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Structural Properties of k-Distance Closed Domination Critical Graphs for k = 3 and 4

T.N. Janakiraman[1], P. J. A. Alphonse[2] and V. Sangeetha[3]

[1,3]*Department of Mathematics, NIT, Tiruchirapalli, Tamilnadu, India.*

E.mail: janaki@nitt.edu, sangeetha77_2005@yahoo.co.in

[2]*Department of Computer Applications, NIT, Tiruchirapalli, Tamilnadu, India.*

E.mail: alphonse@nitt.edu

Abstract: *In a graph $G = (V, E)$, a set $S \subset V(G)$ is a distance closed set of G if for each vertex $u \in S$ and for each $w \in V - S$, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$. Also, S is said to be a distance closed dominating set of G if (i) <S> is distance closed and (ii) S is a dominating set. The critical concept in graphs plays an important role in the study of structural properties of graphs and hence it will be useful to study any communication model. In this paper, we discuss the critical concept in distance closed domination which deals with those graphs that are critical in the sense that their distance closed domination number drops when any missing edge is added. Also, we analyze some structural properties of those distance closed domination critical graphs.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self centered graph, neighborhood, induced sub graph, distance closed dominating set, distance closed domination critical graph.*

1. Introduction

Graphs discussed in this paper are connected and simple graphs only. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $deg_G(v)$ and minimum degree and maximum degree of G are indicated by $\delta(G)$ and $\Delta(G)$ respectively. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max \{d_G(u, v): \forall u \in V(G)\}$ and the eccentric set $E_G(v) = \{u \in V(G): d(v, u) = e_G(v)\}$. If there is no confusion, we simply use the notation deg (v), d(u, v), e(v) and E(v) to denote degree, distance, eccentricity and eccentric set respectively for a connected graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted by r(G) and d(G) respectively. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r self-centered* graph. Such graphs are 2-connected graphs. A vertex u is said to be an eccentric vertex of v in a graph G, if d(u, v) = e(v) in that graph. For $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v] = N_G(v) \cup \{v\}$ is called the *closed neighborhood* of v.

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One important aspect of the concept of distance and eccentricity is the existence of polynomial time algorithm to analyze them. The concept of distance and related properties are studied in [3], [6] and [14] and the structural properties of radius and diameter critical graphs are studied in [7] and [15]. The new concepts such as ideal sets, distance preserving sub graphs, eccentricity preserving sub graphs, super eccentric graph of a graph, pseudo geodetic graphs are introduced and structural properties of those graphs are studied in [11].

The concept of domination in graphs originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters. A set $D \subseteq V(G)$ is called a dominating set of G if every vertex in $V(G) - D$ is adjacent to some vertex in D and D is said to be a minimal dominating set if $D - \{v\}$ is not a dominating set for any $v \in D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$. Different types of dominating sets have been studied by imposing conditions on dominating sets. A dominating set D is called a *connected* dominating set if the induced sub graph <D> is connected. The list of survey of domination and connected domination papers are in [4], [5], [8], [9], [13] and [16]. Janakiraman and Alphonse [2] introduced and studied the concept of weak convex dominating sets, which mixes the concept of dominating set and distance preserving set. Using these, structural properties of various dominating parameters are studied.

Graphs which are critical with respect to a given property frequently play an important role in the investigation of that property. Not only are such graphs of considerable interest in their own right, but also a knowledge of their structure often aids in the development of general theory. In particular, when investigating any finite structure, a great number of results are proven by induction. Consequently, it is desirable to learn as much as possible about those graphs that are critical with respect to a given property so as to aid and abet such investigation. A graph G is said to be domination critical if for every edge $e \notin E(G)$, $\gamma(G + e) < \gamma(G)$. If G is a domination critical graph with $\gamma(G) = k$, we will say G is k-domination critical or just k-critical. The 1-critical graphs are K_n, for $n \geq 1$. The concept of domination critical graphs and their structural properties are studied in [1], [10] and [17].

