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# On Chromatic Strong Dominating Sets in Graphs

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***Abstract:*** *A subset* $D$ *of* $V$ *is said to be a* ***chromatic strong dominating set*** *(or csd- set) if* $D$ *is strong dominating set and* $\chi(<D>) = \chi(G)$. *The minimum cardinality of chromatic strong dominating set is called chromatic strong domination number and is denoted by* $\gamma_s^c(G)$. *In this paper we initiate a study of this parameter and we characterize* $\gamma_s^c(G) = n$, *where n is the order of* $G$.

***Key words:*** *Domination, Strong domination, Chromatic Number.*

## 1 Introduction

Let $G = (V, E)$ be a finite, simple and undirected graph neither loops nor multiple edges. The order and size of $G$ are denoted by n and m respectively. One of the fastest growing areas within graph theory is the study of domination and related problems. A subset D of the vertex set V of a graph G is said to be a **dominating set** of $G$ if every vertex in $V - D$ is adjacent to a vertex in $D$. The minimum cardinality of a dominating set is called the **domination number** of $G$ and is denoted by $\gamma(G)$. A comprehansive treatment of the fundamentals of domination is given in the book by Haynes et al [1]. Sampathkumar and Pushpa Latha introduced the concept of strong domination in graphs in [2]. A set $D \subseteq V$ is called a **strong dominating** set if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that $uv \in E(G)$ and $\deg(u) \geq \deg(v)$. The minimum cardinality of a strong dominating set is called the **strong domination number** of $G$ and is denoted by $\gamma_s(G)$. A strong dominating set of cardinality $\gamma_s(G)$ is called as $\gamma_s$-set of $G$. For $D \subseteq V$, the subgraph induced by $D$ is denoted by $<D>$. A set $D \subseteq V$ is said to be a **chromatic preserving set** if $\chi(<D>) = \chi(G)$. It was defined by Janakiraman and Poobalaranjani [5]. A set $D \subseteq V$ is said to be a **dom-chromatic set** if $D$ is a dominating set and $\chi(<D>) = \chi(G)$. The minimum cardinality of a dom-chromatic set is called the **dom-chromatic number** and is denoted by $\gamma_{ch}(G)$. This concept is also defined Janakiraman and Poobalaranjani [5]. A **split graph** is a graph $G = (V, E)$ whose vertices can be partitioned into two sets $V_1$ and $V_2$, where the vertices in $V_1$ form a complete

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graph and the vertices in $V_2$ form a null graph. A graph $G$ is a **perfect graph** if $\chi(H) = \omega(H)$ for all induced subgraph $H$ of $G$. The length of the smallest cycle (if any) of a graph $G$ is called girth of G and is denoted by $g(G)$ and the length of the smallest odd cycle (if any) is denoted by $g_0(G)$. A graph $G$ is **planar** if it can be embedded in a plane, a plane graph has already been embedded in the plane. A set $D \subseteq V$ is said to be a **chromatic strong dominating set** if $D$ is a strong dominating set and $\chi(< D >) = \chi(G)$. The minimum cardinality of a chromatic strong dominating set is called the **chromatic strong domination number** of $G$ and is denoted by $\gamma_s^c(G)$. A chromatic strong dominating set of cardinality $\gamma_s^c(G)$ is called as $\gamma_s^c$-set of $G$. A vertex $v$ is said to be a **strong vertex** if $d(v) > d(x)$ for all $x \in N(v)$. A graph $G$ is called a vertex $\chi$**-critical graph** if $\chi(G - v) < \chi(G)$ for all $v \in V(G)$.

## 2 Chromatic Strong Dominating Sets

The chromatic strong dominating set mentioned in the introduction is formally defined below.

**Definition 2.1:** *Let* $G = (V, E)$ *be a graph. A subset* $D$ *of* $V$ *is said to be a* **chromatic strong dominating set** (*or csd- set*) *if D is a strong dominating set and* $\chi(< D >) = \chi(G)$. *The minimum cardinality of a chromatic strong dominating set in a graph* $G$ *is called the* **chromatic strong domination number** (*or csd- number*) *and is denoted by* $\gamma_s^c(G)$.

