6^2}{896}+\frac{3}{112}a_{10}a_2a_6^2-\frac{23a_2^2a_6^2}{896}-\frac{9}{224}a_1a_3a_6^2$$

$$+\frac{3}{112}a_{10}a_3a_6^2-\frac{23}{448}a_2a_3a_6^2-\frac{23a_3^2a_6^2}{896}-\frac{3}{448}a_{10}a_4a_6^2+\frac{27}{448}a_2a_4a_6^2$$

$$+\frac{27}{448}a_3a_4a_6^2-\frac{15a_4^2a_6^2}{448}+\frac{3}{112}a_1a_5a_6^2-\frac{3}{448}a_{10}a_5a_6^2+\frac{27}{448}a_2a_5a_6^2$$

$$+\frac{27}{448}a_3a_5a_6^2-\frac{15}{224}a_4a_5a_6^2-\frac{15a_5^2a_6^2}{448}-\frac{a_{10}a_6^3}{448}+\frac{13a_2a_6^3}{448}+\frac{13a_3a_6^3}{448}$$

$$-\frac{5a_4a_6^3}{224}-\frac{5a_5a_6^3}{224}-\frac{5a_6^4}{896}-\frac{5a_{10}^3a_7}{448}+\frac{15}{448}a_{10}^2a_4a_7-\frac{3}{448}a_{10}a_4^2a_7+\frac{a_4^3a_7}{448}$$

$$-\frac{3}{112}a_1a_{10}a_5a_7+\frac{15}{448}a_{10}^2a_5a_7-\frac{1}{112}a_1a_4a_5a_7-\frac{3}{224}a_{10}a_4a_5a_7$$

$$+\frac{3}{448}a_4^2a_5a_7-\frac{1}{112}a_1a_5^2a_7-\frac{3}{448}a_{10}a_5^2a_7+\frac{3}{448}a_4a_5^2a_7+\frac{a_5^3a_7}{448}$$

$$+\frac{15}{448}a_{10}^2a_6a_7-\frac{3}{112}a_{10}a_2a_6a_7-\frac{3}{112}a_{10}a_3a_6a_7-\frac{3}{224}a_{10}a_4a_6a_7$$

$$-\frac{1}{112}a_2a_4a_6a_7-\frac{1}{112}a_3a_4a_6a_7+\frac{3}{448}a_4^2a_6a_7-\frac{1}{112}a_1a_5a_6a_7$$

$$-\frac{3}{224}a_{10}a_5a_6a_7-\frac{1}{112}a_2a_5a_6a_7-\frac{1}{112}a_3a_5a_6a_7+\frac{3}{224}a_4a_5a_6a_7$$

$$-\frac{3}{448}a_{10}a_6^2a_7-\frac{1}{112}a_2a_6^2a_7-\frac{1}{112}a_3a_6^2a_7+\frac{3}{448}a_4a_6^2a_7+\frac{3}{448}a_5a_6^2a_7$$

$$+\frac{a_6^3a_7}{448}-\frac{15a_{10}^2a_7^2}{896}+\frac{9}{448}a_{10}a_4a_7^2+\frac{a_4^2a_7^2}{896}-\frac{1}{224}a_1a_5a_7^2+\frac{9}{448}a_{10}a_5a_7^2$$

$$+\frac{9}{448}a_4a_5a_7^2+\frac{a_5^2a_7^2}{896}+\frac{9}{448}a_{10}a_6a_7^2-\frac{1}{224}a_2a_6a_7^2-\frac{1}{224}a_3a_6a_7^2$$

$$+\frac{1}{448}a_4a_6a_7^2+\frac{1}{448}a_4a_6a_7^2+\frac{a_6^2a_7^2}{896}-\frac{5a_{10}a_7^3}{448}+\frac{a_4a_7^3}{448}+\frac{a_5a_7^3}{448}+\frac{a_6a_7^3}{448}-\frac{5a_7^4}{1792}$$

$$+\frac{3}{224}a_1a_{10}^2a_8-\frac{5a_{10}^3a_8}{448}-\frac{3}{112}a_1a_{10}a_4a_8+\frac{15}{448}a_{10}^2a_4a_8-\frac{1}{224}a_1a_4^2a_8$$

$$-\frac{3}{448}a_{10}a_4^2a_8+\frac{a_4^3a_8}{448}+\frac{1}{56}a_1^2a_5a_8-\frac{3}{56}a_1a_{10}a_5a_8+\frac{15}{448}a_{10}^2a_5a_8$$

$$-\frac{1}{56}a_1a_4a_5a_8-\frac{3}{224}a_{10}a_4a_5a_8+\frac{3}{448}a_4^2a_5a_8-\frac{3}{224}a_1a_5^2a_8-\frac{3}{448}a_{10}a_5^2a_8$$

$$+\frac{3}{448}a_4a_5^2a_8+\frac{a_5^3a_8}{448}-\frac{3}{112}a_1a_{10}a_6a_8+\frac{15}{448}a_{10}^2a_4a_8+\frac{1}{56}a_1a_2a_6a_8$$

$$-\frac{3}{112}a_{10}a_2a_6a_8+\frac{1}{28}a_1a_3a_6a_8-\frac{3}{112}a_{10}a_3a_6a_8-\frac{1}{112}a_1a_4a_6a_8$$

$$-\frac{3}{224}a_{10}a_4a_6a_8-\frac{1}{112}a_2a_4a_6a_8-\frac{1}{112}a_3a_4a_6a_8+\frac{3}{448}a_4^2a_6a_8$$

$$\frac{1}{56}a_1a_5a_6a_8-\frac{3}{224}a_{10}a_5a_6a_8-\frac{1}{112}a_2a_5a_6a_8-\frac{1}{112}a_3a_5a_6a_8$$

$$+\frac{3}{224}a_4a_5a_6a_8+\frac{3}{448}a_5^2a_6a_8-\frac{1}{224}a_1a_6^2a_8-\frac{3}{448}a_{10}a_6^2a_8-\frac{1}{112}a_2a_6^2a_8$$

$$-\frac{1}{112}a_3a_6^2a_8+\frac{3}{448}a_4a_6^2a_8+\frac{3}{448}a_5a_6^2a_8+\frac{a_6^3a_8}{448}+\frac{3}{112}a_1a_{10}a_7a_8$$

$$-\frac{15}{448}a_{10}^2a_7a_8-\frac{1}{112}a_1a_4a_7a_8+\frac{9}{224}a_{10}a_4a_7a_8+\frac{1}{448}a_4^2a_7a_8-\frac{1}{56}a_1a_5a_7a_8$$

$$+\frac{9}{224}a_{10}a_5a_7a_8+\frac{1}{224}a_4a_5a_7a_8+\frac{1}{448}a_5^2a_7a_8-\frac{1}{112}a_1a_6a_7a_8$$

$$+\frac{9}{224}a_{10}a_6a_7a_8-\frac{1}{112}a_2a_6a_7a_8-\frac{1}{112}a_3a_6a_7a_8+\frac{1}{224}a_4a_6a_7a_8$$

$$+\frac{1}{224}a_5a_6a_7a_8+\frac{1}{448}a_6^2a_7a_8+\frac{3}{224}a_1a_7^2a_8-\frac{15}{448}a_{10}a_7^2a_8+\frac{3}{448}a_4a_7^2a_8$$

$$+\frac{3}{448}a_5a_7^2a_8+\frac{3}{448}a_6a_7^2a_8-\frac{5a_7^3a_8}{448}-\frac{a_1^2a_8^2}{112}+\frac{3}{112}a_1a_{10}a_8^2-\frac{15a_{10}^2a_8^2}{896}$$

$$-\frac{1}{112}a_1a_4a_8^2+\frac{9}{448}a_{10}a_4a_8^2+\frac{a_4^2a_8^2}{896}-\frac{3}{224}a_1a_5a_8^2+\frac{9}{448}a_{10}a_5a_8^2$$

$$+\frac{1}{448}a_4a_5a_8^2+\frac{a_5^2a_8^2}{896}-\frac{1}{112}a_1a_6a_8^2+\frac{9}{448}a_{10}a_6a_8^2-\frac{1}{224}a_2a_6a_8^2$$

$$-\frac{1}{224}a_3a_6a_8^2+\frac{1}{448}a_4a_6a_8^2+\frac{1}{448}a_5a_6a_8^2+\frac{a_6^2a_8^2}{896}+\frac{3}{112}a_1a_7a_8^2$$

$$-\frac{15}{448}a_{10}a_7a_8^2+\frac{3}{448}a_4a_7a_8^2+\frac{3}{448}a_5a_7a_8^2+\frac{3}{448}a_6a_7a_8^2-\frac{15a_7^2a_8^2}{896}$$

$$+\frac{3a_1 a_8^3}{224}-\frac{5a_{10} a_8^3}{448}+\frac{a_4 a_8^3}{448}+\frac{a_5 a_8^3}{448}+\frac{a_6 a_8^3}{448}-\frac{5a_7 a_8^3}{448}-\frac{5a_8^4}{1792}-\frac{5a_{10}^3 a_9}{448}$$

$$+\frac{3}{224}a_{10}^2 a_2 a_9-\frac{1}{56}a_1 a_{10} a_3 a_9+\frac{3}{224}a_{10}^2 a_3 a_9+\frac{15}{448}a_{10}^2 a_4 a_9$$

$$-\frac{3}{112}a_{10} a_2 a_4 a_9+\frac{1}{56}a_1 a_3 a_4 a_9-\frac{3}{112}a_{10} a_3 a_4 a_9-\frac{3}{448}a_{10} a_4^2 a_9$$

$$-\frac{1}{224}a_2 a_4^2 a_9-\frac{1}{224}a_3 a_4^2 a_9+\frac{a_4^3 a_9}{448}-\frac{3}{112}a_1 a_{10} a_5 a_9+\frac{15}{448}a_{10}^2 a_5 a_9$$

$$+\frac{1}{56}a_1 a_2 a_5 a_9-\frac{3}{112}a_{10} a_2 a_5 a_9+\frac{1}{28}a_1 a_3 a_5 a_9-\frac{3}{112}a_{10} a_3 a_5 a_9$$

$$-\frac{1}{112}a_1 a_4 a_5 a_9-\frac{3}{224}a_{10} a_4 a_5 a_9-\frac{1}{112}a_2 a_4 a_5 a_9-\frac{1}{112}a_3 a_4 a_5 a_9$$

$$+\frac{3}{448}a_4^2 a_5 a_9-\frac{1}{112}a_1 a_5^2 a_9-\frac{3}{448}a_{10} a_5^2 a_9-\frac{1}{224}a_2 a_5^2 a_9-\frac{1}{224}a_3 a_5^2 a_9$$

$$+\frac{3}{448}a_4 a_5^2 a_9+\frac{a_5^3 a_9}{448}+\frac{15}{448}a_{10}^2 a_6 a_9-\frac{3}{56}a_{10} a_2 a_6 a_9+\frac{1}{56}a_2^2 a_6 a_9$$

$$+\frac{1}{28}a_1 a_3 a_6 a_9-\frac{3}{56}a_{10} a_3 a_6 a_9+\frac{1}{28}a_2 a_3 a_6 a_9+\frac{1}{56}a_3^2 a_6 a_9-\frac{3}{224}a_{10} a_4 a_6 a_9$$

$$-\frac{1}{56}a_2 a_4 a_6 a_9-\frac{1}{56}a_3 a_4 a_6 a_9+\frac{3}{448}a_4^2 a_6 a_9-\frac{1}{112}a_1 a_5 a_6 a_9$$

$$-\frac{3}{224}a_{10} a_5 a_6 a_9-\frac{1}{56}a_2 a_5 a_6 a_9-\frac{1}{56}a_3 a_5 a_6 a_9+\frac{3}{224}a_4 a_5 a_6 a_9$$

$$-\frac{3}{448}a_5^2 a_6 a_9-\frac{3}{448}a_{10} a_6^2 a_9-\frac{3}{224}a_2 a_6^2 a_9-\frac{3}{224}a_3 a_6^2 a_9$$

$$+\frac{3}{448}a_4 a_6^2 a_9+\frac{3}{448}a_5 a_6^2 a_9+\frac{a_6^3 a_9}{448}-\frac{15}{448}a_{10}^2 a_7 a_9+\frac{3}{112}a_{10} a_2 a_7 a_9$$

$$-\frac{1}{56}a_1 a_3 a_7 a_9+\frac{3}{112}a_{10} a_3 a_7 a_9+\frac{9}{224}a_{10} a_4 a_7 a_9-\frac{1}{112}a_2 a_4 a_7 a_9$$