In this paper, we introduced a new domination critical graph called distance closed domination critical graphs through which the structural properties of those graphs are studied. Since this concept deals the reduction in the cardinality of distance closed dominating set for any addition of one new link in the original structure, it will be useful to study the communication model, which reduces it dominating parameters by simple addition of a link, which doesn't exist in the system. Hence this critical concept can be directly applied to the construction of a fault tolerant communication model.

2. Prior Results

The concept of ideal set is defined and studied in the doctoral thesis of Janakiraman [11] and the concept of ideal sets in graph theory is due to the related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to the multiplicative closure property with the elements of that ring. Similarly, the ideal set in a graph is defined with respect to the distance property between the ideal set and the vertices of the graph. Thus, the ideal set of a graph G is defined as follows:

Let I be a vertex subset of G. Then I is said to be an *ideal set* of G if

(i) For each vertex $u \in I$ and for each $w \in V - I$, there exists at least one vertex $v \in I$ such that $d_{<I>}(u, v) = d_G(u, w)$.

(ii) I is the minimal set satisfying (i).

Also, a graph G is said to be a 0-ideal graph if it has no non-trivial ideal set other than G. The ideal set without the minimality condition is taken as a distance closed set in the present work. Hence, the distance closed set of a graph G is defined as follows:

A vertex subset S of G is said to be a *distance closed set* of G if for each vertex $u \in S$ and for each $w \in V - S$, there exists at least one vertex $v \in S$ such that $d_{<S>}(u, v) = d_G(u, w)$. For example, in the graph given in Figure 2.1, S = {4, 1, 2, 5} is a distance closed set.

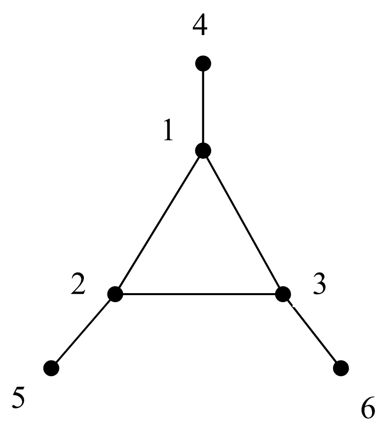

Figure 2.1 - ***An example of distance closed set***

Thus, the distance closed dominating set of a graph G is defined as follows:

A subset $S \subseteq V(G)$ is said to be a *distance closed dominating* (*D.C.D*) *set*, if

(i) <S> is distance closed;

(ii) S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the *distance closed domination number* of G and is denoted by γ_{dcl}.

Clearly from the definition, $1 \leq \gamma_{dcl} \leq p$ and graph with $\gamma_{dcl} = p$ is called a 0-distance closed dominating graph. Also, if S is a D.C.D set of G then the complement V – S need not be a D.C.D set of G. The definition and the extensive study of the above said distance closed domination in graphs are studied in [12]. The following results given in [12] are used to prove many results in the present work.

Theorem 2.1 [12]: $\gamma_{dcl}(G) = 1$ if and only if $G = K_1$.

Theorem 2.2 [12]: Let G be a graph of order p. Then

(i) $\gamma_{dcl}(G) = 2$ if and only if G has at least two vertices of degree p – 1.

(ii) $\gamma_{dcl}(G) = 3$ if and only if G has exactly one vertex of degree p – 1.

Theorem 2.3 [12]: If a graph G is connected and $d(G) \geq 3$, then $\gamma_{dcl}(\overline{G}) = 4$.

3. Main Results

Continuing the above, we studied the critical concept of the distance closed domination in graphs while adding an edge in that graph. The distance closed domination critical graph is defined as follows.

A graph G is said to be a *distance closed domination critical* if for every edge $e \notin E(G)$, $\gamma_{dcl}(G + e) < \gamma_{dcl}(G)$. If G is a D.C.D critical graph with $\gamma_{dcl}(G) = k$, then G is said to be k-D.C.D critical.

Theorem 3.1: There is no 2-D.C.D critical graph.

$\gamma_{dcl}(G) = 1$ if and only if $G \cong K_1$. Also, there is no graph G in which $(G + e) \cong K_1$. Hence the result.

