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Graceful Labeling of a Family of Special Tree with Hanging Stars having non-decreasing number of Branches in Random Order

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Abstract: *In this paper, it is shown graceful a labeling of special type of tree obtained from a family of n hanging stars one at a vertex of a basic path on n vertices arranged in non-decreasing order with random number of branches having one of the branch points in each of those stars merged with one basic point of that path.*

Key words: *graceful labeling, random trees, growing stars*

Mathematics subject classification 2010: 05C78

1. Introduction

Let G = (V, E) be a simple, undirected and finite graph with p vertices and q edges. Graph labeling was first introduced in the late 1960's. Many studies in graph labeling refer to Rosa's research 1967. Rosa [7] called a function a β**-valuation** of a graph G with q edges if f is an injection from the vertices of G to the set {0, 1, 2,…, q} such that when each edge xy is assigned the label $f^*(xy) = |f(x) - f(y)|$, the resulting edge labels are distinct, Golomb [3] studied the same type of labeling and called this as graceful, which is now, the popular term.

Gallian [1] gives the extensive survey of contributions to graceful labeling of variety of graphs. The notation and terminology used in this paper are taken from [1]. There are many works relating to graceful labeling of trees. Huang, Kotzig, and Rosa [4] give a new class of graceful trees, Sethuraman and Jesintha [8, 9] have given graceful labeling of two different families of trees. The above contributions and earlier papers [5, 6] motivated us to give a graceful labeling of some special type of tree with hanging stars having non-decreasing number of branches in random order is denoted by $T_{S(n)}$.

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Let P_n (s_1-s_2-…-s_n) be basic path of $T_{S(n)}$ tree. The vertices of the path P_n are also termed as supporting vertices of $T_{S(n)}$ tree. In $T_{S(n)}$ at each s_i, a star S_i hangs with i branches having centre c_i with one of the branch vertices of S_i is merged with s_i. The special tree with hanging stars whose branches satisfy the condition $|E(S_1)| \leq |E(S_2)| \leq \ldots \leq |E(S_n)|$ (non-decreasing number of branches in the hanging stars).

2. Main result

In the paper [5], it is given a graceful labeling of a tree with hanging stars in which the number of branches growing in arithmetic progression.

In the paper [6], it is given a graceful labeling of a tree with hanging stars in which the number of branches growing in geometric progression.

Definition: 2.1:

In this paper, we give a graceful labeling for special tree with hanging stars that whose branches satisfy the condition $|E(S_1)| \leq |E(S_2)| \leq \ldots \leq |E(S_n)|$ (non-decreasing number of branches in the hanging stars).

Theorem 2.2

The tree $T_{S(n)}$ with n hanging stars having branches, which are non-decreasing in order is graceful.

Proof: Let $T_{S(n)}$ be a tree with n hanging stars $S_1, S_2, \ldots, S_n$ be stars with $|V(S_i)| = i$ for i = 1, 2, 3, . . ., n. Let the support points of the hanging stars $S_1, S_2, \ldots, S_n$ be $s_1, s_2, s_3, \ldots, s_n$ respectively and free leaves of each of the stars S_i are denoted by $f_1^{(i)}, f_2^{(i)}, \ldots, f_{i-1}^{(i)}$ for i = 1, 2 ,…, n.

Let $c_1, c_2, \ldots, c_n$ be the central values of the stars $S_1, S_2, S_3, \ldots, S_n$ respectively.

It can be verified that $|V(T_{S(n)})| = |V(S_1)| + |V(S_2)| + \ldots + |V(S_n)|$ and $|E(T_{S(n)})| = |E(S_1)| + |E(S_2)| + \ldots + |E(S_n)| + (n-1)$.

Let $|E(S_i)| = q_i$ for i = 1, 2,…, n and $q = q_1 + q_2 + q_3 + \ldots + q_n + (n-1)$.

A general random tree is drawn in the following Figure 1.

