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Complementary Tree Domination in Grid Graphs

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Abstract: *Let G(V, E) be a nontrivial, simple, finite and undirected graph. A dominating set of the graph G is a subset D of V such that every vertex not in D is adjacent to some vertex in D. The minimum cardinality of a dominating set is the domination number $\gamma(G)$. A dominating set D is called a complementary tree dominating set if the subgraph <V - D> induced by V - D is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$.*

In this paper, we determine the complementary tree domination numbers of some grid graphs (Cartesian product of two paths P_m and P_n).

Keywords: *domination, complementary tree domination.*

Mathematics Subject Classification: 05C69.

1. Introduction

The graphs considered here are nontrivial, simple, finite and undirected. Let G be a graph with vertex set V(G) and edge set E(G). The concept of domination was first studied by Ore [7] and Berge [1]. A set D $\subseteq$ V is said to be a dominating set of G, if every vertex in V- D is adjacent to some vertex in D. The minimum cardinality of a dominating set is called the domination number of G and is denoted by γ(G). The concept of complementary tree domination was introduced by S. Muthammai, M. Bhanumathi and P. Vidhya in [6]. A dominating set D $\subseteq$ V is called a complementary tree dominating (ctd) set, if the subgraph <V - D> induced by V – D is a tree. The minimum cardinality of a complementary tree dominating set is called the complementary tree domination number of G and is denoted by γ_{ctd}(G).

The Cartesian product of two graphs G_1 and G_2 is the graph, denoted by $G_1 \times G_2$, with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ (where $\times$ denotes the Cartesian product of sets) and two vertices u = (u_1, u_2) and v = (v_1, v_2) in $V(G_1 \times G_2)$ are adjacent in $G_1 \times G_2$ whenever [$u_1 = v_1$

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and $(u_2, v_2) \in E(G_2)$] or [$u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$]. If each G_1 and G_2 is a path P_m and P_n (respectively), then we will call $P_m \times P_n$, a $m \times n$ Grid graph. For notational convenience, we denote $P_m \times P_n$ by $P_{m, n}$. The reader is referred to [4] for survey of results on domination. The inverse domination number for the grid graphs $P_m \times P_n$ $(1 \leq m \leq 5)$ was determined by T. Tamizh Chelvam and G.S. Grace Prema [8]

In this paper, we determine the complementary tree domination numbers of $P_{m, n}$ where m = 2, 3, 4, 5 and 6. $P_{1, n}$ is nothing but the path P_n on n vertices. S. Muthammai, M. Bhanumathi and P. Vidhya [6] have established $\gamma_{ctd}(P_n) = n - 2, n \geq 4$.

Notation:

Let 1, …, m and 1, …, n be the vertices of P_m and P_n respectively. Then the vertices of $P_{m, n}$ are denoted $x_{i,j}$, when i = 1, …, m and j = 1, …, n.

Theorem 1.1. [6]

A complementary tree dominating set D of G is minimal if and only if for each vertex $v \in D$, one of the following conditions is satisfied

(i) there exists a vertex $u \in V(G) - D$ such that $N(u) \cap D = \{v\}$

(ii) v is an isolated vertex in <D>

(iii) $N(v) \cap (V(G)-D) = \phi$

(iv) The subgraph $\langle (V(G)-D) \cup \{v\} \rangle$ of G induced by $(V(G)-D) \cup \{v\}$ either contains a cycle or is disconnected.

2. Complementary Tree Domination Numbers of $P_{2, n}$, $n \geq 1$

In this section, we give the complementary tree domination numbers of $2 \times n$ grid graphs $P_{2, n}$.

Theorem 2.1: For all $n \geq 1$, $\gamma_{ctd}(P_{2, n}) = \left\lceil \frac{n+2}{2} \right\rceil$

Proof:

A minimal complementary tree dominating set of $P_{2, n}$ is given as follows.

Let n = 4q + r, where $1 \leq r \leq 4$. We split the set of columns of $P_{2, n}$ into blocks $B_i \cong P_{2, 4}$ for i = 1, …, q. The vertices ● enclosed within the round symbol in each of the blocks in the figures represent the vertices to be included for a minimal complementary tree dominating set D. The vertices ● with symbol $\times$ in the blocks indicate those vertices that are not dominated by a complementary tree dominating set D constructed upto this stage and to be considered while concatenation.