$$-\frac{1}{112}a_3 a_4 a_7 a_9+\frac{1}{448}a_4^2 a_7 a_9-\frac{1}{112}a_1 a_5 a_7 a_9+\frac{9}{224}a_{10} a_5 a_7 a_9$$

$$-\frac{1}{112}a_2 a_5 a_7 a_9-\frac{1}{112}a_3 a_5 a_7 a_9+\frac{1}{224}a_4 a_5 a_7 a_9+\frac{1}{448}a_5^2 a_7 a_9$$

$$+\frac{9}{224}a_{10}a_6a_7a_9-\frac{1}{56}a_2a_6a_7a_9-\frac{1}{56}a_3a_6a_7a_9+\frac{1}{224}a_4a_6a_7a_9$$
$$+\frac{1}{224}a_5a_6a_7a_9+\frac{1}{448}a_6^2a_7a_9-\frac{15}{448}a_{10}a_7^2a_9+\frac{3}{224}a_2a_7^2a_9+\frac{3}{224}a_3a_7^2a_9$$
$$+\frac{3}{448}a_4a_7^2a_9+\frac{3}{448}a_5a_7^2a_9+\frac{3}{448}a_6a_7^2a_9-\frac{5a_7^3a_9}{448}+\frac{3}{112}a_1a_{10}a_8a_9$$
$$-\frac{15}{448}a_{10}^2a_8a_9-\frac{1}{56}a_1a_2a_8a_9+\frac{3}{112}a_{10}a_2a_8a_9-\frac{1}{28}a_1a_3a_8a_9$$
$$+\frac{3}{112}a_{10}a_3a_8a_9-\frac{1}{112}a_1a_4a_8a_9+\frac{9}{224}a_{10}a_4a_8a_9-\frac{1}{112}a_2a_4a_8a_9$$
$$-\frac{1}{112}a_3a_4a_8a_9+\frac{1}{448}a_4^2a_8a_9-\frac{1}{56}a_1a_5a_8a_9+\frac{9}{224}a_{10}a_5a_8a_9+\frac{a_5^2a_9^2}{896}$$
$$+\frac{9}{448}a_{10}a_6a_9^2-\frac{3}{224}a_2a_6a_9^2-\frac{3}{224}a_3a_6a_9^2+\frac{1}{448}a_4a_6a_9^2+\frac{1}{448}a_5a_6a_9^2$$
$$+\frac{a_6^2a_9^2}{896}-\frac{15}{448}a_{10}a_7a_9^2+\frac{3}{112}a_2a_7a_9^2+\frac{3}{112}a_3a_7a_9^2+\frac{3}{448}a_4a_7a_9^2$$
$$+\frac{3}{448}a_5a_7a_9^2+\frac{3}{448}a_6a_7a_9^2-\frac{15a_7^2a_9^2}{896}+\frac{3}{224}a_1a_8a_9^2-\frac{15}{448}a_{10}a_8a_9^2$$
$$+\frac{3}{112}a_2a_8a_9^2+\frac{3}{112}a_3a_8a_9^2+\frac{3}{448}a_4a_8a_9^2+\frac{3}{448}a_5a_8a_9^2+\frac{3}{448}a_6a_8a_9^2$$
$$-\frac{15}{448}a_7a_8a_9^2-\frac{15a_8^2a_9^2}{896}-\frac{5a_{10}a_9^3}{448}+\frac{3a_2a_9^3}{224}+\frac{3a_3a_9^3}{224}+\frac{a_4a_9^3}{448}+\frac{a_5a_9^3}{448}$$
$$+\frac{a_6a_9^3}{448}-\frac{5a_7a_9^3}{448}-\frac{5a_9^4}{1792}=\frac{1}{120}$$

(2.4 –vii)

$h^5f^2f_y^2f_{yy}$:

$$\frac{3a_1^4}{224}+\frac{3a_{10}^4}{448}-\frac{a_1^3a_2}{224}-\frac{a_1^2a_2^2}{224}-\frac{a_1a_2^3}{224}+\frac{3a_2^4}{224}+\frac{3a_1^3a_3}{224}-\frac{1}{112}a_1^2a_2a_3$$

$$-\frac{15}{224}a_1a_2^2a_3+\frac{3a_2^3a_3}{56}+\frac{39a_1^2a_3^2}{224}-\frac{27}{224}a_1a_2a_3^2+\frac{9a_2^2a_3^2}{112}-\frac{13a_1a_3^3}{224}+\frac{3a_2a_3^3}{56}$$

$$+\frac{3a_3^4}{224}-\frac{a_{10}^3a_4}{32}-\frac{a_2^3a_4}{224}+\frac{1}{112}a_1^2a_3a_4+\frac{1}{56}a_1a_2a_3a_4-\frac{3}{224}a_2^2a_3a_4$$

$$+\frac{1}{56}a_1a_3^2a_4-\frac{3}{224}a_2a_3^2a_4-\frac{a_3^3a_4}{224}+\frac{13a_{10}^2a_4^2}{224}-\frac{a_2^2a_4^2}{224}+\frac{1}{112}a_1a_3a_4^2$$

$$-\frac{1}{112}a_2a_3a_4^2-\frac{a_3^2a_4^2}{224}+\frac{3a_{10}a_4^3}{224}-\frac{a_2a_4^3}{224}-\frac{a_3a_4^3}{224}+\frac{3a_4^4}{224}+\frac{1}{16}a_1^2a_{10}a_5$$

$$+\frac{13}{112}a_1a_{10}^2a_5-\frac{a_{10}^3a_5}{32}+\frac{1}{112}a_1^2a_2a_5+\frac{1}{112}a_1a_2^2a_5-\frac{a_2^3a_5}{224}+\frac{1}{56}a_1^2a_3a_5$$

$$+\frac{1}{28}a_1a_2a_3a_5-\frac{3}{224}a_2^2a_3a_5+\frac{3}{112}a_1a_3^2a_5-\frac{3}{224}a_2a_3^2a_5-\frac{a_3^3a_5}{224}-\frac{1}{56}a_1^2a_4a_5$$

$$+\frac{1}{8}a_1a_{10}a_4a_5+\frac{13}{12}a_{10}^2a_4a_5+\frac{1}{56}a_1a_2a_4a_5-\frac{1}{112}a_2^2a_3a_5+\frac{1}{28}a_1a_3a_4a_5$$

$$-\frac{1}{56}a_2a_3a_4a_5-\frac{1}{112}a_3^2a_4a_5-\frac{3}{56}a_1a_4^2a_5+\frac{9}{224}a_{10}a_4^2a_5-\frac{3}{224}a_2a_4^2a_5$$

$$-\frac{3}{224}a_3a_4^2a_5+\frac{3a_4^3a_5}{56}+\frac{5a_1^2a_5^2}{56}+\frac{1}{8}a_1a_{10}a_5^2+\frac{13a_{10}^2a_5^2}{224}+\frac{1}{56}a_1a_2a_5^2-\frac{a_2^2a_5^2}{224}$$

$$+\frac{3}{112}a_1a_3a_5^2-\frac{1}{112}a_2a_3a_5^2-\frac{a_3^2a_5^2}{224}-\frac{3}{28}a_1a_4a_5^2+\frac{9}{224}a_{10}a_4a_5^2-\frac{3}{224}a_2a_4a_5^2$$

$$-\frac{3}{224}a_3a_4a_5^2+\frac{9a_4^2a_5^2}{112}-\frac{3a_1a_5^3}{56}+\frac{3a_{10}a_5^3}{224}-\frac{a_2a_5^3}{224}-\frac{a_3a_5^3}{224}+\frac{3a_4a_5^3}{56}+\frac{3a_5^4}{224}$$

$$-\frac{a_{10}^3a_6}{32}+\frac{13}{112}a_{10}^2a_2a_6+\frac{1}{16}a_{10}a_2^2a_6+\frac{3a_2^3a_6}{224}+\frac{13}{112}a_{10}^2a_3a_6+\frac{13}{56}a_1a_2a_3a_6$$

$$+\frac{1}{8}a_{10}a_2a_3a_6+\frac{9}{224}a_2^2a_3a_6+\frac{13}{56}a_1a_3^2a_6+\frac{1}{16}a_{10}a_3^2a_6+\frac{9}{224}a_2a_3^2a_6$$

$$+\frac{3a_3^3a_6}{224}+\frac{13}{112}a_{10}^2a_4a_6+\frac{1}{8}a_{10}a_2a_4a_6-\frac{1}{112}a_2^2a_4a_6+\frac{13}{56}a_1a_3a_4a_6$$

$$-\frac{1}{112}a_2a_3a_5^2-\frac{a_3^2a_5^2}{224}-\frac{3}{28}a_1a_4a_5^2+\frac{9}{224}a_{10}a_4a_5^2-\frac{3}{224}a_2a_4a_5^2$$

$$-\frac{3}{224}a_3a_4a_5^2+\frac{9a_4^2a_5^2}{112}-\frac{3a_1a_5^3}{56}+\frac{3a_{10}a_5^3}{224}-\frac{a_2a_5^3}{224}-\frac{a_3a_5^3}{224}+\frac{3a_4a_5^3}{56}$$

$$+\frac{3a_5^4}{224}-\frac{a_{10}^3a_6}{32}+\frac{13}{112}a_{10}^2a_2a_6+\frac{1}{16}a_{10}a_2^2a_6+\frac{3a_2^3a_6}{224}+\frac{13}{112}a_{10}^2a_3a_6$$

$$+\frac{13}{56}a_1a_2a_3a_6+\frac{1}{8}a_{10}a_2a_3a_6+\frac{9}{224}a_2^2a_3a_6+\frac{13}{56}a_1a_3^2a_6+\frac{1}{16}a_{10}a_3^2a_6$$

$$+\frac{9}{224}a_2a_3^2a_6+\frac{3a_3^3a_6}{224}+\frac{13}{112}a_{10}^2a_4a_6+\frac{1}{8}a_{10}a_2a_4a_6-\frac{1}{112}a_2^2a_4a_6$$

$$+\frac{13}{56}a_1a_3a_4a_6+\frac{1}{8}a_{10}a_3a_4a_6-\frac{1}{56}a_2a_3a_4a_6-\frac{1}{112}a_3^2a_4a_6+\frac{9}{224}a_{10}a_4^2a_6$$

$$-\frac{15}{224}a_2a_4^2a_6-\frac{15}{224}a_3a_4^2a_6+\frac{3a_4^3a_6}{56}-\frac{1}{56}a_1^2a_5a_6+\frac{1}{8}a_1a_{10}a_5a_6$$

$$+\frac{13}{112}a_{10}^2a_5a_6+\frac{13}{56}a_1a_2a_5a_6+\frac{1}{8}a_{10}a_2a_5a_6-\frac{1}{112}a_2^2a_5a_6+\frac{13}{28}a_1a_3a_5a_6$$

$$+\frac{1}{8}a_{10}a_3a_5a_6-\frac{1}{56}a_2a_3a_5a_6-\frac{1}{112}a_3^2a_5a_6-\frac{3}{28}a_1a_4a_5a_6+\frac{9}{112}a_{10}a_4a_5a_6$$

$$-\frac{15}{112}a_2a_4a_5a_6-\frac{15}{112}a_3a_4a_5a_6+\frac{9}{56}a_4^2a_5a_6-\frac{3}{28}a_1a_5^2a_6+\frac{9}{112}a_{10}a_4a_5a_6$$

$$-\frac{15}{112}a_2a_4a_5a_6-\frac{15}{112}a_3a_4a_5a_6+\frac{9}{56}a_4^2a_5a_6-\frac{3}{28}a_1a_5^2a_6+\frac{9}{224}a_{10}a_5^2a_6$$

$$-\frac{15}{224}a_2a_5^2a_6-\frac{15}{224}a_3a_5^2a_6+\frac{9}{56}a_4a_5^2a_6+\frac{3a_5^3a_6}{56}+\frac{13a_{10}^2a_6^2}{224}+\frac{1}{8}a_{10}a_2a_6^2$$

$$+\frac{23a_2^2a_6^2}{224}+\frac{25}{112}a_1a_3a_6^2+\frac{1}{8}a_{10}a_3a_6^2+\frac{23}{112}a_2a_3a_6^2+\frac{23a_3^2a_6^2}{224}+\frac{9}{224}a_{10}a_4a_6^2$$