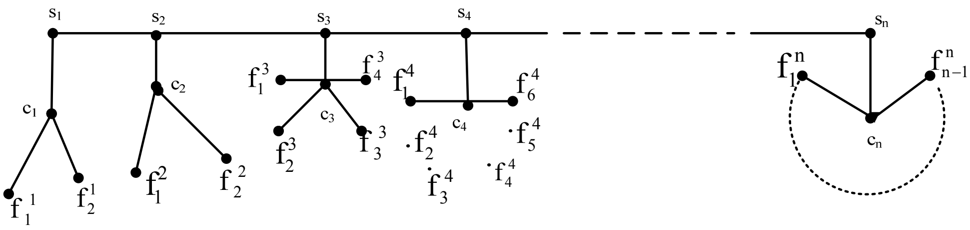

Figure 1

We denote the labeling of node v in the tree as l (v). Here, for the tree $T_{S(n)}$, we assign the labeling in three different cases due to condition of number of branches of S_i, i = 1, 2,…, n.

1. $T_{S(n)}$ with each of the stars S_i having $|E(S_i)| = 1$(i.e. without free leaves in the stars S_i for i = 1, 2,…, n.), which is equivalent to comb graph P_n □ L_1.
2. Without free leaves up to i^{th} star and $(i+1)^{th}$ star S_{j+1} onwards have free leaves.
3. The star S_1 itself contains free leaves.

Case (1):

In this case, labeling is given as follows.

R (1): $l(s_1) = 0$, $l(c_1) = q$, $l(s_2) = q-1$, $l(c_2) = 1$. R (2): $l(c_{2j+1}) = q - \left(\sum_{i=1}^{j} q_{2i} + j\right)$, $1 \le j \le n$.

R (3): $l(c_{2j}) = \sum_{i=0}^{j-1} q_{2i+1} + j - 1$, $1 \le j \le n$.

R (4): $l(s_{2j}) = q - \left(\sum_{i=0}^{j-1} q_{2i+1} + j - 1\right)$, $1 \le j \le n$.

R (5): $l(s_{2j+1}) = \sum_{i=1}^{j} q_{2i} + j$, $1 \le j \le n$.

It can be seen that the above type graph also taken as comb graph, which has been independently labeled with graceful labeling by ganajothi [2].

Case (2):

Up to i^{th} stars S_i, all $S_1, S_2, \ldots, S_i$ are with $|E(S_i)| = 1$ and remaining stars satisfying the condition $2 \le |E(S_{i+1})| \le |E(S_{i+2})| \le \ldots \le |E(S_n)|$.

R (1): $l(s_1) = 0$, $l(c_1) = q$, $l(s_2) = q-1$, $l(c_2) = 1$. (Let n = 2j+1 for odd, and n = 2j for even)

R (2): $l(c_{2j+1}) = q - (\sum_{i=1}^{j} q_{2i} + j)$, $1 \le j \le n$. R (3): $l(c_{2j}) = \sum_{i=0}^{j-1} q_{2i+1} + j - 1$, $1 \le j \le n$.

R (4): $l(s_{2j}) = q - (\sum_{i=0}^{j-1} q_{2i+1} + j - 1)$, $1 \le j \le n$. R (5): $l(s_{2j+1}) = \sum_{i=1}^{j} q_{2i} + j$, $1 \le j \le n$.

From (i+1)th star to nth star the induced labeling of free leaves of the stars S_{2m} and S_{2m+1} stars are given as follows.

The labeling of free leaves of odd stars of S_{2m+1} based on its supporting vertex s_{2m+1} as follows.

R (6): labeling of free leaves of S_{2m+1} is {integers starting from $l(c_{2m}) + 1$ to the number of leaves in the particular star assigned in ascending order except the value $l(s_{2m+1})$ } .

The labeling of leaves of even stars S_{2m} based on its supporting vertex s_{2m} as follows.

R (7): labeling of leaves of S_{2m} is {integers starting from $l(c_{2m-1}) - 1$ to the number of leaves in the particular branch assigned in descending order except the value $l(s_{2m})$. }

The assignment of vertex labeling for case (2) shown in Figure 2.