Let $P_i = \{x_{1, 4i-3}, x_{2, 4i-1}\}$, i = 1, …, q. (Figure 1)

Let $D = \bigcup_{i=1}^{q} P_i$. Therefore, $|D| = 2\left\lfloor \frac{n}{4} \right\rfloor$.

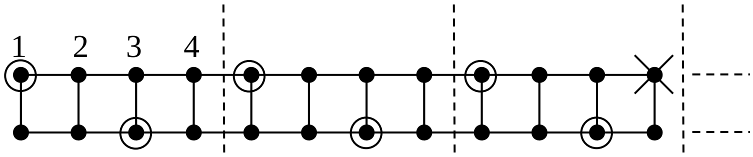


Figure 1

Case i: n ≡ 1(mod 4)

Consider the set $D_1 = D \cup \{x_{1, n}\}$. (Figure 2(a)). This set is a minimal complementary tree dominating set of $P_{2, n}$.

$$|D_1| = 2\left\lfloor \frac{n}{4} \right\rfloor + 1 = \left\lfloor \frac{n+2}{2} \right\rfloor$$

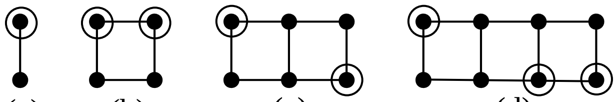


Figure 2

Case ii: n ≡ 2(mod 4)

Here the set $D_2 = D \cup \{x_{1, n-1}, x_{1, n}\}$ (Figure 2(b)) is a minimal complementary tree dominating set of $P_{2, n}$. Hence,

$$|D_2| = 2\left\lfloor \frac{n}{4} \right\rfloor + 2 = \left\lfloor \frac{n+2}{2} \right\rfloor$$

Case iii: n ≡ 3(mod 4)

In this case, the set $D_3 = D \cup \{x_{1, n-2}, x_{2, n}\}$. (Figure 2(c)) is a minimal complementary tree dominating set of $P_{2, n}$ and

$$|D_3| = 2\left\lfloor \frac{n}{4} \right\rfloor + 2 = \left\lfloor \frac{n+2}{2} \right\rfloor$$

Case iv: n ≡ 0(mod 4)

Let n = 4q + 4, q = 0, 1, ….

In this case, the set $D_4 = D \cup \{x_{1, n-3}, x_{2, n-1}, x_{2, n}\}$. (Figure 2(d)) is a minimal complementary tree dominating set of $P_{2, n}$.

$|D_4| = 2\left\lfloor \frac{n-1}{4} \right\rfloor + 3 = \left\lfloor \frac{n+2}{2} \right\rfloor$

From all the cases, $\gamma_{ctd}(P_{2,n}) = \left\lfloor \frac{n+2}{2} \right\rfloor$, for all $n \geq 1$.

3. Complementary Tree Domination Numbers of $P_{3,n}$, $n \geq 4$.

In this section, we give complementary tree domination numbers of 3×n grid graphs $P_{3,n}$, $n \geq 4$. Here we split the columns of $P_{3,n}$ into blocks $P_{3,4}$.

Theorem 3.1: For $n \geq 4$, $\gamma_{ctd}(P_{3,n}) = n + i$ for $n \equiv i \pmod 4$, i = 0, 1, 2, 3

Proof:

We give a minimal complementary tree dominating (ctd) set D of $P_{3,n}$ as follows. Let $n \geq 4$.

Case i: For n = 4q, $\gamma_{ctd}(P_{3,n}) = n$.

We split the set of columns of $P_{3,n}$ into blocks $B_i \cong P_{3,4}$ for i = 1, 2,…, q.
Let $P_i = \{x_{1,4i-1}, x_{1,4i}, x_{2,4i-3}, x_{3,4i-1}\}$. P_i dominates all the four columns of B_i such that $<B_i - P_i>$ is a tree for i = 1, …, q. Let $D = \bigcup_{i=1}^{q} P_i$ (Figure 3).