**Proposition 2.2:** *Let* $G$ *be a graph of order* $n$. *Then* $1 \leq \gamma_s^c(G) \leq n$

**Proposition 2.3:** *Let* $G$ *be a graph. Then* $\gamma_s^c(G) = 1$ *if and only if* $G = K_1$

**Proposition 2.4** *For* $n \geq 1$, $\gamma_s^c(K_n) = n$.

**Proof:** Let D be any proper subset of $V(K_n)$. Then $\chi(< D >) < n$. Therefore, D is not a chromatic strong dominating set. Therefore, $V(K_n)$ is the only chromatic strong dominating set. Therefore, $\gamma_s^c(G) = n$ .

**Proposition 2.5:** *For* $2 \leq r \leq s$, $\gamma_s^c(D_{r,s}) = 2$, *where* $D_{r,s}$ *denotes the double star.*

**Proof:** $\gamma_s(D_{r,s}) = 2$. Let D be a $\gamma_s$- set of $D_{r,s}$. Then $D = \{u_1, u_2\}$, where $d(u_1) \geq 2$ and $d(u_2) \geq 2$. Since $u_1u_2 \in E(D_{r,s})$, $\chi(< D >) = 2$. Therefore, D is a chromatic strong dominating set of $D_{r,s}$. $2 = \gamma_s(D_{r,s}) \leq \gamma_s^c(D_{r,s}) \leq |D| = 2$.

Therefore, $\gamma_s^c(D_{r,s}) = 2$

**Proposition 2.6:** *For* $n \geq 4$, $\gamma_s^c(F_n) = 3$, *where* $F_n$ *denotes a fan.*

**Proof:** $\gamma_s(F_n) = 1$ and $\chi(F_n) = 3$. Let $u$ be a vertex of $F_n$ with $\deg(u) = n-1$. Let $x, y \in V(F_n) - \{u\}$ such that $xy \in E(F_n)$. Let $D = \{u, x, y\}$. Then $\chi(<D>) = 3 = \chi(F_n)$. Therefore, $D$ is a chromatic strong dominating set of $F_n$. Therefore, $3 = \chi(F_n) \leq \gamma_s^c(F_n) \leq |D| = 3$. Therefore, $\gamma_s^c(F_n) = 3$

**Proposition 2.7:** *For* $n \geq 5$,

$$\gamma_s^c(W_n) = \begin{cases} n & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

**Proof:** Let $D$ be a $\gamma_s^c$ - set of $W_n$. Let $u$ be a vertex of $W_n$ with $\deg(u) = n-1$. Then $u \in D$.

**Case (i):** Let $n$ be even. Then $< V(W_n) - \{u\} >$ is a cycle on $n-1$ vertices. Since $n$ is even, $n-1$ is odd. Therefore, $\chi(W_n - \{u\}) = 3$. Therefore, $\chi(W_n) = 4$. Since $\chi((W_n) - \{x\}) \leq 3$ for all $x \in V(W_n)$, $V(W_n)$ is the only $\gamma_s^c$ - set of $W_n$. Therefore, $\gamma_s^c(W_n) = n$.

**Case (ii):** Let $n$ be odd. Then $< V(W_n) - \{u\} >$ is an even cycle. Therefore, $\chi(< V(W_n) - \{u\} >) = 2$. Therefore, $\chi(W_n) = 3$. Let $x, y \in V(W_n) - \{u\}$ and $xy \in E(W_n)$. Then $\{u, x, y\}$ is a chromatic strong dominating set. Therefore, $3 = \chi(W_n) \leq \gamma_s^c(W_n) \leq |\{u, x, y\}| = 3$. Therefore, $\gamma_s^c(W_n) = 3$

**Proposition 2.8:** *For* $m, n \geq 2$,

$$\gamma_s^c(K_{m,n}) = \begin{cases} \min\{m, n\} + 1 & \text{if } m \neq n \\ 2 & \text{if } m = n \end{cases}$$

**Proof:** Let $D$ be a $\gamma_s$ - set of $K_{m,n}$.

**Case (i):** Let $m = n$. Then $\gamma_s(K_{m,n}) = 2$. Let $D = \{u, v\}$ where $uv \in E(K_{m,n})$. Since $K_{m,n}$ is bipartite, $\chi(K_{m,n}) = 2$. Since $uv \in E(K_{m,n})$, $D$ is a chromatic strong dominating set of $K_{m,n}$. Therefore, $2 = \chi(K_{m,n}) \leq \gamma_s^c(K_{m,n}) \leq |D| = 2$. Therefore, $\gamma_s^c(K_{m,n}) = 2$.