Theorem 3.2: G is 3-D.C.D critical if and only if

(i) G is connected.

(ii) G has $\gamma_{dcl}(G) = 3$.

(iii) G has exactly one vertex with eccentricity equal to 1.

(iv) For every pair of non adjacent vertices at least one of them is of degree p – 2.

Proof: Results (i) and (ii) are trivial.

(iii) By (ii), $\gamma_{dcl}(G) = 3$. Then by Theorem 2.2, G must have exactly one vertex with eccentricity equal to 1.

(iv) To prove, for every pair of non adjacent vertices at least one of them is of degree p – 2, let us take x, y be any two non adjacent vertices. As G is a 3-D.C.D critical graph, the inclusion of the edge xy reduces the domination number to 2. Therefore, at least one of the vertices x or y must dominate all the vertices of (G + xy). That is any one of x or y must have degree p – 1 in (G + xy). Hence, d(x) or d(y) must be equal to p – 2 in G.

Proof of the converse is trivial.

Corollary 3.1: If G is a 3-D.C.D critical graph, then $\Delta(G) = p - 1$ and $1 \leq \delta(G) \leq p - 2$.

Lemma 3.1: Any 3-D.C.D critical graph has diameter equal to two.

Proof: Let G be a 3-D.C.D critical graph. Then by result (iii) of Theorem 3.2, G must have exactly one vertex with eccentricity equal to 1. Hence, the radius of G is 1 and hence the diameter of G must be equal to two.

Lemma 3.2: Any 3-D.C.D critical graph with $\delta \geq 2$ is a block.

Proof: If G is a 3-D.C.D critical graph with $\delta(G) = 1$, then the vertex with eccentricity 1 must be a cut vertex of G. Hence, G cannot be a block. But if $\delta(G) \geq 2$, then G must be a block.

Lemma 3.3: If G is a 3-D.C.D critical graph, then any minimum D.C.D set of G can have at most one vertex of degree less than or equal to p – 3.

Proof: Let G be a 3-D.C.D critical graph and let D = {u, v, w} be the minimum D.C.D set of G. Then clearly d(v) = p – 1 and also u and w are non adjacent. Suppose that, if both d(u) and d(w) are of degree less than or equal to p – 3, then u and w are any two non adjacent vertices of degree less than or equal to p – 3 in G, a contradiction to result (iv) of Theorem 3.2. Hence, any minimum D.C.D set D of G can have at most one vertex of degree less than or equal to p – 3.

Lemma 3.4: Any 3-D.C.D critical graph has at least two p – 2 degree vertices.

Proof: Let G be a 3-D.C.D critical graph and let D = {u, v, w} be the minimum D.C.D set of G. Then by Lemma 3.3, D can have at most one vertex of degree less than or equal to p – 3. Let d(u) ≤ p – 3 and d(v) = p – 1. Then degree of w must be equal to p – 2, as it is non adjacent to u. Hence, D must have at least one vertex of degree p – 2.

As d(u) ≤ p – 3 and it is non adjacent to w, it must be non adjacent to at least one vertex of V – D, say x. Then d(x) = p – 2, as it is non adjacent to a vertex of degree less than or equal to p – 3. Hence, V – D has at least one vertex of degree p – 2 and hence G has at least two vertices of degree equal to p – 2.

Theorem 3.3: If G is a 3-D.C.D critical graph and if $A = \{u \in V(G) \mid deg(u) \leq p - 3\}$ and $B = \{v \in V(G) \mid v \in E(u), u \in A\}$, then we have the following:

(i) <A> is a clique

(ii) deg (v) = p – 2, for every $v \in B$ and <B> is a clique

Proof: Let G be a 3-D.C.D critical graph and let A = {u∈V(G) | deg(u) ≤ p – 3} and B = {v∈V(G) | v∈E(u), u∈A}.

(i) By result (iv) of Theorem 3.2, for every pair of non adjacent vertices, at least one of them is of degree p – 2 in G. Hence, no two vertices of degree less than or equal to p – 3 are non adjacent and hence <A> is a clique.