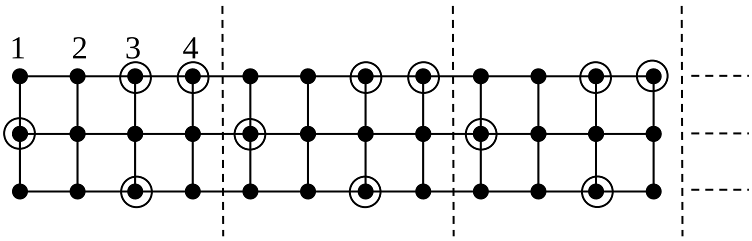
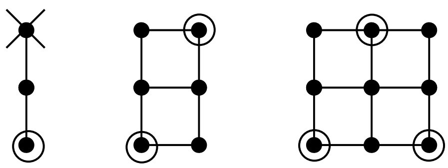


Figure 3



(a) (b) (c)

Figure 4

Then D is a minimal dominating set. Moreover $<V(P_{3,n}) - D>$ is a tree and hence D is a minimal complementary tree dominating set of $P_{3,n}$ and $\gamma_{ctd}(P_{3,n}) = 4\left\lfloor \frac{n}{4} \right\rfloor = n$.

Case ii: For $n \equiv 1 \pmod 4$, $\gamma_{ctd}(P_{3,n}) = n + 1$.

Consider $D_1 = D \cup \{x_{3,n}\}$. D_1 is a minimal ctd set of $P_{3,n}$ and $|D_1| = n + 1$. (To obtain the minimal ctd set we need the vertices of D together with the vertex $x_{3,n}$). (Figure 4(a)). Hence, $\gamma_{ctd}(P_{3,n}) = n + 1$.

Case iii: For $n \equiv 2 \pmod 4$, $\gamma_{ctd}(P_{3,n}) = n + 2$.

Consider $D_2 = D \cup \{x_{1,n}, x_{3,n-1}\}$. (Figure 4(b)).This D_2 is a minimal ctd set of $P_{3,n}$ and $|D_2| = n + 2$ and hence $\gamma_{ctd}(P_{3,n}) = n + 2$.

Case iv: For $n \equiv 3 \pmod 4$, $\gamma_{ctd}(P_{4,n}) = n + 3$.

Consider $D_3 = D \cup \{x_{1,n-1}, x_{3,n-2}, x_{3,n}\}$. (Figure 4(c)). This D_3 is a minimal ctd set of $P_{3,n}$ and $|D_3| = n + 3$ and hence $\gamma_{ctd}(P_{3,n}) = n + 3$.

From all the four cases, we conclude that for $n \geq 4$,

$\gamma_{ctd}(P_{3,n}) = n + i$, for $n \equiv i \pmod 4$, $i = 0, 1, 2, 3$.

Remark 3.2:

$\gamma_{ctd}(P_{3,1}) = 2$

$\gamma_{ctd}(P_{3,n}) = 3$ if $n = 2, 3$.

4. Complementary Tree Domination Numbers of $P_{4,n}$, $n \geq 5$

In this section, we give complementary tree domination numbers of $4 \times n$ grid graphs $P_{4,n}$, $n \geq 5$. Here we split the columns of $P_{4,n}$ into blocks $P_{4,5}$.

It is to be noted that $\gamma_{ctd}(P_{4,6}) = \gamma_{ctd}(P_4) \times \gamma_{ctd}(P_6)$

Theorem 4.1: For $n \geq 5$, $\gamma_{ctd}(P_{4,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$.

Proof:

A minimal complementary tree dominating set of $P_{4,n}$ ($n \geq 5$) is presented as follows.

Case i: For $n = 5q$, $\gamma_{ctd}(P_{4,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$

Here we split the set of columns of $P_{4,n}$ into blocks B_i, where $B_i \cong P_{4,5}$ for $i = 1, \ldots, q$. Let $P_i = \{x_{1,5i-4}, x_{1,5i}, x_{2,5i-2}, x_{2,5i}, x_{3,5i-3}, x_{4,5i-4}, x_{4,5i-1}\}$ $(i = 1, \ldots, q)$.

This set dominates all the five columns of each block B_i such that $\langle B_i - P_i \rangle$ is a tree for $i = 1, \ldots, q$. (Figure 5).

Let $D = \bigcup_{i=1}^{q} P_i$. Then $<V(P_{4,\,n}) - D>$ is a tree and D is a minimal ctd set and hence

$\gamma_{ctd}(P_{4,\,n}) = 7\left\lfloor \frac{n}{5} \right\rfloor = \left\lfloor \frac{7n}{5} \right\rfloor$

Case ii : For $n \equiv 1 \pmod 5$, $\gamma_{ctd}(P_{4,\,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$

Consider the set $D_1 = D \cup \{x_{4,\,n}\}$ is a minimal ctd set of $P_{4,\,n}$. (Figure 6(a)).

