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Dom-Chromatic sets of graphs

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Abstract: *Let G be a simple graph with vertex set V and edge set E. A subset S of V is said to be a dom-chromatic set (or dc-set) if S is a dominating set and $\chi(\langle S \rangle) = \chi(G)$. The minimum cardinality of a dom-chromatic set in a graph G is called the dom-chromatic number (or dc-number) and is denoted by $\gamma_{ch}(G)$. In this paper, bounds for the dom-chromatic numbers are found for standard graphs and some classes of graphs. Further, some Nordhaus Gaddum type of results are obtained.*

Keywords: *Dominating set, domination number, dom-chromatic set, dom-chromatic number, chromatic number, chromatic preserving set, chromatic preserving number.*

1. Introduction

Throughout this paper a graph G = (V, E) always means a finite, simple undirected graph. A set $S \subseteq V$ is a ***dominating set*** of G if each $u \in V - S$, is adjacent to some vertex $v \in S$. The minimum cardinality of a dominating set in G is called the ***domination number*** and is denoted by $\gamma(G)$. We have defined (dc of bipartite and Phd thesis) a new dominating set called dom-chromatic set (shortly dc-set) which preserves the chromatic number of the graph. In other words, a dom-chromatic set S is a dominating set with the property $\chi(\langle S \rangle) = \chi(G)$. The minimum cardinality of a dom-chromatic set is called the d0m-chromatic number(or dc-number) of a graph. To find a dominating set having the same chromatic number as G, it is necessary to find a set of smallest cardinality having the same chromatic number as that of the graph. The ***chromatic preserving set*** (cp-set) introduced by the authors in [8] serves this purpose. Thus, a dom-chromatic set is a dominating cp-set. It is a dominating set with at least one vertex in each color class.

One of the areas of interest for many researchers is an inequality chain of graph parameters. This idea then extended to domination also. In 1978, Cockayne, Hedetniemi and Miller [7, pp 73] obtained an inequality chain called as domination chain. One of the first results in this inequality chain was an example by Slater [7, pp 74]. After this development, a focus was made on the conditions under which any two domination parameters are equal. Allan and Laskar ([1], [7]: pp 78) did a study on the conditions under which domination number and independent domination numbers are equal and

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Favoron ([4], [7]: pp 78]) discussed the equality conditions of the parameters, independence domination number, domination number and irredundance number in a graph. Brigham et.al [2] explored the graphs having equal domination and codomination numbers. In this paper, apart from obtaining upper bounds for dom-chromatic number for graphs, we have obtained inequality chain on the dom-chromatic number under stated conditions together with graphs having two equal domination parameters in the chain.

The definitions that are needed for the understanding of this paper are given in the next section.

2. Definitions and terminology

A graph is a ***(p, q)-graph*** if it has p vertices and q edges. The length of the smallest odd cycle is called ***girth*** and is denoted by $g_o(G)$. A ***clique*** of a graph G is a maximal complete sub graph. The cardinality of a maximum clique is called the ***clique number*** and is denoted by $\omega(G)$. An ***open neighborhood*** of a vertex u is the set of all vertices adjacent to u, and is denoted $N(u)$. A ***closed neighborhood*** of a vertex u is the set $N(u) \cup \{u\}$ and is denoted by $N[u]$. The length of the smallest odd cycle is called ***girth*** and is denoted by $g_o(G)$. A set $S \subseteq V$ is called an independent set of vertices of no two vertices of S are adjacent. The maximum cardinality among such independent sets is called the ***independence number*** of G and is and denoted by $\beta_o(G)$. A matching in a graph is a set of independent edges and a perfect matching is a set of independent edges such that each vertex is an end vertex of some edge.

The ***chromatic number*** $\chi(G)$ is the minimum k such that G is k-colorable. If $\chi(G) = k$, then G is said to be ***k-chromatic***. If $\chi(G) = k$, but $\chi(G) < k$ for every proper sub graph H of G, then G is said to be a ***k-color-critical graph*** or ***k-critical graph***. A graph G is said to be a ***vertex-color-critical graph*** if $\chi(G - u) < \chi(G)$ for every $u \in V$ and called ***edge-critical*** if $\chi(G - e) < \chi(G)$ for every $e \in E$. In general, any element t of the set $V(G) \cup E(G)$ is critical if $\chi(G - t) < \chi(G)$. A graph is called color-critical graph if which each vertex and edge are critical. It is to be noted that no k-critical graph can be infinite and the only k-critical graphs for k = 1, 2 and 3 are K_1, K_2 and odd cycles, respectively. For $k \geq 4$, the k-critical graphs have not been characterized. A set $S \subseteq V$ is said to be a ***chromatic preserving set*** or a ***cp-set*** if $\chi(\langle S \rangle) = \chi(G)$ and the minimum cardinality of a cp-set in G is called the ***chromatic preserving number*** or ***cp-number*** of G and is denoted by cpn(G). A cp-set of cardinality cpn(G) called cpn-set.