(ii) If a vertex v∈B, then v is an eccentric point of a vertex u of degree less than or equal to p – 3. Therefore, v is non adjacent to u and degree of v must be equal to p – 2. Also, v must be adjacent to all the vertices of V(G) – {u}, as d(v) = p – 2. This is the case for every vertex v in B. Hence, <B> is a clique.

Remark 3.1: From the above theorem, it is clear that the vertex set of any 3-D.C.D critical graph V(G) – {v} (where v is a vertex of degree p – 1) is partitioned into three disjoint subsets A, B and C, where A = {u∈V(G) | deg(u) ≤ p – 3}; B = {v∈V(G) | deg(v) = p – 2 and v∈E(u), u∈A} and C = V(G) – A ∪ B = {w∈V(G) | deg(w) = p – 2}. Also, the structure of any 3-D.C.D critical graph is given as follows.

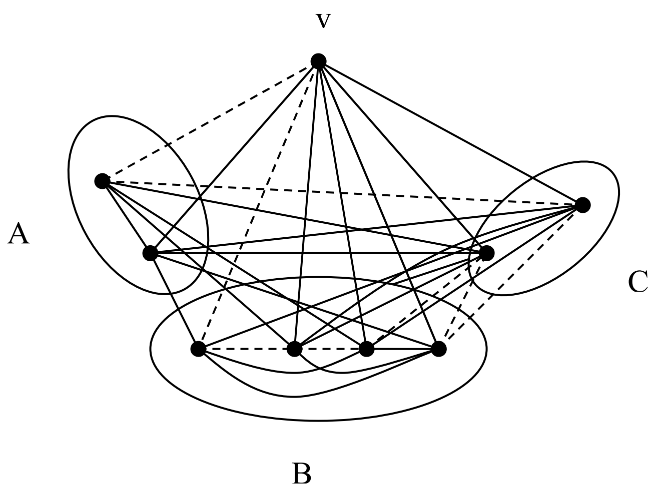

Figure 3.1 - ***Structural design of 3-D.C.D critical graphs***

From the above structure, we have the following:

1. <A> and <B> are cliques.
2. Every node w in C has its eccentric node in C itself. Also, every node w in C is non adjacent to exactly one vertex of C and it must be adjacent to all the vertices of A and B. Therefore, |C| is always even.
3. Every node v in B has its eccentric node in A and also every node in B is adjacent to all the vertices of C.

4. Every node u in A has its eccentric set in B. That is, $E(u) \subseteq B$, for every u in A. Also, every node in A is adjacent to all the vertices C.
5. Any 3-D.C.D critical graph with $\delta \geq 2$ is Hamiltonian.
   If G is a 3-D.C.D critical graph, then G has the structure given in Figure 3.1 and also G has a cycle $(v \rightarrow A \rightarrow C \rightarrow B \rightarrow v)$ that covers all the vertices of G (refer Figure 3.1). Hence, G has a Hamiltonian cycle and hence G is Hamiltonian.

Proposition 3.1: There exists no graph G for which both G and $\overline{G}$ are 3-D.C.D critical.
Proof: Without loss of generality, let us assume that G is a 3-D.C.D critical graph. Then any vertex in G is of degree at least 1. Hence, the degree of any vertex in $\overline{G}$ is of degree at most p – 2 and hence $\overline{G}$ cannot be 3-D.C.D critical.

Proposition 3.2: If G is a 3-D.C.D critical graph, then $q \leq \frac{1}{2}(p-1)^2$.
Proof: Let G be a 3-D.C.D critical graph. Then, we know that G has exactly a vertex of degree (p – 1) and $\delta \leq p - 2$.

Therefore, $2q = \sum_{u \in V(G)} d(u)$

$$= 1(p-1) + (p-1)(\delta)$$
$$\leq (p-1) + (p-1)(p-2)$$
$$\leq (p-1)(p-1)$$

Hence, $q \leq \frac{1}{2}(p-1)^2$.