$|D_1| = 7\left\lfloor \frac{n}{5} \right\rfloor + 1 = \left\lfloor \frac{7n}{5} \right\rfloor$ and hence $\gamma_{ctd}(P_{4,\,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$

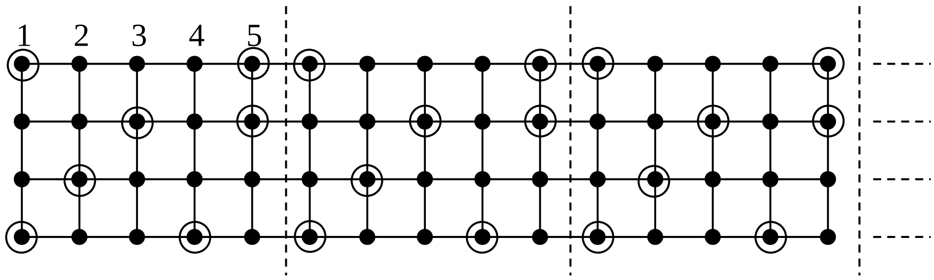


Figure 5

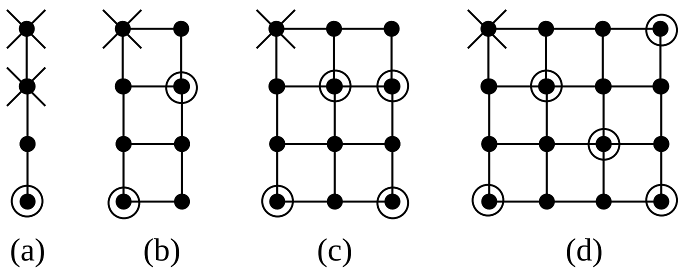


Figure 6

Case iii: For $n \equiv 2 \pmod 5$, $\gamma_{ctd}(P_{4,\,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$

Consider the set $D_2 = D \cup \{ x_{2,\,n}, x_{4,\,n-1}\}$ (Figure 6(b)).

This set D_2 is a minimal ctd set of $P_{4,n}$ and $|D_2| = 7\left\lfloor \frac{n}{5} \right\rfloor + 2 = \left\lfloor \frac{7n}{5} \right\rfloor$ and hence

$\gamma_{ctd}(P_{4,\,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$

Case iv: For $n \equiv 3 \pmod 5$, $\gamma_{ctd}(P_{4,\,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$

The set $D_3 = D \cup \{x_{2,n-1}, x_{2,n}, x_{4,n-2}, x_{4,n}\}$ is a minimal ctd set of $P_{4,n}$. (Figure 6(c)).

$|D_3| = 7\left\lfloor \frac{n}{5} \right\rfloor + 4 = \left\lfloor \frac{7n}{5} \right\rfloor$ and hence $\gamma_{ctd}(P_{4,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$

Case v: For $n \equiv 4 \pmod 5$, $\gamma_{ctd}(P_{4,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$

Here, the set $D_4 = D \cup \{x_{1,n}, x_{2,n-2}, x_{3,n-1}, x_{4,n-3}, x_{4,n}\}$ is a minimal ctd set of $P_{4,n}$. (Figure 6(d)).

$|D_4| = 7\left\lfloor \frac{n}{5} \right\rfloor + 5 = \left\lfloor \frac{7n}{5} \right\rfloor$ and hence $\gamma_{ctd}(P_{4,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$

From the above cases, we see that $\gamma_{ctd}(P_{4,n}) = \left\lfloor \frac{7n}{5} \right\rfloor$, for $n \geq 5$.

Remark 4.2: For $1 \leq n \leq 4$, $\gamma_{ctd}(P_{4,n}) = \begin{cases} n+1, & \text{if } n = 1, 2, 3 \\ n+2, & \text{if } n = 4 \end{cases}$

Remark 4.3: Theorem 4.1. implies the following recurrence relation

$\gamma_{ctd}(P_{4,n}) = \gamma_{ctd}(P_{4,n-5}) + 7$, $n \geq 10$.