A set $S \subseteq V$ is a ***dominating set*** of G if for each $u \in V - S$, there exists a vertex $v \in S$ such that u is adjacent to v. The cardinality of a minimum dominating set in G is called the ***domination number*** and is denoted by $\gamma(G)$. A dominating set $S \subseteq V$ of G is a ***total dominating set*** if <S> has no isolated vertices. A set $S \subseteq V$ of G is a ***global dominating set*** if S is a dominating set of both G and $\overline{G}$ .The minimum cardinality of a global dominating set in G is called the ***global domination number*** and is denoted by $\gamma_g(G)$. A ***domatic partition*** of a graph G is a partition of V into dominating sets. The maximum cardinality of a partition V into dominating sets is called the ***domatic number*** and is denoted by d(G).

The intersection graph of F denoted by $\Omega(F)$, is a graph with vertex set F and two members of F namely S_i and S_j adjacent whenever $S_i \cap S_j \neq \phi$, $i \neq j$. A graph G is said to be an ***intersection graph*** if there exists a family F of subsets of a set S for which $G = \Omega(F)$. The ***independence graph*** $I(G)$ of a graph G is defined to be the intersection graph on the set of all vertex independent sets(not necessarily maximal) in G. In I(G), singleton, doubleton sets in G denote the independent sets of vertices of cardinality one and two respectively. The ***splitting graph*** $S(G)$ of a graph G is obtained by taking a new vertex u' for each $u \in V$ and joining u' to each vertex of N(u). The ***end line graph*** G^+ of a (p, q)-graph G with vertex set $\{u_1, u_2, \ldots, u_p\}$ is defined as follows. Add to G, p vertices v_i and p edges u_iv_i, $i = 1, 2, \ldots, p$. The resultant graph is a (2p, p + q)-graph with $V(G^+) = \{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p\}$ and the edge set $E(G^+) = E(G) \cup \{u_1v_1, u_2v_2, \ldots, u_pv_p\}$. The ***subdivision graph*** $s(G)$ of G is the graph obtained from G by placing a vertex on each edge of G. The ***line graph*** $L(G)$ of a graph G is the intersection graph $\Omega(E(G))$. Thus, the vertices of L(G) are adjacent whenever the corresponding edges of G are incident. A graph G is called a ***split graph*** if the vertex set can be partitioned into two sets V_1 and V_2 such that V_1 induces a null graph and V_2 induces a complete graph.
The notations defined above and which are not defined but used in the subsequent sections could be referred in Harary [5], Haynes [7], and Jensen [10]. Unless otherwise mentioned, in this paper, graph G is a (p, q)-graph.

3. Prior Results

Theorem 3.1 [7, pp 41]. If a graph G has no isolated vertices, then $\gamma(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$.

Theorem 3.2 [7, pp 50]. For any graph G, $\left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor \leq \gamma(G) \leq p - \Delta(G)$ where $\Delta(G)$ denotes the maximum degree of G.

Proposition 3.3. If G is a connected split graph and $\Delta(G) < p - 1$, then $\overline{G}$ is a connected split graph.

Properties 3.4 [10].

i. In the independence graph I(G) of a graph G, no two singleton sets of G intersect.
ii. I(G) is bipartite if and only if $\beta_o(G) \le 2$ and no two doubleton sets of G intersect.

Observation 3.5. For any graph G,

i. s(G) is a bipartite graph.
ii. If G is a (p, q)- graph, then s(G) is a (p + q, 2q)- graph.

Proposition 3.6. If G is a split graph, then $cpn(G) = \omega(G) = \chi(G)$.
Proof: If G is complete, then the result follows trivially. Suppose G is not complete. As G is a split graph, its vertex set can be partitioned into two sets X and Y such that <X> is complete and <Y> is a null graph. Without loss of generality X can be assumed to be maximal, for otherwise there exists some vertex u in Y adjacent to all vertices of X and then $X \cup \{u\}$ induces a complete graph. Since <X> is a maximal clique, each $y \in Y$ is not adjacent to at least one vertex $x \in X$ and hence, may be given the color of x. Hence, $\chi(G) = \chi(\langle X \rangle) = |X|$. Therefore, $cpn(G) = \left| X \right| = \omega(G)$ and hence, $cpn(G) = \chi(G) = \omega(G)$.

Properties 3.7. If G is a split graph, then G is a perfect graph.
Proof: Every sub graph H of G is a split graph and hence, $\chi(H) = \omega(H)$.

Properties 3.8 [11, pp 291]. A graph G is perfect if and only if its complement $\overline{G}$ is perfect.

Theorem 3.9 [9]. If G is a bipartite graph with no isolated vertices, then $\gamma_{ch}(G) \le \frac{p}{2} + 1$ and $\gamma_{ch}(G) = \frac{p}{2} + 1$ if and only if $G = \frac{p}{2} K_2$.

4. Dom-chromatic sets in graphs

Dom-chromatic sets of bipartite graphs were discussed and bounds were obtained in [9]. In this paper, we obtain the bounds for dom-chromatic number of any graph.
The dom-chromatic set mentioned in the introduction is formally defined below.