Theorem 3.4: A graph G is 4-D.C.D critical if and only if
(i) G is connected.
(ii) G has $\gamma_{dcl}(G) = 4$.
(iii)For any two non adjacent vertices at least one of them is of degree p – 2.
Proof: Proof for results (i) and (ii) are trivial.
(iii) Let G be a 4-D.C.D critical graph on p vertices and let x, y be any two non adjacent vertices of G. Now the inclusion of the edge xy reduces the domination number to either 3 or 2.
Case (i): ***The D.C.D number of G is reduced to 2***
If the D.C.D number of G is reduced to 2, then it means that the set {x, y} forms the D.C.D set for (G + xy). That is, both the vertices x and y must dominate all the vertices of (G + xy). Hence, both x and y are of degree p – 1 in (G + xy) and hence in this case both the vertices x and y are of degree p – 2 in G.
Case (ii): ***The D.C.D number of G is reduced to 3***

If the D.C.D number of G is reduced to 3, then it means that one of the vertices x or y must dominate all the vertices of (G + xy). Let it be x. That is x is of degree p – 1 in (G + xy). Now x $\cup$ {u, v}, where u, v$\in N_1$(x) (u can be y also) forms a D.C.D set for (G + xy). Hence, x is of degree p – 1 and y is of degree p – 2 in (G + xy) and hence in this case any one of the vertices (x or y) is of degree p – 2 and the other one (x or y) is of degree p – 3 in G. Therefore, if G is a 4-D.C.D critical graph, then for any two non adjacent vertices, at least one of them is of degree p – 2 in G.

Proof of the converse is trivial.

Corollary 3.2.: If G is a 4-D.C.D critical graph, then Δ(G) = p – 2 and 2 ≤ δ(G) ≤ p – 2.

Lemma 3.5: Any 4-D.C.D critical graph is self centered of diameter 2.

Proof: Let G be a 4-D.C.D critical graph and let x and y be any two non adjacent vertices of G. Then at least one of the vertices x or y must be of degree equal to p – 2. Without loss of generality, let x be of degree p – 2 in G. Then clearly, x is of degree p – 1 in (G + xy) and also x must be in every minimum D.C.D set of (G + xy). Hence, the neighbors of y will be adjacent to x in (G + xy) and hence in G also. Therefore, the diameter of G must be equal to 2. Also, Δ(G) = p – 2 implies that G is a self centered graph of diameter 2.

Lemma 3.6: Any 4-D.C.D critical graph is a block.

Proof: Let G be a 4-D.C.D critical graph. Then, G is a self centered graph of diameter 2 and δ(G) ≥ 2. Since, any self centered graph of diameter 2 is 2-connected, G must be a block.

Theorem 3.5: If G is a 4-D.C.D critical graph and if S is a minimum D.C.D set of G, then <S> is a cycle.

Proof: Let G be a 4-D.C.D critical graph and let S = {x, u, v, y} be a minimum D.C.D set of G.

Claim: ***<S> is a cycle C_4***

Case (i): ***<S > does not contain a C_3***

Suppose that <S> contains a C_3. Then, it will not satisfy the distance closed property (as radius of <S> is 1). Hence, it does not contain a C_3.

Case (ii): ***<S> cannot be a tree***

If <S> is a tree on 4 vertices, then <S> must be isomorphic to $K_{1,3}$. Hence, in this also the radius of <S> is 1 and hence it does not satisfy the distance closed property. Therefore, <S> cannot be a tree.

Case (iii): ***<S> cannot be a path***

Suppose that <S> is a path xuvy. Then addition of an edge xy will not reduce the distance closed domination number of G, a contradiction to G is critical. Hence, <S> cannot be a path.

Thus from cases (i), (ii) and (iii), <S> must be cycle C_4.

Lemma 3.7: If G is a 4-D.C.D critical graph (which is not (p – 2) regular) then, any minimum D.C.D set of G can have at most two vertices of degree less than or equal to p – 3.

Proof: Let G be a 4-D.C.D critical graph. Then for every pair of non adjacent vertices at least one of them is of degree p – 2 and $\delta(G) \le p - 3$. Also, if D is a minimum D.C.D set of G, then <D> is a cycle C_4 (by Theorem 3.5).