5. Complementary Tree Domination Numbers of $P_{5,n}$, $n \geq 6$

In this section, we give complementary tree domination numbers of $5\times n$ grid graphs $P_{5,n}$, $n \geq 6$. Here we split the columns of $P_{5,n}$ into blocks $P_{5,6}$.

Theorem 5.1: For $n \geq 6$,

$$\gamma_{ctd}(P_{5,n}) = \begin{cases} \frac{5n}{3} & \text{if } n \equiv 0, 3 \pmod 6 \\ \frac{5n+1}{3} & \text{if } n \equiv 1 \pmod 6 \\ \left\lfloor \frac{5n-1}{3} \right\rfloor & \text{if } n \equiv 2, 4, 5 \pmod 6. \end{cases}$$

Proof:

We determine the minimal ctd set of $P_{5,n}$ ($n \geq 6$) as follows.

Case i: For $n = 6q$, $\gamma_{ctd}(P_{5,n}) = \left\lfloor \frac{5n}{3} \right\rfloor$.

The set of columns of $P_{5,n}$ can be split into blocks B_i, where $B_i \cong P_{5,6}$ for $i = 1, 2, \ldots, q$.

Consider the set $P_i = \{x_{1, 6i-5}, x_{1, 6i-2}, x_{1, 6i}, x_{2, 6i-3}, x_{3, 6i-5}, x_{3, 6i-1}, x_{4, 6i-3}, x_{4, 6i}, x_{5, 6i-5}, x_{5, 6i-2}\}$

This set P_i dominates all the six columns of the block B_i such that $<B_i - P_i>$ is a tree ($i = 1, 2, \ldots, q$). (Figure 7).

Let $D = \bigcup_{i=1}^{q} P_i$. Then $<V(P_{5, n}) - D>$ is a tree and D is a minimal ctd set of $P_{5, n}$ and hence $\gamma_{ctd}(P_{5, n}) = 5\left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{5n}{3} \right\rfloor = \frac{5n}{3}$.

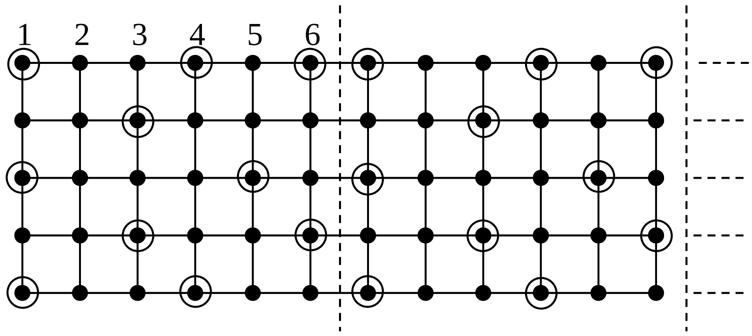


Figure 7

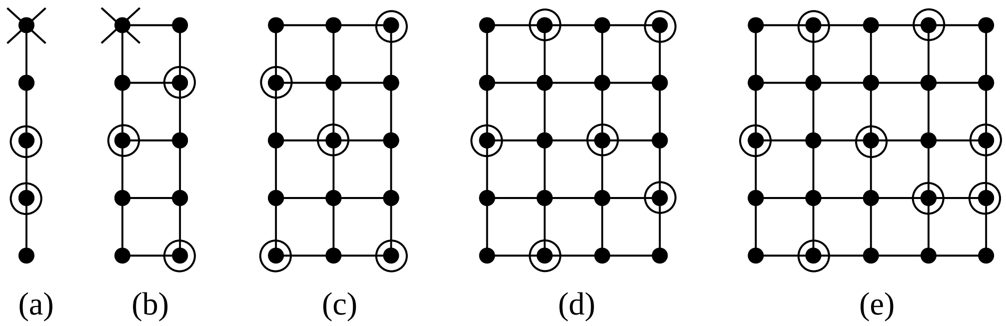


Figure 8

Case ii: For $n \equiv 1 \pmod 6$, $\gamma_{ctd}(P_{5, n}) = \frac{5n+1}{3}$.

Consider the set $D_1 = D \cup \{x_{3, n}, x_{4, n}\}$ (Figure 8(a)). This set is a minimal ctd set of $P_{5, n}$ and $|D_1| = 5\left\lfloor \frac{n}{3} \right\rfloor + 2 = \frac{5n+1}{3}$. Hence, $\gamma_{ctd}(P_{5, n}) = \frac{5n+1}{3}$.