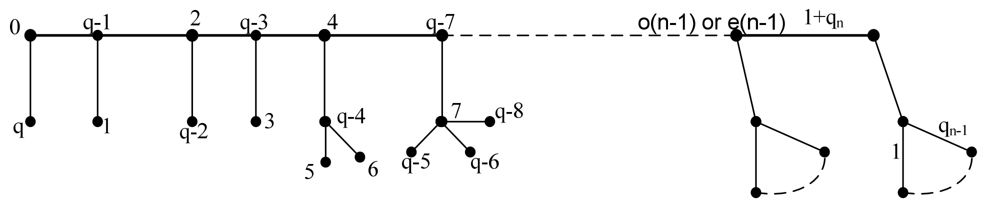

Figure 2

The vertex labeling of case (2):

Labeling of $T_{S(n)}$	Labeling of vertices
s_1	0
c_1	q
s_2	q-1
c_2	1
s_3	2
c_3	q-2
s_4	q-3
c_4	3
s_i (odd star of S_i)	i-1

c_i (odd star of S_i)	q-(i-1)
s_i (even star of S_i)	q-(i-1)
c_i (odd star of S_i)	i-1
s_{2m}	q-l(c_{2m})
c_{2m}	$(q_1+q_3+q_5+\ldots+q_{2m-1}+m-1) = o(m)$
Remaining free leaves	The relation R (7) assigns the values.
s_{2m+1}	$(q_2+q_4+\ldots+q_{2m}+m) = e(m)$
c_{2m+1}	q-l(s_{2m+1})
Remaining free leaves	The relation R (6) assigns the values.
s_{2n}	q- l(c_{2n})
c_{2n}	l(c_{2n-2}) +q_{2n-2} + 1
Remaining free leaves	The relation R (7) assigns the values.
s_{2n+1}	l(s_{2n-1}) + q_{2n-1} +1
c_{2n+1}	q- l(s_{2n+1})
Remaining free leaves	The relation R (6) assigns the values.

Table 2.1

The above labeling of vertices in $T_{S(n)}$ induces edge assignment as follows.

Labeling of $T_{S(n)}$	Labeling of edge values
edge s_1c_1	q
edge s_1s_2	q-1
edge s_2c_2	q-2
edge s_2s_3	q-3
edge s_3c_3	q-4
edge s_ic_i (odd star S_i)	q-2(i-1)
edge s_ic_i (even star S_i)	q-2(i-1)
edge $s_{2m-1}s_{2m}$	$q-(q_1+q_2+\ldots+q_{2m-1}+2m-2)$
edge $s_{2m}c_{2m}$	$q-2(q_1+q_2+\ldots+q_{2m-1}+m-1)$
free leaves of S_{2m}	$\{q-(q_1+q_2+\ldots+q_{2m-1}+2m-1)$ to $q-(q_1+q_2+\ldots+q_{2m}+2m-2)$ except labeling value of the edge $s_{2m}c_{2m} = l(s_{2m}c_{2m})\}$
edge $s_{2m}s_{2m+1}$	$q-(q_1+q_2+\ldots+q_{2m}+2m-1)$
edge $s_{2m+1}c_{2m+1}$	$q-2(q_1+q_2+\ldots+q_{2m}+m)$
free leaves of S_{2m+1}	$\{q-(q_1+q_2+\ldots+q_{2m}+2m)$ to $q-(q_1+q_2+\ldots+q_{2m+1}+2m-1)$ except labeling value of the edge $s_{2m+1}c_{2m+1} = l(s_{2m+1}c_{2m+1})\}$
edge $s_{n-1}s_n$	$1+ \mid E(S_n)\mid$
edge s_nc_n	$\mid l(s_n)-l(c_n)\mid$
free leaves of S_n	$\{\mid E(S_n)\mid$ to 1 except labeling value of the edge $s_nc_n = l(s_nc_n)\}$

Table 2.2

Case (3): $2 \leq |E(S_1)| \leq |E(S_2)| \leq \ldots \leq |E(S_n)|$ and $|E(S_i)| = q_i$, for i = 1, 2,…n.