**Case (ii):** Let $m \neq n$. Without lose of generality assume that $m > n$. Let $D$ be a independent set of cardinality $n$. Then $\chi(<D>) = 1$. Therefore, $D$ is a not a chromatic strong dominating set. But $D \cup \{u\}$ is a chromatic strong dominating set where $u \in V(K_{m,n}) - D$. Therefore, $|D| < \gamma_s^c(K_{m,n}) \leq |D \cup \{u\}| = |D| + 1$. Therefore,

$\gamma_s^c(K_{m,n}) = |D| + 1 = n + 1 = \min\{m,n\} + 1$.

**Proposition 2.9:** *For* $n \geq 2$, $\gamma_s^c(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n \equiv 1 \pmod 3 \\ \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n \equiv 0 \text{ or } 2 \pmod 3 \end{cases}$

**Proof:** Let $D$ be a $\gamma_s$ - set of $P_n$.

**Case (i)** Let $n \equiv 1 \pmod 3$. Let $P_n : u_1, u_2, \ldots, u_n$ be a path on $n$ vertices. Consider a path $P_{n-1} : u_1, u_2, \ldots, u_{n-1}$. Since $n \equiv 1 \pmod 3$; $n - 1 \equiv 0 \pmod 3$. Therefore, $D = \{u_2, u_5, u_8, \ldots, u_{n-2}\}$ is a unique $\gamma_s$ - set of $P_{n-1}$. Therefore, $D \cup \{u_{n-1}\}$ is a $\gamma_s$ - set of $P_n$. Since $D \cup \{u_{n-1}\}$ is not independent $\chi(D \cup \{u_{n-1}\}) = 2 = \chi(P_n)$. Therefore, $D \cup \{u_{n-1}\}$ is a chromatic strong dominating set.

Therefore, $\left\lceil \frac{n}{3} \right\rceil = \gamma_s(P_n) \leq \gamma_s^c(P_n) \leq |D \cup \{u_{n-1}\}| = \left\lceil \frac{n}{3} \right\rceil$. Therefore, $\gamma_s^c(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

**Case (ii):** Let $n \equiv 0 \pmod 3$. Let $n = 3k$. Then $\gamma_s(P_{3k}) = k$ and $P_n$ has unique $\gamma_s$ - set. Let $D = \{u_2, u_5, u_8, \ldots, u_{3k-1}\}$. Then $D$ is independent. Therefore, $\chi(< D >) = 1$. Therefore, $D$ is not a chromatic strong dominating set. Since $\chi < D \cup \{u_1\} >= 2$, $D \cup \{u_1\}$ is a chromatic strong dominating set, $\gamma_s < \gamma_s^c(P_n) = |D \cup \{u_1\}| = |D| + 1 = \left\lceil \frac{n}{3} \right\rceil + 1$

**Case (iii):** Let $n \equiv 2 \pmod 3$. Let $n = 3k + 2$, then $\gamma_s(P_{3k+2}) = k + 1$. Let $D$ be a $\gamma_s$ - set. Suppose $D$ is not independent. Let $v_i, v_{i+1} \in D$.

**Sub Case(i):** Let $i = 3l$. Consider the path $P_1 : u_1, u_2, \ldots, u_{3l-2}$ and the path $P_2 : u_{3l+3}, \ldots, u_{3k+2}$. Therefore, $\gamma_s(P_1) = \left\lceil \frac{3l-2}{3} \right\rceil = l$ and $\gamma_s(P_2) = \left\lceil \frac{3(k-l)}{3} \right\rceil = k - l$.

Therefore, $\gamma_s(P_n) = \gamma_s(P_1) + 2 + \gamma_s(P_2)$

$\gamma_s(P_n) = l + 2 + k - l = k + 2$, which is a contradiction. Therefore, $D$ is independent for every $\gamma_s$ - set of $P_n$. Since $\chi(P_n) = 2$, $\gamma_s < \gamma_s^c$. Let $u \in D$ and $v \in N(u)$. Then $D \cup \{v\}$ is a strong dominating set, $\chi(D \cup \{v\}) = 2$, $D \cup \{v\}$ is a chromatic strong dominating set. Therefore, $\gamma_s^c \leq |D \cup \{v\}| = |D| + 1 = \gamma_s + 1$. Since $\gamma_s < \gamma_s^c$, $\gamma_s^c = \gamma_s + 1 = \left\lceil \frac{n}{3} \right\rceil + 1$. Therefore, $\gamma_s^c(P_n) = \left\lceil \frac{n}{3} \right\rceil + 1$.