Case iii: For $n \equiv 2 \pmod 6$, $\gamma_{ctd}(P_{5, n}) = \left\lfloor \frac{5n-1}{3} \right\rfloor$. The set $D_2 = D \cup \{x_{2, n}, x_{3, n-1}, x_{5, n}\}$ is a minimal ctd set of $P_{5, n}$. (Figure 8(b)).

$|D_2| = 5\left\lfloor \frac{n}{3} \right\rfloor + 3 = \left\lfloor \frac{5n-1}{3} \right\rfloor$. Therefore, $\gamma_{ctd}(P_{5, n}) = \left\lfloor \frac{5n-1}{3} \right\rfloor$.

Case iv: For $n \equiv 3 \pmod 6$, $\gamma_{ctd}(P_{5,n}) = \left\lfloor \frac{5n-1}{3} \right\rfloor$. In this case, the set $D_3 = D \cup \{x_{1,n}, x_{2,n-2}, x_{3,n-1}, x_{5,n-2}, x_{5,n}\}$ is a minimal ctd set of $P_{5,n}$. (Figure 8(c)).

$|D_3| = 5\left\lfloor \frac{n}{3} \right\rfloor + 5 = \left\lfloor \frac{5n}{3} \right\rfloor = \frac{5n}{3}$.

Case v: For $n \equiv 4 \pmod 6$, $\gamma_{ctd}(P_{5,n}) = \left\lfloor \frac{5n-1}{3} \right\rfloor$. Here, the set $D_4 = D \cup \{x_{1,n-2}, x_{1,n}, x_{3,n-3}, x_{3,n-1}, x_{4,n}, x_{5,n-2}\}$ is a minimal ctd set of $P_{5,n}$. (Figure 8(d)). $|D_4| = 5\left\lfloor \frac{n}{3} \right\rfloor + 6 = \left\lfloor \frac{5n-1}{3} \right\rfloor$ and hence $\gamma_{ctd}(P_{5,n}) = \left\lfloor \frac{5n-1}{3} \right\rfloor$.

Case vi: For $n \equiv 5 \pmod 6$, $\gamma_{ctd}(P_{5,n}) = \left\lfloor \frac{5n-1}{3} \right\rfloor$. In this case, the set $D_5 = D \cup \{x_{1,n-3}, x_{1,n-1}, x_{3,n-4}, x_{3,n-2}, x_{3,n}, x_{4,n-1}, x_{4,n}, x_{5,n-3}\}$ is a minimal ctd set of $P_{5,n}$.

$|D_5| = 5\left\lfloor \frac{n}{3} \right\rfloor + 8 = \left\lfloor \frac{5n-1}{3} \right\rfloor$

From the above cases, we conclude that,

$$\gamma_{ctd}(P_{5,n}) = \begin{cases} \frac{5n}{3} & \text{if } n \equiv 0, 3 \pmod 6 \\ \frac{5n+1}{3} & \text{if } n \equiv 1 \pmod 6 \\ \left\lfloor \frac{5n-1}{3} \right\rfloor & \text{if } n \equiv 2, 4, 5 \pmod 6. \end{cases}$$

Remark 5.2: For $n \leq 5$, $\gamma_{ctd}(P_{5,1}) = 3$

$\gamma_{ctd}(P_{5,n}) = 2n - 1$, if $n = 2, 3$,

$\gamma_{ctd}(P_{5,n}) = 2n - 2$, if $n = 4, 5$.

Remark 5.3: A recurrence relation in $5 \times n$ grid graphs is.

$$\gamma_{ctd}(P_{5,n}) = \gamma_{ctd}(P_{5,n-6}) + 10, \text{ for } n \geq 12.$$

6. Complementary Tree Domination Numbers of $P_{6,n}$, $n \geq 7$

In this section, we give complementary tree domination numbers of $6 \times n$ grid graphs $P_{6,n}$, $n \geq 7$. Here we split the columns of $P_{6,n}$ into blocks $P_{6,7}$.

Theorem 6.1: Let $n \geq 7$. Then $\gamma_{ctd}(P_{6,n}) = 2n$.