$$-\frac{27}{224}a_2a_4a_6^2-\frac{27}{224}a_3a_4a_6^2+\frac{9a_4^2a_6^2}{112}-\frac{3}{56}a_1a_5a_6^2+\frac{9}{224}a_{10}a_5a_6^2$$

$$-\frac{27}{224}a_2a_5a_6^2-\frac{27}{224}a_3a_5a_6^2+\frac{9}{56}a_4a_5a_6^2+\frac{9a_5^2a_6^2}{112}+\frac{3a_{10}a_6^3}{224}-\frac{13a_2a_6^3}{224}$$

$$-\frac{13a_3a_6^3}{224}+\frac{3a_4a_6^3}{56}+\frac{3a_5a_6^3}{224}+\frac{3a_6^4}{224}+\frac{3a_{10}^3a_7}{112}-\frac{15}{224}a_{10}^2a_4a_7-\frac{a_4^3a_7}{224}$$

$$+\frac{1}{112}a_1^2a_5a_7+\frac{1}{8}a_1a_{10}a_5a_7-\frac{15}{224}a_{10}^2a_5a_7+\frac{1}{56}a_1a_4a_5a_7-\frac{3}{224}a_4^2a_5a_7$$

$$+\frac{1}{56}a_1a_5^2a_7-\frac{3}{224}a_4a_5^2a_7-\frac{a_5^3a_7}{224}-\frac{15}{224}a_{10}^2a_6a_7+\frac{1}{8}a_{10}a_2a_6a_7$$

$$+\frac{1}{112}a_2^2a_6a_7+\frac{1}{8}a_{10}a_3a_6a_7+\frac{1}{56}a_2a_3a_6a_7+\frac{1}{112}a_3^2a_6a_7+\frac{1}{56}a_2a_4a_6a_7$$

$$+\frac{1}{56}a_3a_4a_6a_7-\frac{3}{224}a_4^2a_6a_7+\frac{1}{56}a_1a_5a_6a_7+\frac{1}{56}a_2a_5a_6a_7+\frac{1}{56}a_3a_5a_6a_7$$

$$-\frac{3}{112}a_4a_5a_6a_7-\frac{3}{224}a_5^2a_6a_7+\frac{1}{56}a_2a_6^2a_7+\frac{1}{56}a_3a_6^2a_7-\frac{3}{224}a_4a_6^2a_7$$

$$-\frac{3}{224}a_5a_6^2a_7-\frac{a_6^3a_7}{224}+\frac{9a_{10}^2a_7^2}{224}-\frac{9}{224}a_{10}a_4a_7^2-\frac{a_4^2a_7^2}{224}+\frac{1}{112}a_1a_5a_7^2$$

$$-\frac{9}{224}a_{10}a_5a_7^2-\frac{1}{112}a_4a_5a_7^2-\frac{a_5^2a_7^2}{224}-\frac{9}{224}a_{10}a_6a_7^2+\frac{1}{112}a_2a_6a_7^2$$

$$+\frac{1}{112}a_3a_6a_7^2-\frac{1}{112}a_4a_6a_7^2-\frac{1}{112}a_5a_6a_7^2-\frac{a_6^2a_7^2}{224}-\frac{3a_{10}a_7^3}{112}-\frac{a_4a_7^3}{224}-\frac{a_5a_7^3}{224}$$

$$-\frac{a_6a_7^3}{2242}+\frac{3a_7^4}{448}-\frac{1}{112}a_1^2a_{10}a_8-\frac{3}{112}a_1a_{10}^2a_8+\frac{3a_{10}^3a_8}{112}+\frac{1}{112}a_1^2a_4a_8$$

$$+\frac{1}{8}a_1a_{10}a_4a_8-\frac{15}{224}a_{10}^2a_4a_8+\frac{1}{112}a_1a_4^2a_8-\frac{a_4^3a_8}{224}+\frac{1}{56}a_1^2a_5a_8+\frac{1}{4}a_1a_{10}a_5a_8$$

$$-\frac{15}{224}a_{10}^2a_5a_8+\frac{1}{28}a_1a_4a_5a_8-\frac{3}{224}a_4^2a_5a_8+\frac{3}{112}a_1a_5^2a_8-\frac{3}{224}a_4a_5^2a_8$$

$$-\frac{a_5^3a_8}{224}+\frac{1}{112}a_1^2a_6a_8+\frac{1}{8}a_1a_{10}a_6a_8-\frac{15}{224}a_{10}^2a_6a_8+\frac{1}{8}a_{10}a_2a_6a_8$$

$$+\frac{1}{112}a_2^2a_6a_8+\frac{1}{8}a_{10}a_3a_6a_8+\frac{1}{56}a_2a_3a_6a_8+\frac{1}{112}a_3^2a_6a_8+\frac{1}{56}a_1a_4a_6a_8$$

$$+\frac{1}{56}a_2a_4a_6a_8+\frac{1}{56}a_3a_4a_6a_8-\frac{3}{224}a_4^2a_6a_8+\frac{1}{28}a_1a_5a_6a_8+\frac{1}{56}a_2a_5a_6a_8$$

$$+\frac{1}{56}a_3a_5a_6a_8-\frac{1}{112}a_4a_5a_6a_8-\frac{3}{224}a_5^2a_6a_8+\frac{1}{112}a_1a_6^2a_8+\frac{1}{56}a_2a_6^2a_8$$

$$+\frac{1}{56}a_3a_6^2a_8-\frac{3}{224}a_4a_6^2a_8-\frac{3}{224}a_5a_6^2a_8-\frac{a_6^3a_8}{224}-\frac{1}{112}a_1^2a_7a_8$$

$$-\frac{3}{56}a_1a_{10}a_7a_8+\frac{9}{112}a_{10}^2a_7a_8+\frac{1}{56}a_1a_4a_7a_8-\frac{9}{112}a_{10}a_4a_7a_8$$

$$-\frac{1}{112}a_4^2a_7a_8+\frac{1}{28}a_1a_5a_7a_8-\frac{9}{112}a_{10}a_5a_7a_8-\frac{1}{56}a_4a_5a_7a_8-\frac{1}{112}a_5^2a_7a_8$$

$$+\frac{1}{56}a_1a_6a_7a_8-\frac{9}{112}a_{10}a_6a_7a_8+\frac{1}{56}a_2a_6a_7a_8+\frac{1}{56}a_3a_6a_7a_8$$

$$-\frac{1}{56}a_4a_6a_7a_8-\frac{1}{56}a_5a_6a_7a_8-\frac{1}{112}a_6^2a_7a_8-\frac{3}{112}a_1a_7^2a_8+\frac{9}{112}a_{10}a_7^2a_8$$

$$-\frac{3}{224}a_4a_7^2a_8-\frac{3}{224}a_5a_7^2a_8-\frac{3}{224}a_6a_7^2a_8+\frac{3a_7^3a_8}{112}+\frac{5a_1^2a_8^2}{112}-\frac{3}{56}a_1a_{10}a_8^2$$

$$+\frac{9a_{10}^2a_8^2}{224}+\frac{1}{56}a_1a_4a_8^2-\frac{9}{224}a_{10}a_4a_8^2-\frac{a_4^2a_8^2}{224}+\frac{3}{112}a_1a_5a_8^2-\frac{9}{224}a_{10}a_5a_8^2$$

$$-\frac{1}{112}a_4a_5a_8^2-\frac{a_5^2a_8^2}{224}+\frac{1}{56}a_1a_6a_8^2-\frac{9}{224}a_{10}a_6a_8^2+\frac{1}{112}a_2a_6a_8^2+\frac{1}{112}a_3a_6a_8^2$$

$$-\frac{1}{112}a_4a_6a_8^2-\frac{1}{112}a_5a_6a_8^2-\frac{a_6^2a_8^2}{224}-\frac{3}{56}a_1a_7a_8^2+\frac{9}{112}a_{10}a_7a_8^2-\frac{3}{224}a_4a_7a_8^2$$

$$-\frac{3}{224}a_5a_7a_8^2-\frac{3}{224}a_6a_7a_8^2+\frac{9a_7^2a_8^2}{224}-\frac{3a_1a_8^3}{112}+\frac{3a_{10}a_8^3}{112}-\frac{a_4a_8^3}{224}-\frac{a_5a_8^3}{224}$$

$$-\frac{a_6a_8^3}{224}+\frac{3a_7a_8^3}{112}+\frac{3a_8^4}{448}+\frac{3a_{10}^3a_9}{112}-\frac{3}{112}a_{10}^2a_2a_9-\frac{1}{112}a_{10}a_2^2a_9+\frac{3}{56}a_1^2a_3a_9$$

$$+\frac{3}{28}a_1a_{10}a_3a_9-\frac{3}{112}a_{10}^2a_3a_9+\frac{3}{28}a_1a_2a_3a_9-\frac{1}{56}a_{10}a_2a_3a_9+\frac{3}{28}a_1a_3^2a_9$$

$$-\frac{1}{112}a_{10}a_3^2a_9-\frac{15}{224}a_{10}^2a_4a_9+\frac{1}{8}a_{10}a_2a_4a_9+\frac{1}{112}a_2^2a_4a_9+\frac{1}{8}a_{10}a_3a_4a_9$$

$$+\frac{1}{56}a_2a_3a_4a_9+\frac{1}{112}a_3^2a_4a_9+\frac{1}{112}a_2a_4^2a_9+\frac{1}{112}a_3a_4^2a_9-\frac{a_4^3a_9}{224}$$

$$+\frac{1}{112}a_1^2a_5a_9+\frac{1}{8}a_1a_{10}a_5a_9-\frac{15}{224}a_{10}^2a_5a_9+\frac{1}{8}a_{10}a_2a_5a_9+\frac{1}{112}a_2^2a_5a_9$$

$$+\frac{1}{8}a_{10}a_3a_5a_9+\frac{1}{56}a_2a_3a_5a_9+\frac{1}{112}a_3^2a_5a_9+\frac{1}{56}a_1a_4a_5a_9+\frac{1}{56}a_2a_4a_5a_9$$

$$+\frac{1}{56}a_3a_4a_5a_9-\frac{3}{224}a_4^2a_5a_9+\frac{1}{56}a_1a_5^2a_9+\frac{1}{112}a_2a_5^2a_9+\frac{1}{112}a_3a_5^2a_9$$

$$-\frac{3}{224}a_2a_5^2a_9-\frac{a_5^3a_9}{224}-\frac{15}{224}a_{10}^2a_6a_9+\frac{1}{4}a_{10}a_2a_6a_9+\frac{1}{56}a_2^2a_6a_9$$

$$+\frac{1}{4}a_{10}a_3a_6a_9+\frac{1}{28}a_2a_3a_6a_9+\frac{1}{56}a_3^2a_6a_9+\frac{1}{28}a_2a_4a_6a_9+\frac{1}{28}a_3a_4a_6a_9$$

$$-\frac{3}{224}a_4^2a_6a_9+\frac{1}{56}a_1a_5a_6a_9+\frac{1}{28}a_2a_5a_6a_9+\frac{1}{28}a_3a_5a_6a_9-\frac{3}{112}a_4a_5a_6a_9$$

$$-\frac{3}{224}a_5^2a_6a_9+\frac{3}{112}a_2a_6^2a_9+\frac{3}{112}a_3a_6^2a_9-+\frac{3}{224}a_4a_6^2a_9-+\frac{3}{224}a_5a_6^2a_9$$

$$-\frac{a_6^3a_9}{224}++\frac{9}{112}a_{10}^2a_7a_9-\frac{3}{56}a_{10}a_2a_7a_9-\frac{1}{112}a_2^2a_7a_9+\frac{3}{28}a_1a_3a_7a_9$$

$$-\frac{3}{56}a_{10}a_3a_7a_9-\frac{1}{56}a_2a_3a_7a_9-\frac{1}{112}a_3^2a_7a_9-\frac{9}{112}a_{10}a_4a_7a_9$$

$$+\frac{1}{56}a_2a_4a_7a_9+\frac{1}{56}a_3a_4a_7a_9-\frac{1}{112}a_4^2a_7a_9+\frac{1}{56}a_1a_5a_7a_9-\frac{9}{112}a_{10}a_5a_7a_9$$

$$+\frac{1}{56}a_2a_5a_7a_9+\frac{1}{56}a_3a_5a_7a_9-\frac{1}{56}a_4a_5a_7a_9-\frac{1}{112}a_5^2a_7a_9-\frac{9}{112}a_{10}a_6a_7a_9$$

