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Global Domination and Neighborhood Numbers in Boolean Function Graph B($\overline{K_p}$, NINC, L(G)) of a Graph

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Abstract: *For any graph G, let V(G) and E(G) denote the vertex set and edge set of G respectively. The Boolean function graph B($\overline{K_p}$, NINC, L(G)) of G is a graph with vertex set V(G)∪E(G) and two vertices in B($\overline{K_p}$, NINC, L(G)) are adjacent if and only if they correspond to two adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_2(G)$. In this paper, global domination number, total global domination number, global point set domination number and neighborhood number are obtained for $B_2(G)$.*

1. Introduction

Graphs discussed in this paper are undirected and simple graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. For a connected graph G, the *eccentricity* $e_G(v)$ of a vertex v in G is the distance to a vertex farthest from v. Thus, $e_G(v) = \{d_G(u, v) : u \in V(G)\}$, where $d_G(u, v)$ is the distance between the vertices u and v. The minimum and maximum eccentricities are the *radius and diameter* of G, denoted r(G) and diam(G) respectively. A set D of vertices in a graph G = (V, E) is a *dominating set*, if every vertex in V−D is adjacent to some vertex in D. Further, D is a *global dominating set*, if it is a dominating set of both G and its complement $\overline{G}$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The *global domination number* γ_g of G is defined similarly [6]. A dominating set D is called a *connected dominating set*, if the induced sub graph <D> is connected. The *connected domination number* $\gamma_c(G)$ of G is the minimum cardinality of a connected dominating set [13]. A *total dominating set* T of G is a dominating set such that the induced sub graph <T> has no isolated vertices. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set. This concept was introduced in Cockayne *et al* [1]. A total dominating set T of G is a *total global dominating set* (t.g.d. set), if T is also a total

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dominating set of $\overline{G}$. The *total global dominating number* $\gamma_{tg}(G)$ of G is the minimum cardinality of a t.g.d. set [8]. $\gamma_t(G)$ is defined for G with $\delta(G) \geq 1$ and $\gamma_{tg}(G)$ is only defined for G with $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$, where $\delta(G)$ is the minimum degree of G.

For a connected graph G =(V, E), a set D of vertices is a *point set dominating set* (psd-set) of G, if for each set $S \subseteq V-D$, there exists a vertex $v \in D$ such that the sub graph $<S \cup \{v\}>$ induced by $S \cup \{v\}$ is connected. The *point set domination number* $\gamma_{ps}(G)$ is the minimum cardinality of a psd-set of G [12]. We say that a graph G is co-connected, if both G and $\overline{G}$ are connected. For a co-connected graph G = (v, E), a set $D \subseteq V$ is said to be *a global psd-set*, if it is a psd-set of both G and $\overline{G}$. The *global point set domination number* γ_{pg} of G is defined as the minimum cardinality of a global psd-set [9]. A γ-set is a minimum dominating set. Similarly, a γ_g-set, γ_t-set. γ_{tg}-set and γ_{pg}-set are defined.

For $v \in V(G)$, the *neighborhood* N(v) of v is the set of all vertices adjacent to v in G. $N[v] = N(v) \cup \{v\}$ is called the closed neighborhood of v. A subset S of V(G) is a *neighborhood set* (n-set) of G, if $G = \cup_{v \in S} <N[v]>$, where $<N[v]>$ is the sub graph of G induced by N[v]. The *neighborhood number* $n_0(G)$ of G is the minimum cardinality of an n-set of G[11].

The *Boolean function graph* $B(\overline{K_p}, NINC, L(G))$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices in $B(\overline{K_p}, NINC, L(G))$ are adjacent if and only if they correspond to two adjacent vertices of G, two adjacent edges of G or to a vertex and an edge not incident to it in G. For brevity, this graph is denoted by $B_2(G)$. In other words, $V(B_2(G)) = V(G) \cup V(L(G))$; and $E(B_2(G)) = [E(T(\overline{G})) - (E(\overline{G}) \cup E(\overline{L(G)}))] \cup E(L(G))$, where $\overline{G}$, L(G) and T(G) denote the complement, the line graph and the total graph of G respectively. The vertices of G and L(G) are referred as point and line vertices respectively.