Proof:

As before, we present a complementary tree dominating (ctd) set of $P_{6,n}$ as follows.

Let $n \geq 7$.

Case i: $n = 7q$.

We split the set of columns of $P_{6,n}$ into blocks B_i, $B_i \cong P_{6,7}$ for $i = 1, 2, \ldots, q$. $P_i = \{x_{1,7i-4}, x_{1,7i}, x_{2,7i-6}, x_{2,7i-2}, x_{3,7i-4}, x_{3,7i-1}, x_{4,7i-6}, x_{4,7i-5}, x_{4,7i-3}, x_{4,7i}, x_{5,7i-2}, x_{6,7i-6}, x_{6,7i-4}, x_{6,7i}\}$ dominates all the seven columns of each block B_i, such that $\langle B_i - P_i \rangle$ is a tree, $i = 1, \ldots, q$. (Figure 9).

Let $D = \bigcup_{i=1}^{q} P_i$. Also $\langle V(P_{6,n}) - D \rangle$ is a tree and D is a minimal ctd set of $P_{6,n}$ and hence $\gamma_{ctd}(P_{6,n}) = 14 \left\lfloor \frac{n}{7} \right\rfloor = 2n$.

Case ii: $n \equiv 1 \pmod 7$.

Let $D_1 = D \cup \{x_{3,n}, x_{6,n}\}$. (Figure 10(a)). This set is a minimal ctd set and $|D_1| = 14 \left\lfloor \frac{n}{7} \right\rfloor + 2 = \left\lfloor \frac{14n}{7} \right\rfloor = 2n$ [To obtain the minimal ctd set, we need the vertices of D together with vertices $x_{3,n}$ and $x_{6,n}$]. Therefore, $\gamma_{ctd}(P_{6,n}) = 2n$.

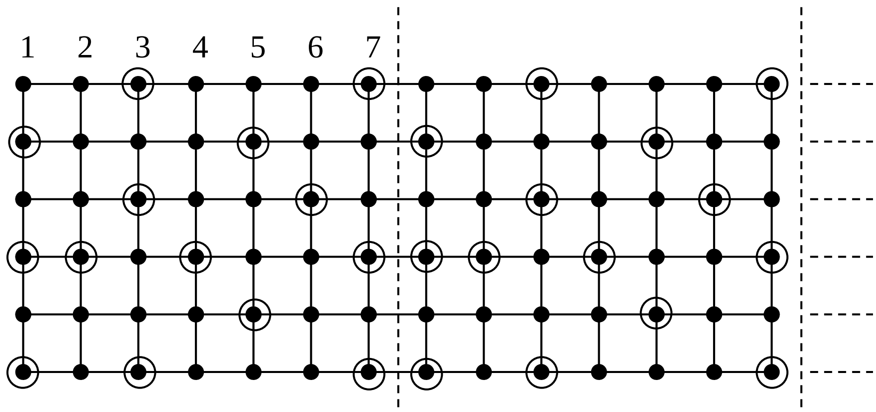


Figure 9

Case iii: $n \equiv 2 \pmod 7$.

Let $D_2 = D \cup \{x_{1,n}, x_{3,n-1}, x_{4,n}\, x_{6,n-1}\}$. This set is a minimal ctd set of $P_{6,n}$. (Figure 10(b)). $|D_2| = 14 \left\lfloor \frac{n}{7} \right\rfloor + 4 = \left\lfloor \frac{14n}{7} \right\rfloor = 2n$. Hence, $\gamma_{ctd}(P_{6,n}) = 2n$.