$$+\frac{1}{28}a_2a_6a_7a_9+\frac{1}{28}a_3a_6a_7a_9-\frac{1}{56}a_4a_6a_7a_9-\frac{1}{56}a_5a_6a_7a_9-\frac{1}{112}a_6^2a_7a_9$$

$$+\frac{9}{112}a_{10}a_7^2a_9-\frac{3}{112}a_2a_7^2a_9-\frac{3}{112}a_3a_7^2a_9-\frac{3}{224}a_4a_7^2a_9-\frac{3}{224}a_5a_7^2a_9$$

$$-\frac{3}{224}a_6a_7^2a_9+\frac{3a_7^3a_9}{112}-\frac{1}{112}a_1^2a_8a_9-\frac{3}{56}a_1a_{10}a_8a_9+\frac{9}{112}a_{10}^2a_8a_9$$

$$+\frac{3}{28}a_1a_2a_8a_9-\frac{3}{56}a_{10}a_2a_8a_9-\frac{1}{112}a_2^8a_8a_9+\frac{3}{14}a_1a_3a_8a_9-\frac{3}{56}a_{10}a_3a_8a_9$$

$$-\frac{1}{56}a_2a_3a_8a_9 - \frac{1}{112}a_3^2a_8a_9 + \frac{1}{56}a_1a_4a_8a_9 - \frac{9}{112}a_{10}a_4a_8a_9 + \frac{1}{56}a_2a_4a_8a_9$$

$$+\frac{1}{56}a_3a_4a_8a_9 - \frac{1}{112}a_4^2a_8a_9 + \frac{1}{28}a_1a_5a_8a_9 - \frac{9}{112}a_{10}a_5a_8a_9 + \frac{1}{56}a_2a_5a_8a_9$$

$$+\frac{1}{56}a_3a_5a_8a_9 - \frac{1}{56}a_4a_5a_8a_9 - \frac{1}{112}a_5^2a_8a_9 + \frac{1}{56}a_1a_6a_8a_9 - \frac{9}{112}a_{10}a_6a_8a_9$$

$$+\frac{1}{28}a_2a_6a_8a_9 + \frac{1}{28}a_3a_6a_8a_9 - \frac{1}{56}a_4a_6a_8a_9 - \frac{1}{56}a_5a_6a_8a_9 - \frac{1}{112}a_6^2a_8a_9$$

$$-\frac{3}{56}a_1a_7a_8a_9 + \frac{9}{56}a_{10}a_7a_8a_9 - \frac{3}{56}a_2a_7a_8a_9 - \frac{3}{56}a_3a_7a_8a_9$$

$$-\frac{3}{112}a_4a_7a_8a_9 - \frac{3}{112}a_5a_7a_8a_9 - \frac{3}{112}a_6a_7a_8a_9 + \frac{9}{112}a_7^2a_8a_9 - \frac{3}{56}a_1a_8^2a_9$$

$$+\frac{9}{112}a_{10}a_8^2a_9 - \frac{3}{112}a_2a_8^2a_9 - \frac{3}{112}a_3a_8^2a_9 - \frac{3}{224}a_4a_8^2a_9 - \frac{3}{224}a_5a_8^2a_9$$

$$-\frac{3}{224}a_6a_8^2a_9 + \frac{9}{112}a_7a_8^2a_9 + \frac{3a_8^3a_9}{112} + \frac{9a_{10}^2a_8^2}{224} - \frac{3}{56}a_{10}a_2a_9^2 + \frac{5a_2^2a_9^2}{112}$$

$$+\frac{3}{28}a_1a_3a_9^2 - \frac{3}{56}a_{10}a_3a_9^2 + \frac{5}{56}a_2a_3a_9^2 + \frac{5a_3^2a_9^2}{112} - \frac{9}{224}a_{10}a_5a_9^2$$

$$+\frac{1}{56}a_2a_4a_9^2 + \frac{1}{56}a_3a_4a_9^2 - \frac{a_4^2a_9^2}{224} + \frac{1}{112}a_1a_5a_9^2 - \frac{a_5^2a_9^2}{224} - \frac{9}{224}a_{10}a_6a_9^2$$

$$+\frac{3}{112}a_2a_6a_9^2 + \frac{3}{112}a_3a_6a_9^2 - \frac{1}{112}a_4a_6a_9^2 - \frac{1}{112}a_5a_6a_9^2 - \frac{a_6^2a_9^2}{224}$$

$$+\frac{9}{112}a_{10}a_7a_9^2 - \frac{3}{56}a_2a_7a_9^2 - \frac{3}{56}a_3a_7a_9^2 - \frac{3}{224}a_4a_7a_9^2 - \frac{3}{224}a_5a_7a_9^2$$

$$-\frac{3}{56}a_6a_7a_9^2 + \frac{9a_7^2a_9^2}{224} - \frac{3}{112}a_1a_8a_9^2 + \frac{9}{112}a_{10}a_8a_9^2 - \frac{3}{56}a_2a_8a_9^2$$

$$-\frac{3}{56}a_3a_8a_9^2 - \frac{3}{224}a_4a_8a_9^2 - \frac{3}{224}a_5a_8a_9^2 - \frac{3}{224}a_6a_8a_9^2 + \frac{9}{112}a_7a_8a_9^2$$

$$+\frac{9a_8^2a_9^2}{224} + \frac{3a_{10}a_9^3}{112} - \frac{3a_2a_9^3}{112} - \frac{3a_3a_9^3}{112} - \frac{a_4a_9^3}{224} - \frac{a_5a_9^3}{224} - \frac{a_6a_9^3}{224} + \frac{3a_7a_9^3}{112}$$

$$+\frac{3a_8a_9^3}{112} + \frac{3a_9^4}{448} = \frac{11}{120} \qquad (2.4 - viii)$$

$h^5 f^3 f_{yy}^2$:

$$
\begin{aligned}
&-\frac{a_1^4}{224}-\frac{a_{10}^4}{448}+\frac{a_1^2 a_2^2}{224}-\frac{a_2^4}{224}+\frac{3}{16}a_1^2 a_2 a_3-\frac{a_2^3 a_3}{56}+\frac{41 a_1^2 a_3^2}{224}-\frac{3a_2^2 a_3^2}{112}-\frac{a_2 a_3^3}{56}\\
&-\frac{a_3^4}{224}+\frac{13 a_{10}^2 a_4^2}{224}+\frac{a_2^2 a_4^2}{224}+\frac{1}{112}a_2 a_3 a_4^2+\frac{a_3^2 a_4^2}{224}-\frac{a_4^4}{224}+\frac{3}{28}a_1^2 a_4 a_5\\
&+\frac{13}{112}a_{10}^2 a_4 a_5+\frac{1}{112}a_2^2 a_4 a_5+\frac{1}{56}a_2 a_3 a_4 a_5+\frac{1}{112}a_3^2 a_4 a_5-\frac{a_4^3 a_5}{56}\\
&+\frac{3a_1^2 a_5^2}{28}+\frac{13 a_{10}^2 a_5^2}{224}+\frac{a_2^2 a_5^2}{224}+\frac{1}{112}a_2 a_3 a_5^2+\frac{a_3^2 a_5^2}{224}-\frac{3a_4^2 a_5^2}{112}-\frac{a_4 a_5^3}{56}\\
&-\frac{a_5^4}{224}+\frac{13}{112}a_{10}^2 a_4 a_6+\frac{13}{112}a_2^2 a_4 a_6+\frac{13}{56}a_2 a_3 a_4 a_6+\frac{13}{112}a_3^2 a_4 a_6-\frac{a_4^3 a_6}{56}\\
&+\frac{3}{28}a_1^2 a_5 a_6+\frac{13}{112}a_{10}^2 a_5 a_6+\frac{13}{112}a_2^2 a_5 a_6+\frac{13}{56}a_2 a_3 a_5 a_6+\frac{13}{112}a_3^2 a_5 a_6\\
&-\frac{3}{56}a_4^2 a_5 a_6-\frac{3}{56}a_4 a_5^2 a_6-\frac{a_5^3 a_6}{56}+\frac{13 a_{10}^2 a_6^2}{224}+\frac{25 a_2^2 a_6^2}{224}+\frac{25}{112}a_2 a_3 a_6^2\\
&+\frac{25 a_3^2 a_6^2}{224}-\frac{3a_4^2 a_6^2}{112}-\frac{3}{56}a_4 a_5 a_6^2-\frac{3a_5^2 a_6^2}{112}-\frac{a_4 a_6^3}{56}-\frac{a_6^4}{224}-\frac{a_{10}^3 a_7}{112}\\
&+\frac{1}{16}a_{10} a_4^2 a_7+\frac{1}{8}a_{10} a_4 a_5 a_7+\frac{1}{16}a_{10} a_5^2 a_7+\frac{1}{8}a_{10} a_4 a_6 a_7+\frac{1}{8}a_{10} a_5 a_6 a_7\\
&+\frac{1}{16}a_{10} a_6^2 a_7-\frac{3a_{10}^2 a_7^2}{224}+\frac{a_4^2 a_7^2}{224}+\frac{1}{112}a_4 a_5 a_7^2+\frac{a_5^2 a_7^2}{224}+\frac{1}{112}a_4 a_6 a_7^2\\
&+\frac{1}{112}a_5 a_6 a_7^2+\frac{a_6^2 a_7^2}{224}-\frac{a_{10} a_7^3}{112}-\frac{a_7^4}{448}+\frac{3}{56}a_1^2 a_{10} a_8-\frac{a_{10}^3 a_8}{112}+\frac{1}{16}a_{10} a_4^2 a_8\\
&+\frac{1}{8}a_{10} a_4 a_5 a_8+\frac{1}{16}a_{10} a_5^2 a_8+\frac{1}{8}a_{10} a_4 a_6 a_8+\frac{1}{8}a_{10} a_5 a_6 a_8+\frac{1}{16}a_{10} a_6^2 a_8\\
&+\frac{3}{56}a_1^2 a_7 a_8-\frac{3}{112}a_{10}^2 a_7 a_8+\frac{1}{112}a_4^2 a_7 a_8+\frac{1}{56}a_4 a_5 a_7 a_8+\frac{1}{112}a_5^2 a_7 a_8\\
&+\frac{1}{56}a_4 a_6 a_7 a_8+\frac{1}{56}a_5 a_6 a_7 a_8+\frac{1}{112}a_6^2 a_7 a_8-\frac{3}{112}a_{10} a_7^2 a_8-\frac{a_7^3 a_8}{112}
\end{aligned}
$$

$$+\frac{3a_1^2a_8^2}{56}-\frac{3a_{10}^2a_8^2}{224}+\frac{a_4^2a_8^2}{224}+\frac{1}{112}a_4a_5a_8^2+\frac{a_5^2a_8^2}{224}+\frac{1}{112}a_4a_6a_8^2+\frac{1}{112}a_5a_6a_8^2$$

$$+\frac{a_6^2a_8^2}{224}-\frac{3}{112}a_{10}a_7a_8^2-\frac{3a_7^2a_8^2}{224}-\frac{a_{10}a_8^3}{112}-\frac{a_7a_8^3}{112}-\frac{a_8^4}{448}-\frac{a_{10}a_9^3}{112}+\frac{3}{56}a_{10}a_2^2a_9$$

$$+\frac{3}{28}a_{10}a_2a_3a_9+\frac{3}{56}a_{10}a_3^2a_9+\frac{1}{16}a_{10}a_4^2a_9+\frac{1}{8}a_{10}a_4a_5a_9+\frac{1}{16}a_{10}a_5^2a_9$$

$$+\frac{1}{8}a_{10}a_4a_6a_9+\frac{1}{8}a_{10}a_5a_6a_9+\frac{1}{16}a_{10}a_6^2a_9-\frac{3}{112}a_{10}^2a_7a_9+\frac{3}{56}a_2^2a_7a_9$$

$$+\frac{3}{28}a_2a_3a_7a_9+\frac{3}{56}a_3^2a_7a_9+\frac{1}{112}a_4^2a_7a_9+\frac{1}{56}a_4a_5a_7a_9+\frac{1}{112}a_5^2a_7a_9$$

$$+\frac{1}{56}a_4a_6a_7a_9+\frac{1}{56}a_5a_6a_7a_9+\frac{1}{112}a_6^2a_7a_9-\frac{3}{112}a_{10}a_7^2a_9-\frac{a_7^3a_9}{112}$$

$$+\frac{3}{56}a_1^2a_8a_9-\frac{3}{112}a_{10}^2a_8a_9-\frac{3}{56}a_2^2a_8a_9+\frac{3}{28}a_2a_3a_8a_9+\frac{3}{56}a_3^2a_8a_9$$

$$+\frac{1}{112}a_4^2a_8a_9+\frac{1}{56}a_4a_5a_8a_9+\frac{1}{112}a_5^2a_8a_9+\frac{1}{56}a_4a_6a_8a_9+\frac{1}{56}a_5a_6a_8a_9$$

$$+\frac{1}{112}a_6^2a_8a_9-\frac{3}{56}a_{10}a_7a_8a_9-\frac{3}{112}a_7^2a_8a_9-\frac{3}{112}a_{10}a_8^2a_9-\frac{3}{112}a_7a_8^2a_9$$

$$-\frac{a_8^3a_9}{112}-\frac{3a_{10}^2a_9^2}{224}+\frac{3a_2^2a_9^2}{56}+\frac{3}{28}a_2a_3a_9^2+\frac{3a_3^2a_9^2}{56}+\frac{a_2^2a_9^2}{224}+\frac{1}{112}a_4a_5a_9^2$$

$$+\frac{a_5^2a_9^2}{224}+\frac{1}{112}a_4a_6a_9^2+\frac{1}{112}a_5a_6a_9^2+\frac{a_6^2a_9^2}{224}-\frac{3}{112}a_{10}a_7a_9^2-\frac{3a_7^2a_9^2}{224}$$

$$-\frac{3}{112}a_{10}a_8a_9^2-\frac{3a_8^2a_9^2}{224}-\frac{a_{10}a_9^3}{112}-\frac{a_7a_9^3}{112}-\frac{a_8a_9^3}{112}-\frac{a_9^4}{448}=\frac{1}{30}$$