**Proposition 2.10:** *For* $n \geq 3$,

$$\gamma_s^c(C_n) = \begin{cases} n & \text{if } n \text{ is odd} \\ \left\lceil \frac{n}{3} \right\rceil & \text{if } n \equiv 1 \pmod 3 \text{ and n is even} \\ \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n \equiv 0 \text{ or } 2 \pmod 3 \text{ and n is even} \end{cases}$$

**Proof:** Let $C_n$ be a cycle on $n$ vertices.

**Case (i):** Let $n$ be odd. Let $D$ be a any proper subset of $V(C_n)$. Then $\chi(<D>) = 2 < \chi(C_n)$. Therefore, $D$ is a chromatic strong dominating set of $C_n$. Therefore, $D = V(C_n)$ is the only chromatic strong dominating set. Therefore, $\gamma_s^c(C_n) = |V(C_n)| = n$.

**Case (ii):** Let $n$ be even and $n \equiv 0 \pmod 3$. Let $n = 3k$, let $D_1 = \{u_2, u_5, u_8, \ldots, u_{3k-1}\}$, $D_2 = \{u_3, u_6, u_9, \ldots, u_{3k}\}$ and $D_3 = \{u_1, u_4, u_7, \ldots, u_{3k-2}\}$. Then $D_1$, $D_2$, $D_3$ are the only $\gamma_s$-sets which are also independent. Since $\chi(<D_i>) = 1 \neq \chi(C_n) = 2$, $i = 1,2,3$ $D_i$ is not a chromatic strong dominating set. Now, $D_1 \cup \{u_1\}$ is a chromatic strong dominating set. Therefore, $\gamma_s^c(C_n) \leq |D| + 1 = \gamma_s + 1$.

Since $\gamma_s < \gamma_s^c$, $\gamma_s^c(C_n) = \gamma_s + 1 = \left\lceil \frac{n}{3} \right\rceil + 1$.

**Case (iii):** Let $n$ be even and $n \equiv 2 \pmod 3$. Let $n = 3k + 2$. Then $\gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{3k+2}{3} \right\rceil = k + 1$. Let $D$ be a $\gamma_s$- set. Suppose $D$ is not independent. Let $v_i, v_{i+1} \in D$. Consider the path $P_1 : v_{i+3}, v_{i+4}, \ldots, v_n, v_1, v_2, v_3, \ldots, v_{i-2}$ of length $n - 4$. Therefore, $\gamma_s(P_1) = \left\lceil \frac{n-4}{3} \right\rceil = \left\lceil \frac{3k+2-4}{3} \right\rceil = \left\lceil \frac{3k-2}{3} \right\rceil = k$.

Therefore, $\gamma_s(C_n) = \gamma_s(P_1) + 2 = k + 2$, which is a contradiction. Therefore, $D$ is independent. Therefore, $\gamma_s^c(C_n) > \gamma_s(C_n)$. Since $D$ is independent and $\chi(C_n) = 2$, $D \cup \{x\}$ is chromatic strong dominating set, where $x \in N(y)$ for some $y \in D$.

Therefore, $\gamma_s^c(C_n) \leq \gamma_s(C_n) + 1$. Hence $\gamma_s^c(C_n) = \gamma_s + 1 = \left\lceil \frac{n}{3} \right\rceil + 1$.

**Case (iv):** Let $n$ be even and $n \equiv 1 \pmod 3$. Let $n = 3k + 1$. Then $D = \{u_2, u_5, \cdots, u_{3k-1}, u_{3k}\}$ is a $\gamma_s$-set. Since $D$ is not independent and

$\chi(<D>) = 2 = \chi(C_n)$, D is a $\gamma_s^c$-set. Therefore, $\gamma_s^c(C_n) = \gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

**Observation 2.11:**

1. Chromatic strong dominating set exists for all graphs since the vertex set $V$ is a trivial chromatic strong dominating set.
2. For a vertex $\chi$ - critical graph, $V$ is the only chromatic strong dominating set .
3. If $D$ is a chromatic strong dominating set of $G$, then each vertex of $V-D$ is not adjacent to at least one vertex of $D$.
4. $\gamma_{ch} \leq \gamma_s^c(G)$