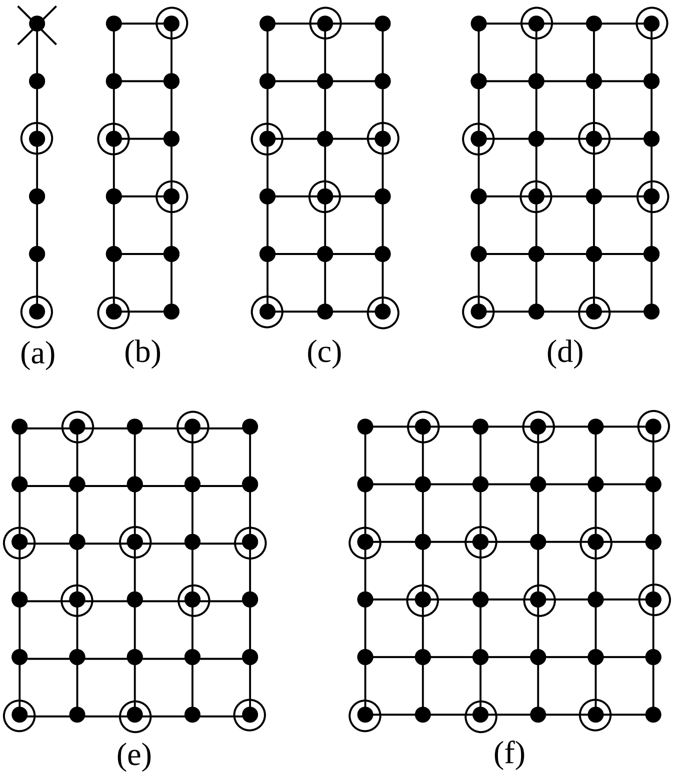


Figure 10

Case iv: $n \equiv 3 \pmod 7$.

Let $D_3 = D \cup \{ x_{1, n-1}, x_{3, n-2}, x_{3, n}, x_{4, n-1}, x_{6, n-2}, x_{6, n}\}$.

(Figure 10(c)). This set is a minimal ctd set of $P_{6, n}$. $|D_3| = 14 \left\lfloor \frac{n}{7} \right\rfloor + 6 = \left\lfloor \frac{14n}{7} \right\rfloor = 2n$

Hence, $\gamma_{ctd}(P_{6, n}) = 2n$.

Case v: $n \equiv 4 \pmod 7$

Let $D_4 = D \cup \{x_{1, n-2}, x_{1, n} x_{3, n-3}, x_{3, n-1}, x_{4, n-2}, x_{4, n}, x_{6, n-3}, x_{6, n-1}\}$. (Figure 10(d)).

This set is a minimal ctd set of $P_{6, n}$. $|D_4| = 14 \left\lfloor \frac{n}{7} \right\rfloor + 8 = \left\lfloor \frac{14n}{7} \right\rfloor = 2n$

Hence, $\gamma_{ctd}(P_{6, n}) = 2n$.

Case vi: $n \equiv 5 \pmod 7$

Let $D_5 = D \cup \{x_{1, n-3}, x_{1, n-1}, x_{3, n-4}, x_{3, n-2}, x_{3, n} x_{4, n-3}, x_{4, n-1}, x_{6, n-4}, x_{6, n-2}, x_{6, n}\}$.

(Figure 10(e)).

This set is a minimal ctd set of $P_{6, n}$.

$|D_5| = 14 \left\lfloor \frac{n}{7} \right\rfloor + 10 = \left\lfloor \frac{14n}{7} \right\rfloor = 2n$. Therefore, $\gamma_{ctd}(P_{6, n}) = 2n$.

Case vii: $n \equiv 6 \pmod 7$

Let $D_6 = D \cup \{x_{1, n-4}, x_{1, n-2}, x_{1, n}, x_{3, n-5}, x_{3, n-3}, x_{3, n-1}, x_{4, n-4}, x_{4, n-2}, x_{4, n}, x_{6, n-5}, x_{6, n-3}, x_{6, n-1}\}$. (Figure 10(f)). This set is a minimal ctd set of $P_{6, n}$.

$|D_6| = 14 \left\lfloor \frac{n}{7} \right\rfloor + 12 = \left\lfloor \frac{14n}{7} \right\rfloor = 2n$. Hence, $\gamma_{ctd}(P_{6, n}) = 2n$.

Therefore $\gamma_{ctd}(P_{6, n}) = 2n$, for $n \geq 7$.

Remark 6.2:

$$\text{For } 2 \leq n \leq 6,\ \gamma_{ctd}(P_{6, n}) = \begin{cases} 4 & \text{if } n = 2 \\ 2n - 1 & \text{if } n = 3, 4, 5 \\ 12 & \text{if } n = 6. \end{cases}$$

Note 6.3:

The above method of splitting the columns of $P_{m, n}$ into blocks $P_{m, m+1}$ doesn't work for $m \geq 7$. When we concatenate two blocks $P_{m, m+1}$, the subgraph induced by the complement of a minimal dominating set either will contain a cycle or will be disconnected.

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