In this case, we give following labeling procedure to give graceful labeling of the $T_{S(n)}$.

R (1): $l(s_1) = 0$; $l(c_1) = q$; $l(c_2) = q_1$ and $l(s_2) = q-q_1$. Let $n = 2j+1$ for odd, and $n = 2j$ for even,

R (2): $l(c_{2j+1}) = q - (\sum_{i=1}^{j} q_{2i} + j)$, $j \geq 1$. R (3): $l(c_{2j}) = \sum_{i=0}^{j-1} q_{2i+1} + j - 1$, $j \geq 1$.

R (4): $l(s_{2j}) = q - (\sum_{i=0}^{j-1} q_{2i+1} + j-1)$, $j \geq 1$. R(5): $l(s_{2j+1}) = \sum_{i=1}^{j} q_{2i} + j$, $j \geq 1$.

Labeling of free leaves in case (3):

Let the free leaves of growing m^{th} star of $T_{S(m)}$ at s_m be $f_1^{\,m}, f_2^{\,m}, \ldots, f_{m-1}^{\,m}$.

Let the free leaves of S_1 are labeled with values 1 to q_1-1. Then for $m \geq 1$

The labeling of free leaves of odd stars of S_{2m+1} based on its supporting vertex s_{2m+1} as follows.

R (6): labeling of free leaves of S_{2m+1} is {integers starting from $l(c_{2m}) + 1$ to the number of leaves in the particular star assigned in ascending order except the value of $l(s_{2m+1})$ } .

The labeling of leaves of even stars S_{2m} based on its supporting vertex s_{2m} as follows.

R (7): labeling of leaves of S_{2m} is {integers starting from $l(c_{2m-1}) - 1$ to the number of leaves in the particular branch assigned in descending order except the value of $l(s_{2m})$. }

We observe that for all cases the labeling of S_i's in which s_{2m+1}, $m \geq 1$ are increasing order and s_{2m}, $m \geq 2$ are decreasing order in relation with q. $l(s_i) + l(c_i) = q$, for any i.

The labeling of vertices in case (3) is shown in Figure 3.

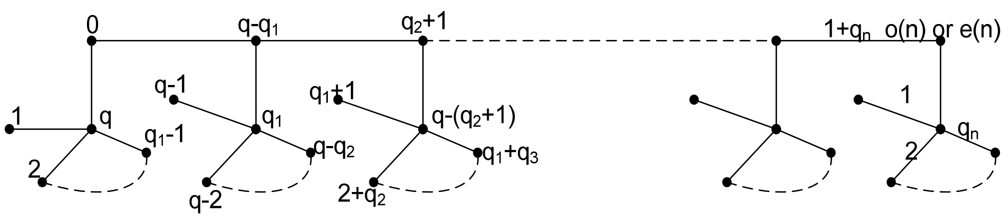

Figure 3

The above labeling of vertices and edges induces a bijective mapping h_E and h_V as follows.

h_E: $E(T_{S(n)}) \longrightarrow \{1, 2,\ldots, q\}$

h_V: $V(T_{S(n)}) \longrightarrow \{0,1,\ldots, q\}$

which could be verified easily that it admits graceful labeling for the given tree $T_{S(n)}$.

The general vertex labeling of case (3):

Labeling of $T_{S(n)}$	Labeling of vertices
s_1	0
c_1	q
Remaining free leaves of S_1	1 to q_1-1
s_2	$q-q_1$
c_2	q_1
Remaining free leaves of S_2	{q-1 to $(q-q_2)$ except labeling value of the vertex $(q-q_1) = l(s_2)$}
s_3	$1+q_2$
c_3	$q-(1+q_2)$
Remaining free leaves of S_3	{$(q_1 +1)$ to $(q_1 + q_3)$ except labeling value of the vertex $(1+q_2) = l(s_3)$}
s_4	$q-(q_1+q_3+1)$
c_4	$q_1+ q_3 + 1$
Remaining free leaves of S_4	{$(q-q_2-2)$ to $(q-(q_2+q_4+1)$ except labeling value of the vertex $q-(q_1+q_3+1) = l(s_4)$}
s_{2m}	$q-l(c_{2m})$
c_{2m}	$(q_1+q_3+q_5+\ldots+q_{2m-1}+m-1) = o(m)$
Remaining free leaves of S_{2m}	The relation R (7) assigns the values.
s_{2m+1}	$(q_2+q_4+\ldots+q_{2m}+m) = e(m)$
c_{2m+1}	$q-l(s_{2m+1})$
Remaining free leaves of S_{2m+1}	The relation R (6) assigns the values.
s_{2n}	$q- l(c_{2n})$
c_{2n}	$l(c_{2n-2}) +q_{2n-2} + 1$
Remaining free leaves of S_{2n}	The relation R (7) assigns the values.