**Proof:** 1 and 2 follow trivially. Suppose $x \in V-D$ is adjacent to each vertex of $D$. Since $D$ contains a chromatic preserving set of $G$, let $S \subseteq D$ be a chromatic preserving set of $D$. Since $x$ is adjacent to each vertex of $S$, $\chi(G) \geq \chi(<S>)+1 = \chi(G)+1$, a contradiction. Hence $(3)$. Every chromatic strong dominating set is a dom-chromatic set. Therefore, $\gamma_{ch} \leq \gamma_s^c(G)$

**Theorem 2.12 :**

1. Let $G$ be a graph without strong vertices. Then $\gamma_s^c(G) = n$ if and only if $G$ is a vertex $\chi$ - critical graph.

2. If $G$ is a disconnected graph, then $\gamma_s^c(G) = n$ if and only if either $G$ is a null graph or has exactly one non trivial component which is vertex $\chi$ - critical and in which there are no strong vertices.

**Proof of 1:** Let $\gamma_s^c(G) = n$. Suppose that $G$ is not a vertex $\chi$ - critical graph. Then There exists a vertex $v \in V$ such that $\chi(G-v) = \chi(G)$. Let $D = V - \{v\}$. Since $G$ has no strong vertices, $D$ is a strong dominating set and $\chi(<D>) = \chi(G-v) = \chi(G)$. Therefore, $D$ is a chromatic strong dominating set of $G$. Therefore, $\gamma_s^c(G) \leq |D| = n-1$, a contradiction. Thus, $G$ is vertex $\chi$ - critical. Conversely, assume that $G$ is vertex $\chi$-critical. For any proper subset $D$ of $V$, $\chi(<D>) < \chi(G)$. Therefore, no proper subset of $V$ can be a chromatic strong dominating set of $G$. Hence $\gamma_s^c(G) = n$

**Proof of 2:** Let $G$ be a disconnected graph. Let $\gamma_s^c(G) = n$. If every component of $G$ is tivial, then $G = \overline{K_n}$. Suppose there exists at least two components of $G$ which are non trivial. Let $G_1$ and $G_2$ be two such components. Without loss of generality $\chi(G_1) \geq \chi(G_2)$. Let $D_1$ be a $\gamma_s^c$-set of $G_1$ and $D_2$ be a $\gamma_s$-set of $G_2$. Then $D_1 \cup D_2$ is a $\gamma_s^c$-set of $G_1 \cup G_2$. $D_2 = \gamma_s(G_2) \leq |V(G_2)| - \Delta(G_2) \leq |V(G_2)| - 1$. Therefore,

$\gamma_s^c(G_1 \cup G_2) < |V(G_1)| + |V(G_2)|$, a contradiction, since $\gamma_s^c(G) = n$. Therefore, there exists exactly one component which is non trivial. Therefore, either every component of $G$ is trivial or there exists a exactly one component of $G$, say $G_1$, which is non trivial. In the first case $G = \overline{K_n}$. In the second case, $n = \gamma_s^c(G) = \gamma_s^c(G_1) + n - k$, where $|V(G_1)| = k$. Therefore, $\gamma_s^c(G_1) = k = |V(G_1)|$. Therefore, $G_1$ is $\chi$-critical and without strong vertices. The convers is obvious.

**Remark 2.13:** *If $G$ is a $\chi$-critical graph, then $\gamma_s^c(G) = n$ and $G$ has no strong vertices.*

**Observation 2.14:** *Any superset of a chromatic strong dominating set is a chromatic strong dominating set.*

**Proof:** Let $D$ be a chromatic strong dominating set $G$ and $D \subseteq D_1$. Then $D_1$ is a chromatic strong dominating set, since any super set of a chromatic strong dominating set is a chromatic strong dominating set. Also $\chi(G) = \chi(<D>) \leq \chi(<D_1>) \leq \chi(G)$. Therefore, $D_1$ is also a chromatic strong dominating set.

**Proposition 2.15:** *A chromatic strong dominating set $D$ is minimal if and only if for each $u \in D$, one of the following conditions hold.*

1. $\chi(<D - \{u\}>) < \chi(G)$. ( that is $<D>$ is $\chi$-critical )
2. $D - \{u\}$ is not a strong dominating set is
   (a) $v$ is a strong isolate of $D$
   (b) There exists $v \in V - D$ such that $N_s(v) \cap D = \{u\}$ where $N_s(v) = \{x \in V(G) : xv \in E(G) \text{ and } deg(x) \geq deg(v)\}$.