(2.4 –ix)

$h^5 f^3 f_y f_{yyy}$:

$$\frac{-a_1^4}{168}-\frac{a_{10}^4}{336}+\frac{a_1^3a_2}{336}+\frac{a_1a_2^3}{336}-\frac{a_2^4}{168}+\frac{a_1^3a_3}{16}+\frac{3}{16}a_1a_2^2a_3-\frac{a_2^3a_3}{42}+\frac{41}{112}a_1a_2a_3^2$$

$$-\frac{a_2^2 a_3^2}{28}+\frac{61 a_1 a_3^3}{336}-\frac{a_2 a_3^3}{42}-\frac{a_3^4}{168}+\frac{19 a_{10}^3 a_4}{336}+\frac{a_2^3 a_4}{336}+\frac{1}{112}a_2^2 a_3 a_4+\frac{1}{112}a_2 a_3^2 a_4$$

$$+\frac{a_3^3 a_4}{336}+\frac{a_{10} a_4^3}{48}+\frac{a_2 a_4^3}{336}+\frac{a_3 a_4^3}{336}-\frac{a_4^4}{168}+\frac{a_1^3 a_5}{28}+\frac{19 a_{10}^3 a_5}{336}+\frac{a_2^3 a_5}{336}+\frac{1}{112}a_2^2 a_3 a_5$$

$$+\frac{1}{112}a_2 a_3^2 a_5+\frac{a_3^3 a_5}{336}+\frac{3}{28}a_1 a_4^2 a_5+\frac{1}{16}a_{10} a_4^2 a_5+\frac{1}{112}a_{10} a_4^2 a_5+\frac{1}{112}a_2 a_4^2 a_5$$

$$+\frac{1}{112}a_3 a_4^2 a_5-\frac{a_4^3 a_5}{42}+\frac{3}{14}a_1 a_4 a_5^2+\frac{1}{16}a_{10} a_4 a_5^2+\frac{1}{112}a_2 a_4 a_5^2+\frac{1}{112}a_3 a_4 a_5^2$$

$$-\frac{a_4^2 a_5^2}{28}+\frac{3 a_1 a_5^3}{28}+\frac{a_{10} a_5^3}{48}+\frac{a_2 a_5^3}{336}+\frac{a_3 a_5^3}{336}-\frac{a_4 a_5^3}{42}-\frac{a_5^4}{168}+\frac{19 a_{10}^3 a_6}{336}+\frac{13 a_2^3 a_6}{336}$$

$$+\frac{13}{112}a_2^2 a_3 a_6+\frac{13}{112}a_2 a_3^2 a_6+\frac{13 a_3^3 a_6}{336}+\frac{1}{16}a_{10} a_4^2 a_6+\frac{13}{112}a_2 a_4^2 a_6$$

$$+\frac{13}{112}a_3 a_4^2 a_6-\frac{a_4^3 a_6}{42}+\frac{3}{14}a_1 a_4 a_5 a_6+\frac{1}{8}a_{10} a_4 a_5 a_6+\frac{13}{56}a_2 a_4 a_5 a_6$$

$$+\frac{13}{56}a_3 a_4 a_5 a_6-\frac{1}{14}a_4^2 a_5 a_6+\frac{3}{14}a_1 a_5^2 a_6+\frac{1}{16}a_{10} a_5^2 a_6+\frac{13}{112}a_2 a_5^2 a_6$$

$$+\frac{13}{112}a_3 a_5^2 a_6-\frac{1}{14}a_4 a_5^2 a_6-\frac{a_5^3 a_6}{42}+\frac{1}{16}a_{10} a_4 a_6^2+\frac{25}{112}a_2 a_4 a_6^2+\frac{25}{112}a_3 a_4 a_6^2$$

$$-\frac{a_4^2 a_6^2}{28}+\frac{3}{28}a_1 a_5 a_6^2+\frac{1}{16}a_{10} a_5 a_6^2+\frac{25}{112}a_2 a_5 a_6^2+\frac{25}{112}a_3 a_5 a_6^2-\frac{1}{14}a_4 a_5 a_6^2$$

$$-\frac{a_5^2 a_6^2}{28}+\frac{a_{10} a_6^3}{48}+\frac{37 a_2 a_6^3}{336}+\frac{37 a_3 a_6^3}{336}-\frac{a_4 a_6^3}{42}-\frac{a_5 a_6^3}{42}-\frac{a_6^4}{168}-\frac{a_{10}^3 a_7}{84}$$

$$+\frac{13}{112}a_{10}^2 a_4 a_7+\frac{a_4^3 a_7}{336}+\frac{13}{112}a_{10}^2 a_5 a_7+\frac{1}{112}a_4^2 a_5 a_7+\frac{1}{112}a_4 a_5^2 a_7+\frac{a_5^3 a_7}{336}$$

$$+\frac{13}{112}a_{10}^2 a_6 a_7+\frac{1}{112}a_4^2 a_6 a_7+\frac{1}{56}a_4 a_5 a_6 a_7+\frac{1}{112}a_5^2 a_6 a_7+\frac{1}{112}a_5 a_6^2 a_7$$

$$+\frac{a_6^3 a_7}{336}-\frac{a_{10}^2 a_7^2}{56}+\frac{1}{16}a_{10} a_4 a_7^2+\frac{1}{16}a_{10} a_5 a_7^2+\frac{1}{16}a_{10} a_6 a_7^2-\frac{a_{10} a_7^3}{84}+\frac{a_4 a_7^3}{336}$$

$$+\frac{a_5 a_7^3}{336}+\frac{a_6 a_7^3}{336}-\frac{a_7^4}{336}+\frac{a_1^3 a_8}{56}+\frac{3 a_1 a_{10}^2 a_8}{56}-\frac{a_{10}^3 a_8}{84}+\frac{13}{112}a_{10}^2 a_4 a_8+\frac{a_4^3 a_8}{336}$$

$$+\frac{13}{112}a_{10}^2a_5a_8+\frac{1}{112}a_4^2a_5a_8+\frac{1}{112}a_4a_5^2a_8+\frac{a_5^3a_8}{336}+\frac{13}{112}a_{10}^2a_6a_8$$

$$+\frac{1}{112}a_4^2a_6a_8+\frac{1}{56}a_4a_5a_6a_8+\frac{1}{112}a_5^2a_6a_8+\frac{1}{112}a_4a_6^2a_8+\frac{1}{112}a_5a_6^2a_8$$

$$+\frac{a_6^3a_8}{336}+\frac{3}{28}a_1a_{10}a_7a_8-\frac{1}{28}a_{10}^2a_7a_8+\frac{1}{8}a_{10}a_4a_7a_8+\frac{1}{8}a_{10}a_5a_7a_8$$

$$+\frac{1}{8}a_{10}a_6a_7a_8+\frac{3}{56}a_1a_7^2a_8-\frac{1}{28}a_{10}a_7^2a_8+\frac{1}{112}a_4a_7^2a_8+\frac{1}{112}a_5a_7^2a_8$$

$$+\frac{1}{112}a_6a_7^2a_8-\frac{a_7^3a_8}{84}+\frac{3}{28}a_1a_{10}a_8^2-\frac{a_{10}^2a_8^2}{56}+\frac{1}{16}a_{10}a_4a_8^2+\frac{1}{16}a_{10}a_5a_8^2$$

$$+\frac{1}{16}a_{10}a_6a_8^2+\frac{3}{28}a_1a_7a_8^2-\frac{1}{28}a_{10}a_7a_8^2+\frac{1}{112}a_4a_7a_8^2+\frac{1}{112}a_5a_7a_8^2$$

$$+\frac{1}{112}a_6a_7a_8^2-\frac{a_7^2a_8^2}{56}+\frac{3a_1a_8^3}{56}-\frac{a_{10}a_8^3}{84}+\frac{a_4a_8^3}{336}+\frac{a_5a_8^3}{336}+\frac{a_6a_8^3}{336}-\frac{a_7a_8^3}{84}$$

$$-\frac{a_8^4}{336}-\frac{a_{10}^3a_9}{84}+\frac{3}{56}a_{10}^2a_2a_9+\frac{a_2^3a_9}{56}+\frac{3}{56}a_{10}^2a_3a_9+\frac{3}{56}a_2^2a_3a_9+\frac{3}{56}a_2a_3^2a_9$$

$$+\frac{a_3^3a_9}{56}+\frac{13}{112}a_{10}^2a_4a_9+\frac{a_4^3a_9}{336}+\frac{13}{112}a_{10}^2a_5a_9+\frac{1}{112}a_4^2a_5a_9+\frac{1}{112}a_4a_5^2a_9$$

$$+\frac{a_5^3a_9}{336}+\frac{13}{112}a_{10}^2a_6a_9+\frac{1}{112}a_4^2a_6a_9+\frac{1}{56}a_4a_5a_6a_9+\frac{1}{112}a_5^2a_6a_9$$

$$+\frac{1}{112}a_4a_6^2a_9+\frac{1}{112}a_5a_6^2a_9+\frac{a_6^3a_9}{336}-\frac{1}{28}a_{10}^2a_7a_9+\frac{3}{28}a_{10}a_2a_7a_9$$

$$+\frac{3}{28}a_{10}a_3a_7a_9+\frac{1}{8}a_{10}a_4a_7a_9+\frac{1}{8}a_{10}a_5a_7a_9+\frac{1}{8}a_{10}a_6a_7a_9-\frac{1}{28}a_{10}a_7^2a_9$$

$$+\frac{3}{56}a_2a_7^2a_9+\frac{3}{56}a_3a_7^2a_9+\frac{1}{112}a_4a_7^2a_9+\frac{1}{112}a_5a_7^2a_9+\frac{1}{112}a_6a_7^2a_9-\frac{a_7^3a_9}{84}$$

$$+\frac{3}{28}a_1a_{10}a_8a_9-\frac{1}{28}a_{10}^2a_8a_9+\frac{3}{28}a_{10}a_2a_8a_9+\frac{3}{28}a_{10}a_3a_8a_9$$

$$+\frac{1}{28}a_{10}a_4a_8a_9+\frac{1}{28}a_{10}a_5a_8a_9+\frac{1}{28}a_{10}a_6a_8a_9+\frac{3}{28}a_1a_7a_8a_9$$

$$-\frac{1}{14}a_{10}a_7a_8a_9+\frac{3}{28}a_2a_7a_8a_9+\frac{3}{28}a_3a_7a_8a_9+\frac{1}{56}a_4a_7a_8a_9+\frac{1}{56}a_5a_7a_8a_9$$
$$+\frac{1}{56}a_6a_7a_8a_9-\frac{1}{28}a_7^2a_8a_9+\frac{3}{28}a_1a_8^2a_9-\frac{1}{28}a_{10}a_8^2a_9+\frac{3}{56}a_2a_8^2a_9$$
$$+\frac{3}{56}a_3a_8^2a_9+\frac{1}{112}a_4a_8^2a_9+\frac{1}{112}a_5a_8^2a_9+\frac{1}{112}a_6a_8^2a_9-\frac{1}{28}a_7a_8^2a_9-\frac{a_8^3a_9}{84}$$
$$-\frac{a_{10}^2a_9^2}{56}+\frac{3}{28}a_{10}a_2a_9^2+\frac{1}{16}a_{10}a_3a_9^2+\frac{1}{16}a_{10}a_4a_9^2+\frac{1}{16}a_{10}a_5a_9^2$$
$$+\frac{1}{16}a_{10}a_6a_9^2-\frac{1}{128}a_{10}a_7a_9^2+\frac{3}{28}a_2a_7a_9^2+\frac{3}{28}a_3a_7a_9^2+\frac{1}{112}a_4a_7a_9^2$$
$$+\frac{1}{112}a_5a_7a_9^2+\frac{1}{112}a_6a_7a_9^2-\frac{a_7^2a_9^2}{56}+\frac{3}{56}a_1a_8a_9^2-\frac{1}{28}a_{10}a_8a_9^2+\frac{3}{28}a_2a_8a_9^2$$
$$+\frac{1}{112}a_4a_8a_9^2+\frac{1}{112}a_5a_8a_9^2+\frac{1}{112}a_6a_8a_9^2-\frac{1}{28}a_7a_8a_9^2-\frac{a_8^2a_9^2}{56}-\frac{a_{10}a_9^3}{84}$$
$$+\frac{3a_2a_9^3}{56}+\frac{3a_3a_9^3}{56}+\frac{a_4a_9^3}{336}+\frac{a_5a_9^3}{336}+\frac{a_6a_9^3}{336}-\frac{a_7a_9^3}{84}-\frac{a_8a_9^3}{84}-\frac{a_9^4}{336}=\frac{7}{120}$$