The mixed relations of incident, non-incident, adjacent and non-adjacent can be used to analyze nature of clustering of elements of communication networks. The concept of domination set can be visualized in each cluster as that cluster representatives and the domination set of whole network can be taken as representatives of entire network. If any clustering or a partition of vertices network such that each cluster having at least one representative or at least one element of dominating set of the given network

In this paper, we obtain the bounds for the global, total global and global point set domination numbers and neighborhood number for this Boolean function graph. The definitions and details not furnished in this paper may found in [2].

2. Prior Results

In this section, we list some results with indicated references, which will be used in the subsequent main results. Let G be any (p, q) graph.

Theorem 2.1[8]: A total dominating set T of G is a total global dominating set (t.g.d. set) if and only if for each vertex $v \in V(G)$ there exists a vertex $u \in T$ such that v is not adjacent to u.

Theorem 2.2[8]: Let G be a graph with $diam(G) \geq 5$. Then $T \subseteq V(G)$ is a total dominating set of G if and only if T is a total global dominating set.

Theorem 2.3[12]: Let G = (V, E) be a connected graph. A set $D \subseteq V(G)$ is a point set dominating set of G if and only if for every independent set W in $V-D$, there exists u in D such that $W \subseteq N(u) \cap (V-D)$ in G.

Theorem 2.4[12]: For a co-connected graph G, a set $D \subseteq V(G)$ is a global point set dominating set if and only if the following conditions are satisfied:

(i). For every independent set W in $V-D$, there exists u in D such that $W \subseteq N(u) \cap (V-D)$ in G; and

(ii). For every set $S \subseteq V-D$ such that <S> is complete in G, there exists v in D such that $S \cap N(v) = \Phi$ in G.

Theorem 2.5[12]: For a co-connected graph of order $p \geq 5$, $3 \leq \gamma_{pg}(G) \leq p-2$.

Proposition 2.6[11]: For a graph G without isolated vertices $\gamma(G) \leq n_0(G) \leq \alpha_0(G)$, where $\alpha_0(G)$ is the point covering number of G.

Observation[7]:

2.7: L(G) is an induced sub graph of $B_2(G)$ and the sub graph of $B_2(G)$ induced by point vertices is totally disconnected.

2.8: The number of vertices in $B_2(G)$ is p + q and if $d_i = deg_G(v_i)$, $v_i \in V(G)$ and the number of edges in $B_2(G)$ is $q(p - 3) + (1/2)\Sigma_{1 \leq i \leq p} d_i^2$.

Theorem 2.9[3]: $B_2(G)$ contains isolated vertices if and only if G is one of the following graphs: nK_1 and $K_{1,m} \cup tK_1$, for $n \geq 1$, $m \geq 1$ and $t \geq 0$.

Theorem 2.10[3]: $B_2(G)$ is disconnected (no component being K_1) if and only if $G \cong 2K_2$.

Theorem 2.11[3]: For any graph G having at least one edge, $\gamma(B_2(G)) \geq 2$.

Theorem 2.12[3]: For any (p, q) graph G, $\alpha_0(B_2(G))$ = q and $\beta_0(B_2(G))$ = p.

3. Main Results

In the following, the bounds for the global domination number γ_g of $B_2(G)$ are obtained. $\overline{B_2(G)}$ denotes the complement of $B_2(G)$.

Theorem 3.1: For any graph G with at least one edge, $\gamma_g(B_2(G)) \leq 3$.

Proof: Let e = (u, v)$\in$E(G), where u, v$\in$V(G) and e' be the line vertex in $B_2(G)$ corresponding to e. Then D = {u, v, e'} $\subseteq$ $V(B_2(G))$ is a dominating set of $B_2(G)$. In $\overline{B_2(G)}$, the vertices u, v dominate all the point vertices and the line vertices in $\overline{B_2(G)}$ corresponding to the edges in G incident with u or v in G. e' dominates the remaining line vertices in $\overline{B_2(G)}$, since $\overline{L(G)}$ is an induced sub graph of $\overline{B_2(G)}$. Hence, D is also a dominating set of $\overline{B_2(G)}$. Thus, $\gamma_g(B_2(G)) \leq 3$.