**Proposition 2.16:** *Let $D$ be any chromatic strong dominating set of $G$. Then $|V - D| = \sum_{u \in D} deg(u)$ if and only if $G = \overline{K_n}$*

**Proof:** If $G = \overline{K_n}$, then $D = V$ and $deg(u) = 0$, for each $u \in D$. Then $|V - D| = 0 = \sum_{u \in D} deg(u)$. Suppose $|V - D| = \sum_{u \in D} deg(u) = k$. Let $k \geq 1$. Then $G$ has an edge and hence $\chi(G) \geq 2$. Let $V - D = \{u_1, u_2, u_3, \ldots, u_k\}$. Since $D$ is a strong dominating set, each $u_i$ is adjacent to a vertex of $D$ and hence, contributes at least one degree to every vertex of $D$.

Since $\chi(<D>) \geq 2$, $D$ contains at least one edge which contributes $2$ degrees to $D$. Hence $\sum_{u \in D} deg(u) \geq k + 2$, a contradiction.

**Corrolary 2.17:** *For any non trivial connected graph with a chromatic strong dominating set* $D$, $\sum_{u\in D}\deg(u) \geq |V-D|+2$

**Proof:** If $G$ is $\chi$-critical, then $V = D$ and $\sum_{u\in D}\deg(u) = 2m \geq 2 = |V-D|+2$. Suppose $G$ is not $\chi$-critical. Since $G$ is not trivial, $\chi(G) \geq 2$. By similar argument in the **Proposition 2.16:** $\sum_{u\in D}\deg(u) \geq |V-D|+2$.

**Proposition 2.18:** *For any graph* $G$, $\left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor \leq \gamma_s^c(G)$ *and equality holds if and only if* $G = \overline{K_n}$.

**Proof:** Since $\left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor \leq \gamma(G) \leq \gamma_s^c(G)$ we get the lower bound. If $G = \overline{K_n}$, then the result follows. Suppose $\left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor = \gamma_s^c(G) = k$ and $D$ is a $\gamma_s^c$- set of $G$. Suppose $G \neq \overline{K_n}$. Then $\gamma_s^c(G) \geq 2$. Then by corollary2.17, $|V-D| < \sum_{u\in D}\deg(u)$. Thus, $n-k < \sum_{u\in D}\deg(u) \leq k\Delta(G)$ and hence, $\frac{n}{\Delta(G)+1} < k$. Hence $k > \frac{n}{\Delta(G)+1} \geq \left\lfloor \frac{n}{\Delta(G)+1} \right\rfloor = k$ a contradiction. Thus, $G = \overline{K_n}$.

**Theorem 2.19:** *Given a positive integer* $k \geq 2$, *then there exists a graph* $G$ *such that* $\gamma_s^c(G) = k$.

**Proof:** Let $G$ be a complete bipartite graph, $K_{k-1,k}$.

Then $\gamma_s^c(G) = \min\{k-1,k\}+1 = k-1+1 = k$.

**Theorem 2.20:** *Given a positive integer* $k \geq 1$, *there exists a graph* $G$ *such that*

(i). $\gamma_s^c(G) - \gamma(G) = k$. (ii). $\gamma_s^c(G) - \gamma_s(G) = k$.

(iii). $\gamma_s^c(G) - \gamma_{ch}(G) = k$.

**Proof of 1:** Let $G = K_{k+1}$. Then $\gamma_s(G) = 1$ and $\gamma_s^c(G) = k+1$. Therefore, $\gamma_s^c(G) - \gamma(G) = k+1-1 = k$.

**Proof of 2:** Let $G = K_{k+1}, k+2$. Then $\gamma_s^c(G) = \min\{k+1, k+2\}+1 = k+1+1 = k+2$ and $\gamma_s G) = 2$. Therefore, $\gamma_s^c(G) - \gamma_s(G) = k+2-2 = k$.

**Proof of 3:** Let $G = K_{k+1,k+2}$. Then $\gamma_{ch}(G) = 2$ and $\gamma_s^c(G) = \min\{k+1, k+2\} + 1$ $= k+2$. Therefore, $\gamma_s^c(G) - \gamma_{ch}(G) = k+2-2 = k$.