(2.4 –x)

$h^5 f^4 f_{yyyy}$:

$$\frac{a_1^4}{112}+\frac{5a_2^4}{336}+\frac{5a_2^3a3}{84}+\frac{5a_2^2a_3^2}{56}+\frac{5a_2a_3^3}{84}+\frac{5a_3^4}{336}+\frac{a_4^4}{112}+\frac{a_4^3a_5}{28}+\frac{3a_4^2a_5^2}{56}+\frac{a_4a_5^3}{28}$$
$$+\frac{a_5^4}{112}+\frac{a_4^3a_6}{28}+\frac{3}{28}a_4^2a_5a_6+\frac{3}{28}a_4a_5^2a_6+\frac{a_5^3a_6}{28}+\frac{3a_4^2a_6^2}{56}+\frac{3}{28}a_4a_5a_6^2$$
$$+\frac{3a_5^2a_6^2}{56}+\frac{a_4a_6^3}{28}+\frac{a_5a_6^3}{28}+\frac{a_6^4}{112}+\frac{3a_{10}^2a_7^2}{112}+\frac{a_{10}a_7^3}{56}+\frac{a_7^4}{224}+\frac{3}{56}a_{10}^2a_7a_8$$
$$+\frac{3}{56}a_{10}a_7^2a_8+\frac{a_7^3a_8}{56}+\frac{3a_{10}^2a_8^2}{112}+\frac{3}{56}a_{10}a_7a_8^2+\frac{3a_7^2a_8^2}{112}+\frac{a_{10}a_8^3}{56}+\frac{a_7a_8^3}{56}$$

$$+\frac{a_8^4}{224}+\frac{3}{56}a_{10}^2a_7a_9+\frac{3}{56}a_{10}a_7^2a_9+\frac{a_7^3a_9}{56}+\frac{3}{56}a_{10}^2a_8a_9+\frac{3}{28}a_{10}a_7a_8a_9$$

$$+\frac{3}{56}a_7^2a_8a_9+\frac{3}{56}a_{10}a_8^2a_9+\frac{3}{56}a_7a_8^2a_9+\frac{a_8^3a_9}{56}+\frac{3a_{10}^2a_9^2}{112}+\frac{3}{56}a_{10}a_7a_9^2$$

$$+\frac{3a_7^2a_9^2}{112}+\frac{3}{56}a_{10}a_8a_9^2+\frac{3}{56}a_7a_8a_9^2+\frac{3a_8^2a_9^2}{112}+\frac{a_{10}a_9^3}{56}+\frac{a_7a_9^3}{56}+\frac{a_8a_9^3}{56}$$

$$+\frac{a_9^4}{224}=1/120$$

(2.4 –xi)

Equations (2.4 -i) to (2.4-xi) are simultaneously solved and obtained the required parameters, which are as follows:

a_1=0.237877235038103, a_2=-0.2511873529868023,
a_3= 0.7098880827120877, a_4= -0.35348824655933775,
a_5= 0.9907332405242792, a_6= 0.3595437671239466,
a_7=1.6181640809236344, a_8= -1.3905838215846875,
a_9= -0.16134445573089054, a_{10}= 0.602096438387009.

(2.5)

By substituting the values of above parameters in equation (2.2), we get a new fifth order Runge-Kutta method based on Heronian Mean as follows:

$$y_{n+1}=y_n+\frac{h}{14}\left[k_1+2\left(k_2+k_3\right)+2\left(k_3+k_4\right)+k_5+\sqrt{\left|k_1k_2\right|}+\sqrt{\left|k_2k_3\right|}+\sqrt{\left|k_3k_4\right|}+\sqrt{\left|k_4k_5\right|}\right] \quad (2.6)$$

where,

$$k_1=f(x_n,y_n)$$

$$k_2=f(x_n+0.237877235038103h,\ y_n+0.237877235038103hk_1)$$

$$k_3=f(x_n+(-0.2511873529868023+0.7098880827120877)h,$$
$$y_n-0.2511873529868023\,hk_1+0.7098880827120877\,hk_2)$$

$$k_4 = f(x_n + (-0.353488246559337775 + 0.9907332405242792 + 0.3595437671239466)h, y_n - 0.353488246559337775hk_1 + 0.9907332405242792hk_2 + 0.3595437671239466hk_3)$$

$$k_5 = f(x_n + (1.6181640809236344 - 1.3905838215846875 - 0.16134445573089054 + 0.602096438387009)h, y_n + 1.6181640809236344hk_1 - 1.3905838215846875hk_2 - 0.16134445573089054hk_3 + 0.602096438387009hk_4)$$

(2.7)

To rationalizing the coefficients of equation (2.5) then we obtained rational form of the equation (2.7) as given in the Appendix 1.

3 Derivation of Local Truncation Error(Lte) for Hem(5,5)

The Local truncation error for fifth order RK method (RKAM) is expressed as

$$y_{n+1}^{AM} = y_n + LTE^{AM}$$

(3.1)

and for the Heronian Mean Runge-Kutta method is,

$$y_{n+1}^{HeM} = y_n + LTE^{HeM}$$

(3.2)

Where y^{AM} and y^{HeM} are the numerical approximations at x_{n+1} obtained by the Arithmetic Mean and Heronian Mean methods respectively and LTE^{AM} and LTE^{HeM} are the Local Truncation Errors of the fifth order Arithmetic Mean Runge-Kutta method and the fifth order Heronian Mean Runge-Kutta method.

An error estimate is obtained by taking the difference between the two methods for the numerical approximations at x_{n+1} by

$$y_{n+1}^{AM} - y_{n+1}^{HeM} = LTE^{AM} - LTE^{HeM}$$

(3.3)

The Local Truncation Error for the fifth order Arithmetic Mean Runge-Kutta method involves y derivatives given by

$$LTE^{AM} = h^6 [\frac{1}{720} ff_y^5 + 0.0018022816 f^2 f_y^3 f_{yy} - 0.0166861138 f^3 f_y f_{yy}^2 + 0.0082646021 f^3 f_y^2 f_{yyy} + 0.0041171137 f^4 f_{yy} f_{yyy} - 0.0023096163 f^4 f_y f_{yyyy} + 0.0000588245 f^5 f_{yyyyy}] \ldots$$

while the Local Truncation Error for the Heronian Mean method is given by

$$LTE^{HeM} = h^6 [- 0.0012557831 f f_y^5 - 0.0072638436 f^2 f_y^3 f_{yy} - 0.0039318538 f^3 f_y f_{yy}^2 + 0.0010892514 f^3 f_y^2 f_{yyy} - 0.0002046344 f^4 f_{yy} f_{yyy} + 0.0021303658 f^4 f_y f_{yyyy} + 0.0004306801 f^5 f_{yyyyy}] \ldots \quad (3.5)$$

The absolute difference between LTE^{AM} and LTE^{HeM} is given by

$$LTE^{AM} - LTE^{HeM} = h^6 [0.00151447 ff_y^5 + 0.00906666 f^2 f_y^3 f_{yy} - 0.0127543 f^3 f_y f_{yy}^2 - 0.00717535 f^3 f_y^2 f_{yyy} + 0.00432175 f^4 f_{yy} f_{yyy} - 0.00443998 f^4 f_y f_{yyyy} - 0.000371856 f^5 f_{yyyyy}]$$

(3.6)

Following Lotkin[3] if the following bounds for f and its partial derivatives hold for x ∈ [a, b] and y ∈ [-∞, ∞] we have,

$$\left|f(x,y)\right| < Q, \quad \left|\frac{\partial^{i+j} f(x,y)}{\partial x^i \partial y^j}\right| < \frac{P^{i+j}}{Q^{i-1}}, \quad i+j \le P \qquad (3.7)$$

where P and Q are positive constants and p is the order of the method. In this case, we have p=5. Hence using equation (3.7), we have

$$\left.\begin{aligned}
\left|ff_y^5\right| &< Q\left(\frac{P^{0+1}}{Q^{I-1}}\right)^5 \\
\left|f^2 f_y^3 f_{yy}\right| &< Q^2\left(\frac{P^1}{Q^0}\right)^3 \frac{P^2}{Q} \\
\left|f^3 f_y f_{yy}^2\right| &< Q^3 P\left(\frac{P^2}{Q}\right) \\
\left|f^3 f_y^2 f_{yyy}\right| &< Q^3 P^2\left(\frac{P^3}{Q^2}\right)^5 \\
\left|f^4 f_{yy} f_{yyy}\right| &< Q^4 \frac{P^2}{Q}\frac{P^3}{Q^2} \\
\left|f^4 f_y f_{yyyy}\right| &< Q^4 P \frac{P^4}{Q^3} \\
\left|f^5 f_{yyy}\right| &< Q^5 \frac{P^5}{Q^4}
\end{aligned}\right\} P^5 Q \ldots \qquad (3.8)$$

From the equation (3.4) - (3.8), we obtain

$$\left|LTE^{AM} - LTE^{HeM}\right| \le 0.004512\ P^5 Q h^6 \qquad (3.9)$$

Hence,

$$\left|y_{n+1}^{AM} - y_{n+1}^{HeM}\right| \le 0.004512\, P^5 Q h^6 \qquad (3.10)$$

or

$$\left|y_{n+1}^{AM} - y_{n+1}^{HeM}\right| \le \frac{141}{31250} P^5 Q h^6$$

By taking the tolerance as TOL, i.e., Ɛ<0.0000001, then by setting

$$\left| y_{n+1}^{AM} - y_{n+1}^{HeM} \right| \leq TOL$$

the error control and step size selection can be determined by equation (3.10) to given formula as

$$0.004512\, P^5 Q h^6 < TOL$$

$$h < \left[\frac{TOL}{0.004512\, P^5 Q} \right]^{\frac{1}{6}} \qquad (3.11)$$

4 Derivation of Global Truncation Error(Gte) For Hem(5,5)

The LTE for HeM(5,5) is rewritten as

$$LTE^{HeM} = \frac{1}{414720}[-521 f f_Y^5 - 3013 f^2 f_y^3 f_{yy} - 1631 f^3 f_y f_{yy}^2$$
$$+ 452 f^3 f_y^2 f_{yyyy} - 85 f^4 f_{yy} f_{yyy} + 884 f^4 f_y f_{yyy}$$
$$+ 178 f^5 f_{yyyyy}] \qquad (4.1)$$

By using the same procedure adapted by Ponalagusamy and Senthilkumar [10] to evaluate the GTE for RKARMS(4,4), we have for the Heromain mean as

$$\varepsilon_{n+1} \leq \varepsilon_n + hL\varepsilon_n + \frac{h^2}{2} L\varepsilon_n + \frac{h^3}{6} L\varepsilon_n + \frac{h^4}{24} L\varepsilon_n + \frac{h^5}{120} L\varepsilon_n + \frac{h^6}{414720} y^{iv}(\varepsilon)$$