Case (i): If all the four vertices of D are of degree less than or equal to (p – 3), then <D> is K_4 (since the set of vertices with degree less than or equal to (p – 3) forms a clique in a 4-D.C.D critical graph). Hence, D cannot be a minimum D.C.D set of G, a contradiction.

Case (ii): If D has three vertices of degree less than or equal to (p – 3), then <D> is $K_4 - e$. Hence, D cannot be a minimum D.C.D set of G, a contradiction.

Therefore, from cases (i) and (ii), any minimum D.C.D set of a 4-D.C.D critical graph can have at most 2 vertices of degree less than or equal to (p – 3).

Lemma 3.8: Any 4-D.C.D critical graph has at least four vertices of degree (p – 2).

Proof: Let G be a 4-D.C.D critical graph and let D = {x, y, u, v} be a minimum D.C.D set of G. Then by Theorem 3.5, <D> is a cycle.

Claim: ***D must contain at least two vertices of degree equal to (p – 2)***

By Lemma 3.7, D can have at most two vertices of degree less than or equal to (p – 3). Also, the set of vertices with degree less than or equal to (p – 3) in G forms a clique. Without loss of generality let $d(x) \le p - 3$ and $d(y) \le p - 3$. Then, clearly x and y are adjacent and also x is non adjacent to u and y is non adjacent to v in D. Hence, u and v must be of degree (p – 2) as they are non adjacent to vertices of degree less than or equal to (p – 3) and hence D must contain at least two vertices of degree equal to (p – 2).

As $d(x) \le p - 3$ and $d(y) \le p - 3$ and they are non adjacent to u and v respectively, they each must be non adjacent to at least one vertex of V – D say u is non adjacent to u^1 and v is non adjacent to v^1. Then $d(u^1) = p - 2$ and $d(v^1) = p - 2$, as they are non adjacent to vertices of degree $\le p - 3$. Hence, V – D has at least two vertices of degree (p – 2) and hence G has at least four vertices of degree equal to (p – 2).

Theorem 3.6: Let G be a graph on p vertices. G is 4-D.C.D critical and a unique eccentric point graph if and only if G is a (p – 2) regular graph.

Proof: Let G be a 4-D.C.D critical and a unique eccentric point graph. Then, $\delta(G) \le p - 2$. If G has a vertex u of degree less than or equal to (p – 3), then $N_2(u)$ must contain more than two vertices. That is, $|E(u)| \geq 2$ (as G a is self centered graph of diameter 2). This is a contradiction to the fact that G is unique eccentric point graph. Hence, G is a $(p - 2)$ regular graph.

Conversely, if G is a $(p - 2)$ regular graph, then clearly G is a 4-D.C.D critical and a unique eccentric point graph.

Theorem 3.7: Let G be a 4-D.C.D critical graph. Let v be any vertex of degree $p - 2$ and let $A = \{u \in N_1(v) \mid deg(u) = p - 2\}$ and $B = \{u \in N_1(v) \mid deg(u) \leq p - 3\}$. Then the following properties hold good.

(i) Any vertex of degree $p - 2$ in $N_2(v)$ must be adjacent to all the vertices of A and B.

(ii) Any vertex in $N_2(v)$ must be adjacent to all the vertices of B.

(iii) Any vertex of degree $p - 3$ in $N_2(v)$ must be non adjacent to exactly one vertex of A.

(iv) $\langle N_2(v) \rangle$ is complete.

Proof: Let G be a 4 D.C.D critical graph. Let v be any vertex of degree $p - 2$ and let $A = \{u \in N_1(v) \mid deg(u) = p - 2\}$ and $B = \{u \in N_1(v) \mid deg(u) = p - 3\}$.

(i) Let w be any vertex in $N_2(v)$ such that $deg(w) = p - 2$. Then, clearly it must be adjacent to all the vertices of $N_2(v)$ and hence to the vertices in both A and B. This proves property (i).