s_{2n+1}	$l(s_{2n-1}) + q_{2n-1} + 1$
c_{2n+1}	$q - l(s_{2n+1})$
Remaining free leaves of S_{2n+1}	The relation R (6) assigns the values.

Table 3.1

The above labeling of vertices in $T_{S(n)}$ induces edge assignment as follows.

Labeling of $T_{S(n)}$	Labeling of edge values
edge s_1c_1	q
free leaves of S_1	$q-1$ to $q-(q_1-1)$
edge s_1s_2	$q-q_1$
edge s_2c_2	$q-2q_1$
free leaves of S_2	$\{q-(q_1+1)$ to $q-(q_1+q_2)$ except labeling value of the edge $s_2c_2 = (q-2q_1)\}$
edge s_2s_3	$q-(q_1+q_2+1)$
edge s_3c_3	$q-2(q_2+1)$
free leaves of S_3	$\{q-(q_1+q_2+2)$ to $q-(q_1+q_2+q_3+1)$ except labeling value of the edge $s_3c_3 = q-2(1+q_2)\}$
edge s_3s_4	$q-(q_1+q_2+q_3+2)$
edge s_4c_4	$q-2(q_1+q_3+1)$
free leaves of S_4	$\{q-(q_1+q_2+q_3+3)$ to $q-(q_1+q_2+q_3+q_4+2)$ except labeling value of the edge $s_4c_4 = q-2(q_1+q_3+1)\}$
edge $s_{2m-1}s_{2m}$	$q-(q_1+q_2+\ldots+q_{2m-1}+2m-2)$
edge $s_{2m}c_{2m}$	$q-2(q_1+q_2+\ldots+q_{2m-1}+m-1)$
free leaves of S_{2m}	$\{q-(q_1+q_2+\ldots+q_{2m-1}+2m-1)$ to $q-(q_1+q_2+\ldots+q_{2m}+2m-2)$ except the labeling value of the edge $s_{2m}c_{2m} = q-2(q_1+q_2+\ldots+q_{2m-1}+m-1)\}$
edge $s_{2m}s_{2m+1}$	$q-(q_1+q_2+\ldots+q_{2m}+2m-1)$
edge $s_{2m+1}c_{2m+1}$	$q-2(q_1+q_2+\ldots+q_{2m}+m)$
free leaves of S_{2m+1}	$\{q-(q_1+q_2+\ldots+q_{2m}+2m)$ to $q-(q_1+q_2+\ldots+q_{2m+1}+2m-1)$ except the labeling value of the edge $s_{2m+1}c_{2m+1} = q-2(q_1+q_2+\ldots+q_{2m}+m)\}$
edge $s_{n-1}s_n$	$1+\|E(S_n)\|$
edge s_nc_n	$\|l(s_n)-l(c_n)\|$

free leaves of S_n	{$\lvert E(S_n)\rvert$ to 1 except the labeling value of the edge s_nc_n = $\lvert l(s_n)-l(c_n)\rvert$}

Table 3.2

Concluding remarks (as generalization)

Using the above scheme, we conclude that any random tree consisting growing stars with the condition $|E(S_1)| \leq |E(S_2)| \leq \ldots \leq |E(S_n)|$ is always graceful.

Now, it leads to an interesting problem of finding a graceful labeling of trees with hanging stars whose branches are selected in random.

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