Where o<Ɛ<1,

$$\left|\varepsilon_{n+1}\right| \leq [1 + hL + \frac{h^2 L}{2} + \frac{h^3 L}{6} + \frac{h^4 L}{24} + \frac{h^5 L}{120}]\left|\varepsilon_n\right| + \frac{h^6}{414720} M \leq [1+A]\left|\varepsilon_n\right| + B$$

Where

$$A = L\left(\sum \frac{h^{p-1}}{p!} \right) h,\ B = \frac{h^6}{414720} M,\ C = 1 + A \ and \left| y^{iv}(\varepsilon) \right| \leq M$$

A simple induction proof leads to

$$\left|\varepsilon_n\right| \leq C^n \left|\varepsilon_0\right| + \left(\sum_{k=0}^{k-1} C^k \right) B$$

i.e., for C ≠ 1 and $\varepsilon_0 = 0$, the term | ε_n| can be written as,

$$\left|\varepsilon_n\right| \leq \left(\frac{C^n - 1}{C - 1}\right) B \qquad (4.2)$$

If we use the inequality $1+x \leq e^x$, we set

$$C^n = (1 + A)^n = [1 + hL(\sum_{p=1}^{5} \frac{h^{p-1}}{p!})]^n$$

$$\leq e^{Lhn(\sum_{p=1}^{5} \frac{h^{p-1}}{p!})}$$

$$= e^{DL(x_n - x_0)}$$

Where $D = \sum_{p=1}^{5} \frac{h^{p-1}}{p!}$ (4.3)

By substituting eqn(2) into eqn(1) for Ɛ, we obtain,

$$\left|\varepsilon\right| \leq (\frac{h^5}{414720\, LD}) M (e^{DL(x_n - x_0)} - 1) \qquad (4.4)$$

It is observed from eqn(4.4) that the GTE for the fifth order RK Heronian mean method is of order h^5. The LTE represents the magnitude of the error involved in the numerical solution obtained by the corresponding numerical technique. From the the above discussion, we can conclude that if the LTE of a numerical method is $O(h^{p+1})$ then the GTE is $O(h^6)$. The estimate of the GTE can not be used for practical error estimation or error control because the value from the GTE is less accurate than the LTE.

4 Stability Polynomial Of The New Fifth Order Rk Method

We discuss the stability region for the new fifth order Runge-Kutta method. To check on the stability, the simple test equation $y' = yo_n$ is used. In this case, equation(2.7) becomes

$$k_1 = \lambda(y_n)$$

$$k_2 = \lambda(y_n + 0.237877235038103hk_1)$$

$$k_3 = \lambda(y_n - 0.251187352 9868023hk_1 + 0.709888082 7120877hk_2)$$

$$k_4 = \lambda(y_n - 0.35348824655933775 h k_1 + 0.99073324052427792 h k_2 + 0.35954376712394 66 h k_3)$$

$$k_5 = \lambda(y_n + 1.61816408092363 44 h k_1 - 1.39058382158468 75 h k_2 - 0.16134445573089054 h k_3 + 0.602096438387009 h k_4) \quad (5.1)$$

Using equations(2.6) and (5.1), the corresponding stability polynomial is obtained as

$$y_{n+1} = y_n + h\lambda y_n + 0.5\,(h\lambda)^2 y_n + 0.162594\,(h\lambda)^3 y_n + 0.034733\,(h\lambda)^4 y_n + 0.004883\,(h\lambda)^5 y_n + (h)^6 \quad (5.2)$$

By substituting h λ= z in equation (5.2), we get,

$$y_{n+1} = y_n + y_n [z + 0.5\, z^2 + 0.162594\, z^3 + 0.034733\, z^4 + 0.004883\, z^5] + O(z)^6 \quad (5.3)$$

Defining

$$\frac{y_{n+1}}{y_n} = Q, \quad (5.4)$$

equation (5.3) becomes

$$Q = 1 + z + 0.5\, z^2 + 0.162594\, z^3 + 0.034733\, z^4 + 0.004883\, z^5 + O(z)^6 \quad (5.5)$$

Similarly, one can obtain the stability polynomial for fifth order Runge-Kutta method based on contraharmonic mean (CoM) and it is expressed as

$$Q = 1 + z + 0.5\, z^2 + 0.166667\, z^3 + 0.04166667\, z^4 + 0.00833333\, z^5 + O(z)^6 \quad (5.6)$$

And also the stability polynomial for fifth order Runge-Kutta method based on Harmonic Mean (HM) [15] is

$$Q = 1 + z + 0.499999\, z^2 + 0.166667\, z^3 + 0.04166667\, z^4 + 0.00833333\, z^5 + O(z)^6 \quad (5.7)$$

To determine the stability region of the fifth order RK formula in the complex plane that satisfies the condition

$$\left|\frac{y_{n+1}}{y_n}\right| = Q < 1$$

5. Stability Region of New Fifth Order Five-Stage Rk Method

With the help of stability polynomials, the stability regions for the fifth order Runge-Kutta formula based on Harmonic Mean (shown as square format), the fifth order Runge-Kutta method based on Contraharmonic Mean (shown as circle format) and the proposed fifth order Runge-Kutta method based on Heronian Mean (shown as triangle format) are depicted in Figure 1. It is observed that the present fifth order RK method based on heronian mean has a wider stability region in comparison with other two fifth order methods.

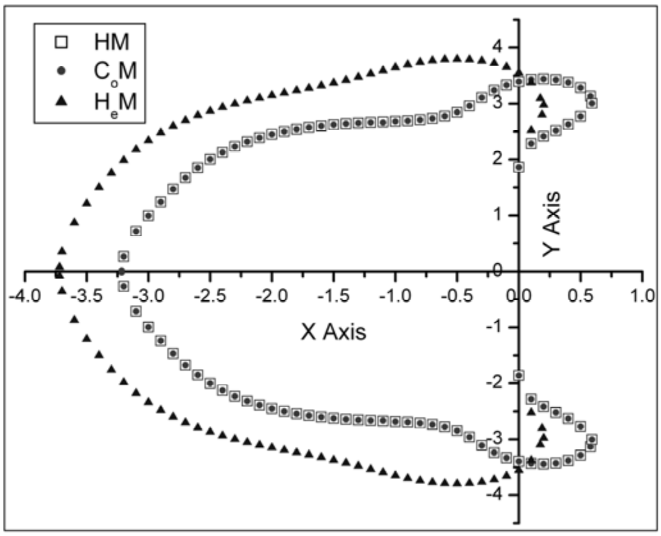

Figure 1. Comparison of Stability regions for the fifth order RK based on HM, C_oM & H_eM

It is concluded from Table 1 that our new fifth order method (HeM) has got the better stability region in the negative real axis and both in the positive and negative imaginary axis as compared to the existing two fifth order methods(HM and CoM).

Fifth order methods	Real-Axis Negative	Real-Axis Positive	Imaginary-Axis Negative	Imaginary-Axis Positive
HM	-3.2170478	0.5	-3.395751592946766	3.395751592946766
C_oM	-3.217	0.5	-3.395751592142904	3.3957515921429042
H_eM	-3.72	0.1	-3.552981719790954	3.552981719790954

Table 1. Ranges of the stability regions for the various methods

6. Numerical Experimentation

Problem No.	Differential Equation	Initial Value	Exact solution
1.	$\frac{dy}{dx} = -y + x + 1$	$x_0 = 0$; $y_0 = 1$	$y = x + \exp(-x)$
2.	$\frac{dy}{dx} = x^2 - y + 1$	$x_0 = 0$; $y_0 = 1$	$y = -2e^{-x} + x^2 - 2x + 3$
3.	$\frac{dy}{dx} = 1 + \sin x - \frac{y}{1+x}$	$x_0 = 0$; $y_0 = 1$	$y = \frac{x + \frac{x^2}{2} - \cos x - x\cos x + \sin x + 2}{1+x}$
4.	$\frac{dy}{dx} = y + 2x;$	$x_0 = 0$; $y_0 = 0$	$y = 2\left[e^{x} - 2 - 1\right]$
5.	$\frac{dy}{dx} = -y - 2x;$	$x_0 = 0$; $y_0 = -1$	$y = 2 - 3e^{-x} - 2x$
6.	$\frac{dy}{dx} = \frac{2x}{y} - xy\ ;$	$x_0 = 0$; $y_0 = 1$	$y = \sqrt{2 - e^{-x^2}}$
7.	$\frac{dy}{dx} = x + y\ ;$	$x_0 = 0$; $y_0 = 1$	$y = 2e^{x} - x - 1$

The numerical results along with the exact solutions of the problems(1-7) are presented respectively in Tables (2-15) for the values of h = 0.1 and 0.2.

step size	Exact sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0	1	0	0	0	0	0	0
0.2	1.01873075	1.01739612	1.02119312	1.01857799	0.0013346	0.00246237	0.0001528
0.4	1.07032005	1.06893258	1.07284691	1.07017541	0.0013875	0.00252686	0.0001446
0.6	1.14881164	1.14751945	1.15114725	1.14868409	0.0012922	0.00233562	0.0001275
0.8	1.24932896	1.24817148	1.25141016	1.24921942	0.0011575	0.0020812	0.0001096
1	1.36787944	1.36686277	1.36970016	1.3677866	0.0010167	0.00182072	9.284E-05

Table 2. Numerical solution of the problem-I with step size h=0.1

step size	Exact sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	1.000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	1.0187307	1.0139507	1.0269076	1.0181700	0.0047799	0.0081769	0.000560
0.4	1.0703200	1.0651415	1.0792400	1.0697788	0.0051785	0.0089200	0.000541
0.6	1.1488116	1.1438923	1.1572747	1.1483282	0.0049192	0.0084631	0.000483
0.8	1.24932896	1.24486672	1.25698932	1.24890978	0.00446225	0.0076603	0.000419
1	1.36787944	1.36392413	1.37465503	1.36752132	0.00395531	0.0067755	0.000358

Table 3. Numerical solution of the problem-I with step size h=0.2

Step Size	Exact Sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	1.0025385	1.0021979	1.0030973	1.0025303	0.0003406	0.0005588	0.0000082
0.4	1.0193599	1.0186946	1.0204534	1.0193490	0.0006653	0.0010935	0.0000109
0.6	1.0623767	1.0614487	1.0638890	1.0623666	0.0009281	0.0015123	0.0000101
0.8	1.1413421	1.1402070	1.1431784	1.1413351	0.0011351	0.0018363	0.0000069
1.0	1.2642411	1.2629454	1.2663252	1.2642388	0.0012958	0.0020841	0.0000023

Table 4. Numerical solution of the problem-II with step size h=0.1

Step Size	Exact Sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	1.0025385	1.0006084	1.0045188	1.0024824	0.0019301	0.0019804	0.0000561
0.4	1.0193599	1.0161283	1.0236357	1.0192792	0.0032317	0.0042758	0.0000807
0.6	1.0623767	1.0580467	1.0684810	1.0622888	0.0043299	0.0061043	0.0000879
0.8	1.1413421	1.1361245	1.1488616	1.1412587	0.0052176	0.0075195	0.0000834
1	1.2642411	1.2583223	1.2728407	1.2641702	0.0059189	0.0085996	0.0000709

Table 5. Numerical solution of the problem-II with step size h=0.2

Step Size	Exact Sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	1.0354912	1.0330049	1.0400949	1.0351831	0.0024863	0.0046038	0.0003082
0.4	1.1285235	1.1259529	1.1332422	1.1282193	0.0025706	0.0047187	0.0003043
0.6	1.2650659	1.2626169	1.2695434	1.2647789	0.0024490	0.0044775	0.0002871
0.8	1.4351578	1.4328751	1.4393247	1.4348879	0.0022827	0.0041669	0.0002699
1.0	1.6304332	1.6283234	1.6342847	1.6301776	0.0021098	0.0038516	0.0002556

Table 6. Numerical solution of the problem-III with step size h=0.1

Step Size	Exact Sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	1.0354912	1.0268877	1.0503668	1.0343800	0.0086035	0.0148756	0.0011112
0.4	1.1285235	1.1193139	1.1445946	1.1274079	0.0092096	0.0160711	0.0011155
0.6	1.2650659	1.2561704	1.2806199	1.2640034	0.0088955	0.0155541	0.0010626
0.8	1.4351578	1.4268125	1.4497798	1.4341521	0.0083453	0.0146221	0.0010057
1.0	1.6304332	1.6226999	1.6440248	1.6294761	0.0077332	0.0135916	0.0009571