(ii) Let $w \in N_2(v)$. Then the following two cases can be considered.

Case 1: $\boldsymbol{deg(w) = p - 2}$

If $deg(w) = p - 2$, then the property (ii) is clear from case 1.

Case 2: $\boldsymbol{deg(w) = p - 3}$

If $w \in N_2(v)$ and $deg(w) = p - 3$, then w must be adjacent to all the vertices of B and also it is not adjacent to v. Hence, w must be non adjacent to exactly one vertex of A. This proves property (iii).

(iii) Let $x, y \in N_2(v)$. If x and y both have degree equal to $p - 2$, then clearly they are adjacent (as they non adjacent to v and $d(x) = d(y) = p - 2$). If any one of them (x or y) is of degree $p - 2$, then also x and y are adjacent. If both are of degree less than or equal to $(p - 3)$, then also x and y are adjacent and hence $\langle N_2(v) \rangle$ is complete. This proves the property (iv).

Proposition 3.3: Let G be a 4-D.C.D critical graph on p vertices. Let v be any vertex of degree less than or equal to $(p - 3)$ and let $A = \{u \in N_1(v), deg(u) = p - 2\}$ and $B = \{u \in N_1(v), deg(u) \leq p - 3\}$. Then any vertex of $N_2(v)$ must be adjacent to all the vertices of A and B.

Proof: Let w be any vertex of $N_2(v)$. Then clearly, $deg(w) = p - 2$ (since all the vertices of degree less than or equal to $(p - 3)$ forms a clique in a 4-D.C.D critical graph and $deg(v) \leq$

p – 3). Hence, v is the unique eccentric point of w and hence w is adjacent to all the vertices of A and B.

Remark 3.2.: From the above theorems, it is clear that the vertex set of any 4-D.C.D critical graph is partitioned into three disjoint subsets A, B and C, where $A = \{u \in V(G) \mid deg(u) \leq p - 3\}$, $B = \{v \in V(G) \mid deg(v) = p - 2 \text{ and } v \in E(u), u \in A\}$ and $C = V(G) - A \cup B = \{w \in V(G) \mid deg(w) = p - 2\}$. Also, the structure of any 4-D.C.D critical graph is same as that of a 3-D.C.D critical graph (refer Figure 3.1), except the vertex v of degree (p – 1). Hence, we have the following theorem without proof.

Theorem 3.8: A graph G with $p \geq 5$ is 3-D.C.D critical if and only if G – {v} is 4-D.C.D critical, where v is a vertex of degree p – 1 in G.

Proposition 3.4: Any 4-D.C.D critical graph is Hamiltonian.
Proof: Let G be a 4-D.C.D critical graph. As the structure of any 4-D.C.D critical graph is same as that of a 3-D.C.D critical graph except that the vertex of degree (p – 1), G is also Hamiltonian.

Proposition 3.5: There exists no graph G for which both G and $\overline{G}$ are 4-D.C.D critical.
Proof: Without loss of generality, assume that G is a 4-D.C.D critical graph. Then for any two non adjacent vertices of G, at least one of them is of degree p – 2 and also $\Delta(G) = p - 2$. Thus, the degree of that vertex in $\overline{G}$ is 1. Therefore, $\delta(\overline{G}) = 1$. Hence, $\overline{G}$ cannot be 4-D.C.D critical.

Proposition 3.6: Let G be a 4-D.C.D critical graph. Then, $q \leq \frac{1}{2} p(p - 2)$.
Proof: By lemma 3.8, any 4-D.C.D critical graph has at least four vertices of degree equal to (p – 2) and $\delta(G) \leq p - 2$.

$$\begin{aligned} \text{Therefore, } 2q &= \sum_{u \in V(G)} d(u) \\ &= 4(p - 2) + (p - 4)(\delta) \\ &\leq 4(p - 2) + (p - 4)(p - 2) \\ &= p(p - 2) \end{aligned}$$

Hence, $q \leq \frac{1}{2} p (p - 2)$.

Open Problem

Analyze the structural properties of distance closed domination critical graphs with respect to edge deletion.

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