**Proposition 2.21:** *If* $G$ *is triangle free graph with* $\chi(G) \geq 3$, *then* $\gamma_s^c(G) \geq 5$

**Proof:** Consider $< D >$, where $D$ is a $\gamma_s^c$-set of $G$. $\chi(< D >) \geq 3$. Since $G$ is triangle free, $< D >$ is also triangle free. Therefore, $< D >$ contains an odd cycle of length $\geq 5$. ($< D >$ contains no odd cycles if and only if $< D >$ is bipartite in which case $\chi(< D >) \leq 2$). Therefore, $|D| \geq 5$. Therefore, $\gamma_s^c(G) = |D| \geq 5$.

**Proposition 2.22:** *If* $G$ *is a vertex* $\chi$ *- critical graph without strong vertices, then* $\alpha_0(G) < \gamma_s^c(G)$.

**Proof:** For any graph $G$, $\alpha_0(G) \leq n-1$. Since $G$ is $\chi$-critical, $\gamma_s^c(G) = n$. Therefore, $\alpha_0(G) < \gamma_s^c(G)$.

## 3 Special Classes of Graphs

**Proposition 3.1:** [5] *If* $G$ *is a connected split graph and* $\Delta(G) < n-1$, *then* $\overline{G}$ *is a connected split graph with* $\Delta(\overline{G}) < n-1$.

**Definition 3.2:** [6] *A set* $D$ *subset of* $V$ *is said to be a chromatic preserving set or a cp-set if* $\chi(< D >) = \chi(G)$. *The minimum cardinality of a chromatic preserving set in* $G$ *is called chromatic preserving number or cp-number of* $G$ *and is denoted by cpn*(G).

**Theorem 3.3:** *If* $G$ *is a connected split graph and* $\Delta(G) < n-1$, *then*

1. $\gamma_s^c(G) = \omega(G) = cpn(G)$.
2. $\gamma_s^c(\overline{G}) = \omega(\overline{G}) = cpn(\overline{G})$

**Proof of 1:** As $G$ is a split graph, its vertex set can be partitioned into two sets X and Y such that $< X >$ is complete and $< Y >$ is a totally disconnected graph. Without lose of generality $< X >$ can be assumed to be maximal clique, Since $< X >$ is a maximal clique, each vertex of $Y$ is not adjacent to at least one vertex of $X$ and as $< Y >$ is totally disconnected graph, $\chi(G) = \chi(< X >)$. Therefore, $cpn(G) = |X| = \omega(G)$. As $G$ is connected each vertex of $Y$ is adjacent to at least one vertex of $X$. For any $u \in Y$, $deg(u) \leq |X| - 1$ and for any let $v \in X$, $deg(v) \geq |X| - 1$. Therefore, $deg(u) \leq deg(v)$. Therefore, $X$ strongly dominates $Y$ and hence $X$ is a $\gamma_s^c$-set of $G$.

**Proof of 2:** If $G$ is a connected split graph, then from proposition 3.1 $\overline{G}$ is also a connected split graph and hence the result follows.

**Theorem 3.4:** *If* G *is a perfect graph, then* $\gamma_s^c(G) \leq \gamma_s(G) + \omega(G)$

**Proof:** Let S be a maximum clique in G and D a $\gamma_s$ - set in G. Since G is perfect, $\omega(G) = \chi(G)$, and hence $\chi(G) = \omega(G) = |S| = \chi(<S>)$.
Therefore, $\chi(<S \cup D>) = \chi(G)$. Thus, $S \cup D$ is a chromatic strong dominating set of G, which implies that $\gamma_s^c(G) \leq |S \cup D| \leq |S| + |D| = \omega(G) + \gamma_s(G)$.

**Remark 3.5:** *If* G *is a graph with* $\Delta(G) = n - 1$, *then* $\gamma_s^c(G)$ *need not be equal to* $\chi(G)$.

**Example:** Let $W_6$ be the wheel on six vertices. Then $\gamma_s^c(W_6) = 6$ and $\chi(W_6) = 4$, $\gamma_s^c(W_6) \neq \chi(W_6)$.