Table 7. Numerical solution of the problem-III with step size h=0.2

Step Size	Exact sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	0.0428055	0.0399179	0.0488471	0.0424025	0.0028877	0.0060416	0.0004031
0.4	0.1836494	0.1793191	0.1925350	0.1830612	0.0043303	0.0088856	0.0005882
0.6	0.4442376	0.4384315	0.4560564	0.4434417	0.0058061	0.0118188	0.0007959
0.8	0.8510819	0.8435889	0.8662663	0.8500352	0.0074929	0.0151844	0.0010467
1.0	1.4365637	1.4270724	1.4557431	1.4352076	0.0094913	0.0191794	0.0013561

Table 8. Numerical solution of the problem-IV with step size h=0.1

Step Size	Exact sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	0.0428055	0.0352094	0.0601718	0.0415496	0.0075961	0.0173662	0.0012559
0.4	0.1836494	0.1715588	0.2104527	0.1817614	0.0120906	0.0268033	0.0018880
0.6	0.4442376	0.4276247	0.4806126	0.4416346	0.0166129	0.0363749	0.0026030
0.8	0.8510819	0.8293522	0.8983431	0.8476104	0.0217297	0.0472613	0.0034715
1.0	1.4365637	1.4088045	1.4966878	1.4320142	0.0277592	0.0601242	0.0045494

Table 9. Numerical solution of the problem-IV with step size h=0.2

Step Size	Exact sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	-1.00000	0.00000	0.00000	0.000000	0.0000000	0.0000000	0.0000000
0.2	-0.85619	-0.85698	-0.85491	-0.85628	0.0007858	0.0012813	0.0000918
0.4	-0.81096	-0.81401	-0.80564	-0.81148	0.0030518	0.0053159	0.0005174
0.6	-0.84643	-0.84582	-0.84715	-0.82695	0.0006099	0.0007115	0.0194904
0.8	-0.94799	-0.94688	-0.94960	-0.87295	0.0011053	0.0016168	0.0750387
1.0	-1.10364	-1.10241	-1.10549	-0.94724	0.0012229	0.0018614	0.1563955

Table 10. Numerical solution of the problem-V with step size h=0.1

Step Size	Exact sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	-1.00000	0.00000	0.00000	0.00000	0.0000000	0.0000000	0.0000000
0.2	-0.85619	-0.85959	-0.85112	-0.85659	0.0033959	0.0050676	0.0003973
0.4	-0.81096	-0.82236	-0.79328	-0.81335	0.0113997	0.0176760	0.0023898
0.6	-0.84643	-0.84452	-0.84775	-0.82897	0.0019129	0.0013155	0.0174639
0.8	-0.94798	-0.9438	-0.95345	-0.87372	0.0041991	0.0054595	0.0742689
1.0	-1.10364	-1.09881	-1.11040	-0.94603	0.0048323	0.0067670	0.1576028

Table 11. Numerical solution of the problem-V with step size h=0.2

Step Size	Exact sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	1.0194168	1.0181218	1.0219868	1.0192496	0.0012950	0.0025701	0.0001672
0.4	1.0713805	1.0700833	1.0739487	1.0712084	0.0012972	0.0025681	0.0001721
0.6	1.1411939	1.1401701	1.1432604	1.1410398	0.0010239	0.0020665	0.0001542
0.8	1.2135516	1.2128597	1.2150059	1.2134238	0.0006919	0.0014542	0.0001278
1.0	1.2775447	1.2771388	1.2784619	1.2774443	0.0004059	0.0009172	0.0001005

Table 12. Numerical solution of the problem-VI with step size h=0.1

Step Size	Exact sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	1.0194168	1.0153846	1.0276819	1.0188484	0.0040322	0.0082652	0.0005684
0.4	1.0713805	1.0671548	1.0800682	1.0707878	0.0042257	0.0086877	0.0005927
0.6	1.1411937	1.1378950	1.1483109	1.1406569	0.0032989	0.0071169	0.0005370
0.8	1.2135516	1.2114529	1.2185842	1.2131015	0.0020987	0.0050326	0.0004502
1.0	1.2775447	1.2764843	1.2807063	1.2771879	0.0010604	0.0031616	0.0003568

Table 13. Numerical solution of the problem-VI with step size h=0.2

Step Size	Exact sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	1.2428055	1.2426216	1.2431501	1.2427678	0.0001839	0.0003446	0.0000377
0.4	1.5836494	1.5832652	1.5843689	1.5835630	0.0003842	0.0007195	0.0000864
0.6	2.0442376	2.0436242	2.0453857	2.0440875	0.0006134	0.0011481	0.0001501
0.8	2.6510819	2.6501993	2.6527335	2.6508478	0.0008826	0.0016516	0.0002341
1.0	3.4365637	3.4353598	3.4388159	3.4362191	0.0012039	0.0022523	0.0003445

Table 14. Numerical solution of the problem-VII with step size h=0.1

Step Size	Exact sol	HM	CoM	HeM	Error by HM	Error by CoM	Error by HeM
0.0	1.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.2	1.2428055	1.2421474	1.2441178	1.2426565	0.0006582	0.0013122	0.0001491
0.4	1.5836494	1.5822732	1.5863892	1.5833059	0.0013762	0.0027398	0.0003435
0.6	2.0442376	2.0420397	2.0486088	2.0436373	0.0021979	0.0043712	0.0006004
0.8	2.6510819	2.6479181	2.6573689	2.6501415	0.0031638	0.0062871	0.0009404
1.0	3.4365636	3.4322473	3.4451357	3.4351742	0.0043163	0.0085721	0.0013895

Table 15. Numerical solution of the problem-VII with step size h=0.2

Our proposed fifth order Runge-Kutta method based on heronian mean is an effective numerical tool in the sense that it computes more efficient results with considerable less error compared to other exiting numerical techniques based on various means considered in the present paper. Hence, the proposed fifth order Runge-Kutta method based on heronian mean gives another alternative approach for solving initial value problems of ordinary differential equations.

7 Conclusion

In the present paper, we have developed a new fifth order Runge-Kutta technique based on heronian mean and obtained the stability polynomial. Several practically applicable problems have been considered to test the suitability, adoptability and accuracy of the proposed method. It is noticed from the discussion that the new fifth order Runge-Kutta method based on heronian mean is more efficient than the well known fifth order Runge-Kutta method based Harmonic mean and fifth order weighted Runge-Kutta method based on contra harmonic mean. A remarkable result is that the new fifth order Runge-Kutta method based on heronian mean guarantees the most efficient numerical solution in all of the problems tested.

Reference

[1] Alexander, R.K. and Coyle, J.J. (1990) 'Runge-Kutta methods for differential-algebraic systems', *SIAM Journal of Numerical Analysis*, Vol. 27, No. 3, pp.736–752.

[2] Evans, D.J. (1991) 'A new 4th order Runge-Kutta method for initial value problems with error control', *International Journal of Computer Mathematics*, Vol. 139, pp.217–227.

[3] Hung, C. (2000) 'Dissipitivity of Runge-Kutta methods for dynamical systems with delays', *IMA Journal of Numerical Analysis*, Vol. 20, pp.153–166.

[4] Shampine, L.F. and Gordon, M.K. (1975) *Computer Solutions of Ordinary Differential Equations*, p.23, W.H. Freeman, San Francisco, CA.

[5] Shampine, L.F. and Watts, H.A. (1977) 'The art of a Runge-Kutta code. Part I', M*athematical Software*, Vol. 3, pp.257–275.

[6] L.F.Shampine. and H.A.Watts (1985), Some Practical Runge-Kutta Formulas, Mathematics of Computations, 46. 135-150.

[7] L.F. Shampine and H.A. Watts (1979), The Art of writing a Runge-Kutta Code. Part II, Applied Mathematical Computations, 5. 93-121.

[8] R.Ponalagusamy and S.Senthilkumar (2010), System of second order Robot Arm Problem by an Efficient Numerical Integration Algorithm, Int. J. Computational Materials Science and Surface Engineering , 3(2/3), 237-250.

[9] R.Ponalagusamy and S.Senthilkumar (2007), A Comparison of RK-Fourth Orders of Variety of Means and Embedded Means on Multilayer Raster CNN Simulation, J. Theoretical and Applied Information Technology, 3(4), 07-14.

[10] R. Ponalagusamy and S. Senthilkumar (2009), A New Fourth Order Embedded RKAHeM(4,4) Method with Error Control on Multilayer Raster Cellular Neural Network, Signal, Image and video Processing, 3. 1-11.

[11] R. Ponalagusamy and S. Senthilkumar (2009), A New Method of Embedded Fourth Order with Four Stages to Study Raster CNN Simulation, Intern. J. Automation and Computing, 6(3). 285-294.

[12] D. J. Evans and A. R. Yaakub (1995), A New Fourth Order Runge-Kutta Formula Based on the Contraharmonic Mean, Intern. J. Computer Math. 57. 249-256.

[13] N. Yaacob and B. Sanugi (1998), A New Fourth-order Embedded Method Based on the Harmonic Mean, Mathematika, 14. 1-6.

[14] J.C. Butcher (1964), On Runge Processes of Higher Order, J. Australian Mathematical Society, 4. 179-189.

[15] D. J. Evans and A. R. Yaakub (2002), A Fifth Order Runge-Kutta RK(5,5) Method with Error Control, Intern. J. Computer Math., 79. 1179-1185.

[16] N. Razali, R.R.Ahmad, M. Darus and A.S. Rambely (2008), Fifth Order Mean Runge-Kutta Methods Applied to the Lorenz System, Proc. Of the 13th WSEAS Intern. Conf. on Applied Mathematics, 333-338.

[17] D. J. Evans and A. R. Yaakub (1996), A New Fifth Order Weighted Runge-Kutta Formula, Intern. J. Computer Math. 59. 227-243.

[18] A. R. Yaakub and D. J. Evans (1996), A New Fifth Order Contraharmonic Mean Method Weighted Runge-Kutta Formula For Initial Value Problems, Report No. 993, Loughborough University of Technology, Department of Computer Studies.

Appendix 1

These are the rationalized coefficients for the equation (7), we get

$$a_1 = 0.237877235038103 = \frac{38789865}{163066739}, \qquad a_2 = -0.2511873529868023 = -\frac{38926273}{154969080}$$

$a_3 = 0.7098880827120877 = \frac{49239911}{69362921}$, $a_4 = -0.35348824655933775 = -\frac{46861542}{132568883}$

$a_5 = 0.9907332405242792 = \frac{415527214}{419413821}$, $a_6 = 0.3595437671239466 = \frac{38385758}{106762407}$

$a_7 = 1.6181640809236344 = \frac{311954026}{192782691}$, $a_8 = -1.3905838215846875 = -\frac{40359572}{29023473}$

$a_9 = -0.16134445573089054 = -\frac{25308289}{156858746}$, $a_{10} = 0.602096438387009 = \frac{45864583}{76174812}$

and this new fifth order Runge-Kutta method can be written in rational form as

$$y_{n+1} = y_n + \frac{h}{14}\left[k_1 + 2\left(k_2 + k_3\right) + 2\left(k_3 + k_4\right) + k_5 + \sqrt{\left|k_1 k_2\right|} + \sqrt{\left|k_2 k_3\right|} + \sqrt{\left|k_3 k_4\right|} + \sqrt{\left|k_4 k_5\right|} \right]$$

Where,

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{38789865}{163066739}h, y_n + \frac{38789865}{163066739}hk_1)$$

$$k_3 = f(x_n + (-\frac{38926273}{154969080} + \frac{49239911}{69362921})h, y_n - \frac{38926273}{154969080}hk_1 + \frac{49239911}{69362921}hk_2)$$

$$k_4 = f(x_n + (-\frac{46861542}{132568883} + \frac{415527214}{419413821} + \frac{38385758}{106762407})h, y_n - \frac{46861542}{132568883}hk_1 + \frac{415527214}{419413821}hk_2 + \frac{38385758}{106762407}hk_3)$$

$$k_5 = f(x_n + (\frac{311954026}{192782691} - \frac{40359572}{29023473} - \frac{25308289}{156858746} + \frac{45864583}{76174812})h, y_n + \frac{311954026}{192782691}hk_1 - \frac{40359572}{29023473}hk_2 - \frac{25308289}{156858746}hk_3 + \frac{45864583}{76174812}hk_4)$$