Remark 3.1: $\gamma_g(B_2(G)) = 2$ if and only if $G \cong 2K_2$. Thus, it follows that $\gamma_g(B_2(G)) = 3$ if and only if $G \neq 2K_2$.

The next theorem relates global domination number of $B_2(G)$ with the point covering number α_0 of G.

Theorem 3.2: $\gamma_g(B_2(G)) \leq \alpha_0(G) + 1$, if there exists a point cover D of G with $|D| = \alpha_0(G)$ such that D is not independent, where $\alpha_0(G)$ is the point covering number of G.

Proof: Let D be a point cover of G with $|D| = \alpha_0(G)$. Since D is not independent, there exists an edge, say e in D. If e' is the line vertex corresponding to the edge e, then D$\cup$\{e'\} is a dominating set of both $B_1(G)$ and its complement $\overline{B_1(G)}$. Hence, $\gamma_g(B_1(G)) \leq \alpha_0(G)+1$.

The following theorem relates global domination number of $B_2(G)$ with domination number γ of G.

Theorem 3.3: For any graph not totally disconnected, $\gamma_g(B_2(G)) \leq \gamma(G) + 2$.

Proof: Let D be a dominating set of G and e = (u, v) be an edge in G with $u \in D$, $v \in V(G)$. If e' is the line vertex in $B_2(G)$ corresponding to the edge e, then $D \cup \{v, e'\} \subseteq V(B_2(G))$ is a global dominating set of $B_2(G)$. Hence, $\gamma_g(B_1(G)) \leq \gamma(G)+2$.

Next, a necessary and sufficient condition that a global dominating set of L(G) is a global dominating set of $B_2(G)$ is given.

Theorem 3.4: Let D be a global dominating set of L(G). Then D is also a global dominating set of $B_2(G)$ if and only if

(i). D is a line cover for G; and

(ii). For every $v_i \in V(G)$, let $A_{vi} = \{e_i \in E(G) : e_i$ is incident with $v_i\}$. Then $A_{vi} \cap D \neq A_{vi}$.

Proof: Let D be a global dominating set of both L(G) and $B_2(G)$. Since D dominates all the point vertices of $\overline{B_2}(G)$, the set of edges in G corresponding to the line vertices in $B_2(G)$ is a line cover for G. Let $v_i \in V(B_2(G))$ be a point vertex. Since D dominates v_i, there exists at least one vertex, say e_i' in D such that the corresponding edge e_i is not incident with v_i in G. This implies that, $A_{vi} \cap D \neq A_{vi}$. Conversely, assume conditions (i) and (ii). Since D is a global dominating set of L(G), it dominates all the line vertices of both $B_2(G)$ and $\overline{B_2}(G)$. By (i) and (ii), D dominates all the point vertices of $B_2(G)$ and $\overline{B_2}(G)$ respectively. Hence, D is a global dominating set of $B_2(G)$.

Similarly, the following theorem can be proved.

Theorem 3.5: Let D be a dominating set of L(G). Then D is a global dominating set of $B_2(G)$ if and only if

(i). D is a line cover for G;

(ii). For every $v_i \in V(G)$, if $A_{vi} = \{e_i \in E(G) : e_i$ is incident with $v_i\}$, then $A_{vi} \cap D \neq A_{vi}$; and

(iii). $N_{L(G)}(e') \cup D \neq N_{L(G)}(e')$, for every $e' \in V(B_2(G)) - D$.

In the following, the total global point set domination number γ_{tg} of $B_2(G)$ is obtained by using Theorem 2.1.

Theorem 3.6: Let G be any graph with at least five vertices and $\beta_1(G) \geq 2$. Then $\gamma_{tg}(B_2(G)) = 4$, if G contains $2K_2 \cup K_1$ as an induced sub graph.