**Proposition 3.6:** *If* G *is a perfect graph with a full degree vertex, then* $\gamma_s^c(G) = \chi(G)$.

**Proof:** Let u be a vertex in V(G) of degree $n - 1$. Let S be a maximum clique in G. Clearly S contains u. For: otherwise, $S \cup \{v\}$ is a clique, a contradiction. Therefore, S is a chromatic strong dominating set of G. Therefore, $\gamma_s^c(G) \leq |S| = \chi(G)$ (G is a perfect graph). But $\chi(G) \leq \gamma_s^c(G)$. Therefore, $\gamma_s^c(G) = \chi(G)$.

**Theorem 3.7:** *Let* G *be a graph without isolated vertices. If* G *has a vertex* u *which is not a strong isolate and* $\chi(G) > \chi(G - v)$, *for all* $v \in V(G) - \{u\}$ *and* $\chi(G) = \chi(G - u)$, *then* $\gamma_s^c(G) = n - 1$.

**Proof:** Let D be a $\gamma_s^c(G)$-set of G. Then $\chi(<D>) = \chi(G)$. Suppose $|D| < n - 1$. Then D does not contain at least one vertex $v \neq u$. Therefore, $D \subseteq V(G) - v$. Therefore, $\chi(<D>) \leq \chi(<V(G) - \{v\}>) < \chi(G)$, a contradiction. Therefore, $|D| = n - 1$. Therefore, $\gamma_s^c(G) = n - 1$. Therefore, $|D| \geq n - 1$. Consider $S = V(G) - \{u\}$. Then S strongly dominates u and $\chi(<S>) = \chi(G)$. Therefore, S is a chromatic strong dominating set of G and so $\gamma_s^c(G) \leq |S| = n - 1$. Therefore, $\gamma_s^c(G) = n - 1$.

**Theorem 3.8:** *If* G *is a planar graph with* $diam(G) = 2$, $\chi(G) = 3$ *and* $\gamma_s(G) = 2$, *then* $3 \leq \gamma_s^c(G) \leq 5$

**Proof:** Lower bound is trivial. Let $S = \{a, b\}$ be a $\gamma_s$ - set of G. Since $diam(G) = 2$, $g_0(G) = 3$ or 5

**case (i):** $g_0(G) = 3$. Let C be a 3-cycle xyzx. If $a, b \notin C$, then two vertices of C are adjacent to $a$ and one vertex is adjacent to $b$ or vice versa, for otherwise $K_4$ is induced, a contradiction. Let $x$ and $y$ be adjacent to $a$, and $z$ be adjacent to $b$. Then $axya$ is a $3$-cycle. Hence $\{a, x, y, b\}$ is a chromatic strong dominating set of $G$. If $a$ or $b$ is in the $3$-cycle, then the $3$-cycle together with the remaining vertex of $S$ is a chromatic strong dominating set of $G$.

**Case (ii):** $g_0(G) = 5$. Let $C$ be a $5$-cycle $uvwxyu$. If $a, b \notin C$, then as $S$ is a strong dominating, vertices of $C$ are adjacent to $a$ or $b$ and not to both. Otherwise a $3$-cycle is induced. Also no two consecutive vertices of $C$ can be both adjacent to $a$ or $b$, otherwise a $3$-cycle is induced. Then $S$ can strong dominates at most $4$-vertices of $C$, a contradiction. Hence, $a$ or $b \in C$. Let $a \in C$ and $b \notin C$, let $u = a$. Then $x$ and $w$ are adjacent to $b$ and hence a $3$-cycle is induced, a contradiction. Therefore,, both $a, b \in C$ and hence, $C$ is a $\gamma_s^c$- set of $G$. From cases (i) and (ii), the upper bound is proved.

**Theorem 3.9:** *For any integer* $N \geq 0$*.There exist a connected graph* $G$ *and a graph* $G'$ *such that,* $G'$ *is obtained from* $G$ *by adding exactly one vertex and* $\gamma_s^c(G) - \gamma_s^c(G') = N$.

**Proof:** Let $G = P_{3(N+2)}$, a path an $3(N+2)$ vertices. Then $\gamma_s^c(G) = \left\lfloor \frac{3(N+2)}{3} \right\rfloor + 1 = N + 3$. Now, $G'$ is a graph obtained from $G$ by adding a new vertex $v$ and joining $v$ to all the vertices of $G$. Therefore, $\gamma_s^c(G') = 3$. Hence $\gamma_s^c(G) - \gamma_s^c(G') = N + 3 - 3$. Therfore $\gamma_s^c(G) - \gamma_s^c(G') = N$.

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