Proof: Assume G contains $2K_2 \cup K_1$ as an induced sub graph. Let e_1 and e_2 be the edges of $2K_2$ and u be a vertex incident with e_1 and v be the vertex of K_1. Let e_1' and e_2' be the line vertices in $B_2(G)$ corresponding to e_1 and e_2 respectively. Then $D=\{u, v, e_1', e_2'\} \subseteq V(B_2(G))$ is a total dominating set of $B_2(G)$. Also any point vertex in $V(B_2(G))–D$ is not adjacent to both u and v. Since $2K_2 \cup K_1$ is an induced sub graph of G, there exists no edge in G joining the vertices of $2K_2$ other than e_1 and e_2. Hence, for any line vertex e' in $V(B_2(G))$–D, there exists a vertex in D not adjacent to e' in $V(B_2(G))–D$. Since $\langle D\rangle \cong P_4$ in $B_2(G)$, D is a total global dominating set of $B_2(G)$. Hence, $\gamma_{tg}(B_2(G)) = 4$.

Example 3.1:

(i). $\gamma_{tg}(B_2(P_n)) = 4$, if $n \geq 4$. (ii). $\gamma_{tg}(B_2(C_n)) = 4$, if $n \geq 4$.

(iii). $\gamma_{tg}(B_2(K_n)) = 4$, if $n \geq 4$.

Next, the global point set domination number γ_{pg} of $B_2(G)$ is determined by using Theorem 2.4. Here, the graphs G for which both $B_2(G)$ and its complement $\overline{B_2(G)}$ are connected are considered.

Theorem 3.7: For any graph G having at least two edges, $3 \leq \gamma_{pg}(B_2(G)) \leq p + q - 2$.
Proof: By Theorem 2.5., the theorem follows.

Next, we characterize the graphs G for which $\gamma_{pg}(B_2(G))$ is 3.

Theorem 3.8: Let G be any graph not totally disconnected. Then $\gamma_{pg}(B_2(G)) = 3$, if radius of L(G) is equal to 1.

Proof: Since point set domination number of $B_2(G)$ is at least three, $\gamma_{pg}(B_2(G)) \geq 3$. Since $r(L(G)) = 1$, there exists an edge $e = (u, v) \in E(G)$ such that all the edges of G are adjacent to e. Let e' be the corresponding line vertex in $B_2(G)$. Then, $D = \{u, v, e'\}$ is a point set dominating (psd) set of $B_2(G)$. It remains to prove that D is a psd-set of $\overline{B_2(G)}$. Let $S \subseteq V(B_2(G))–D$ be such that $\langle S\rangle$ is complete. Since the sub graph of $B_2(G)$ induced by all the point vertices is disconnected, S will contain at most one point vertex. Hence, either $S \cap N_2(v) = \Phi$ or $S \cap N_2(u) = \Phi$, where $N_2(v)$ is the neighborhood of v in $B_2(G)$. Thus, D is a global psd-set of $B_2(G)$ and $\gamma_{ps}(B_2(G)) \leq 3$. Therefore, $\gamma_{pg}(B_2(G)) = 3$.

In the following, we obtain upper bounds for $\gamma_{pg}(B_2(G))$.

Theorem 3.9: Let G be any graph such that $\beta_1(G) \geq 2$. Then $\gamma_{pg}(B_2(G)) \leq 5$, if $\gamma_c(L(G)) = 2$, where $\gamma_c(L(G))$ is the connected domination number of L(G).

Proof: Since $\gamma_c(L(G)) = 2$, there exist two adjacent edges, say $e_1 = (u_1, v_1)$, $e_2 = (u_1, v_2)$ in E(G) such that each edge in G is adjacent to e_1 or e_2. If e_1' and e_2' is the corresponding line vertices in $B_2(G)$, then $D = \{u_1, v_1, v_2, e_1', e_2'\}$ is a global psd-set of $B_2(G)$. Hence, $\gamma_{pg}(B_2(G)) \leq 5$.

Theorem 3.10: $\gamma_{pg}(B_2(G)) \leq q - \Delta_e(G) + 2$, where $\Delta_e(G)$ is the maximum degree of L(G).

Proof: Let $e' \in V(L(G))$ be such that $\deg_{L(G)}(e') = \Delta_e(G) = m$ and $e = (u, v)$ be the corresponding edge in G, where $u, v \in V(G)$. Let e_i', (i = 1, 2,…, m) be the vertices in L(G) adjacent to e'. Then $D' = \{V(G)-\{u, v\}\} \cup \{e_1', e_2',\ldots,e_m\} \subseteq V(B_2(G))$ and $D = V(B_2(G))- D' = \{u, v, e'\} \cup \{V(L(G))-N_{L(G)}[e']\}$ is a global psd-set of $B_2(G)$. Hence, $\gamma_{pg}(B_2(G)) \leq q - \Delta_e(G) + 2$.

Remark 3.2:

(i). If $r(L(G)) = 1$, then $\gamma_{pg}(B_2(G)) = 3 = q - \Delta_e(G) + 2$.

(ii). If $\beta_1(G) = 2$, then the set of all point vertices is a global psd-set of $B_2(G)$ and hence $\gamma_{pg}(B_2(G)) \leq p$.

Example 3.2:

(i). $\gamma_{pg}(B_2(P_n)) = n-1$, if $n \geq 4$. (ii). $\gamma_{pg}(B_2(C_n)) = n$, if $n \geq 3$.

(iii). $\gamma_{pg}(B_2(K_n)) = (n^2-5n+12)/2$, if $n \geq 3$.

In the following, we obtain lower and upper bounds for the neighborhood number n_0 of $B_2(G)$.

Theorem 3.11: If $B_2(G)$ has no isolated vertices, then $2 \leq n_0(B_2(G)) \leq p$.

Proof: By Theorem 2.6., if $B_2(G)$ has no isolated vertices, then $\gamma(B_2(G)) \leq n_0(B_2(G)) \leq \alpha_0(B_2(G))$. But by Theorem 2.11 and Theorem 2.12, $\gamma(B_2(G)) \geq 2$ and $\alpha_0(B_2(G)) = p$ respectively and hence the theorem follows. The lower bound is attained, if $G \cong 2K_2 \cup 2K_1$ and C_4 and the upper bound is attained, if $G \cong C_3$.

Theorem 3.12: If the connected domination number γ_c of L(G) is 2, then $n_0(B_2(G)) \leq 5$.

Proof: Assume $\gamma_c(L(G)) = 2$. Then there exists two adjacent vertices e_1', e_2' in L(G) such that all the other vertices of L(G) are adjacent to at least one of e_1', e_2'. Let $e_1 = (u_1, v_1)$ and

$e_2 = (u_1, v_2)$ be the corresponding adjacent edges in G, where $u_1, v_1, v_2 \in V(G)$. Then $\{u_1, v_1, v_2, e_1', e_2'\} \subseteq V(B_2(G))$ is an neighborhood set (n-set) for $B_2(G)$. Hence, $n_0(B_2(G)) \leq 5$.

Theorem 3.13: If the independent domination number γ_i of L(G) is 2, then $n_0(B_2(G)) \leq 6$.

Proof: Assume $\gamma_i(L(G)) = 2$. Then there exists two independent edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ in G, such that each edge in G is adjacent to at least one of e_1 and e_2, where $u_1, v_1, u_2, v_2 \in V(G)$. If e_1' and e_2' be the line vertices in $B_2(G)$ corresponding to e_1 and e_2, then $\{u_1, v_1, u_2, v_2, e_1', e_2'\}$ is an n-set for $B_2(G)$. Hence, $n_0(B_2(G)) \leq 6$.

Theorem 3.14: If radius of L(G) is 1, then $n_0(B_2(G)) \leq 3$.

Proof: Let $r(L(G)) = 1$. Then there exists an edge $e = (u, v)$ such that all the edges in G are adjacent to e. Let e' be the corresponding line vertex in $B_2(G)$. Then $\{u, v, e'\}$ is an n-set for $B_2(G)$.Thus, $n_0(B_2(G)) \leq 3$. This bound is attained, if $G \cong C_3$.

Example 3.3:

(i). $n_0(B_2(P_n)) = n - 2$, if $n \geq 4$. (ii). $n_0(B_2(C_n)) = \{n/2\}$, if $n \geq 4$.

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