4.1. Dom-chromatic sets in graphs

Definition 4.1.1. Let G = (V, E) be a graph. A subset S of V is said to be a ***dom-chromatic set*** (or ***dc-set***) if S is a dominating set and $\chi(<S>) = \chi(G)$. The minimum cardinality of a dom-chromatic set in a graph G is called the ***dom-chromatic number*** (or ***dc- number***) and is denoted by $\gamma_{ch}(G)$.

Observation 4.1.2.

For any graph G, cpn(G) ≤ $\gamma_{ch}(G)$.

i. Dom-chromatic set exists for all graphs.
ii. Vertex set V is a trivial dom-chromatic set.
iii. For a vertex-color-critical graph, V is the only dom-chromatic set.
iv. If S is a γ_{ch}-set of G, then each vertex of V – S is not adjacent to at least one vertex of S.
v. A dc-set of a graph is a global dominating set.
vi. $\gamma(\overline{G}) \leq \gamma_g(G) \leq \gamma_{ch}(G)$.

Proof:

(i) to (v) follow trivially.

(vi) Let S be a dc-set of a graph G. From (v), in $\overline{G}$ each vertex of V – S is adjacent to at least one vertex of S. Hence, S is a dominating set of $\overline{G}$, and the result follows.

(vii) Follows from (vi).

Proposition 4.1.3. A dom-chromatic set S is minimal if and only if for each u ∈ S, at least one of the following conditions hold.

i. $\chi(<S - u>) < \chi(G)$.
ii. S - u is not a dominating set.

Dom-chromatic number for some standard graphs are given in the following proposition.

Proposition 4.1.4.

i. $\gamma_{ch}(K_n) = n$;
ii. $\gamma_{ch}(nK_1) = n$;
iii. $\gamma_{ch}(K_{m,n}) = 2$;
iv. $\gamma_{ch}(P_n) = \begin{cases} (n+3)/3, & \text{if } n \equiv 0 \pmod 3 \\ (n+2)/3, & \text{if } n \equiv 1 \pmod 3 \\ (n+4)/3, & \text{if } n \equiv 2 \pmod 3 \end{cases}$
v. a. If n is odd, then $\gamma_{ch}(C_n) = n$.

b. If n is even, then

$$\gamma_{ch}(C_n) = \begin{cases} (n+3)/3, & \text{if } n \equiv 0 \pmod 3 \\ (n+2)/3, & \text{if } n \equiv 1 \pmod 3 \\ (n+4)/3, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

vi. $$\gamma_{ch}(W_n) = \begin{cases} 3, & \text{if n is odd} \\ n, & \text{if n is even.} \end{cases}$$

Proposition 4.1.5. If G is a disconnected graph with k components $G_1, G_2, \ldots, G_k$, then $\gamma_{ch}(G) = \gamma_{ch}(G_m) + \sum_{\substack{i=1 \\ i \neq m}}^{k} \gamma(G_i)$, where $\gamma_{ch}(G_m) = \min_{1 \le i \le k} \{\gamma_{ch}(G_i) \mid \chi(G_i) = \chi(G)\}$, for some $m \in \{1, 2, \ldots, k\}$.

Proposition 4.1.6.

i. If G is a connected graph, then $\gamma_{ch}(G) = p$ if and only if G is a vertex-color- critical or G is color-critical graph.

ii. If G is a disconnected graph, then $\gamma_{ch}(G) = p$ if and only if either G is a null graph or has exactly one non trivial component, which is vertex-color-critical or color-critical.

Proposition 4.1.7. If G is any connected graph, then $\gamma_{ch}(G) = p - q$ if and only if $G = K_1$.

Proof: Necessary condition is trivial. Suppose $\gamma_{ch}(G) = p - q$. Since $\gamma_{ch}(G) \ge 1$, $p - q \ge 1$. As G is connected, $q \ge p - 1$. Thus, $p - q = 1$ and hence, $G = K_1$.

Proposition 4.1.8. Let D be any dc-set of G. Then $|V - D| \le \sum_{u \in D} \deg(u)$.

Proof: As D is a dominating set, the result follows trivially.

Proposition 4.1.9. Let D be any dc-set of G. Then $|V - D| = \sum_{u \in D} \deg(u)$ if and only if $G = pK_1$, $p \ge 1$.

Proof: If $G = pK_1$, then $D = V$ and $\deg(u) = 0$ for each $u \in D$. Then the equality holds. Now suppose $|V - D| = \sum_{u \in D} \deg(u) = k$.

<u>Claim:</u> $k = 0$.

Suppose $k \ge 1$. Two cases arise.

<u>Case i:</u> G is connected.

Then $\chi(G) \geq 2$. Let $V - D = \{u_1, u_2 \ldots u_k\}$. Since D is a dominating set, each u_i is adjacent to a vertex of D and hence contributes at least one degree to D. Since $\chi(\langle D \rangle) \geq 2$, D contains at least one edge which contributes 2 degrees to D. Hence, $\sum_{u \in D} \deg(u) \geq k + 2$, a contradiction.

Case ii: G is disconnected.

If G is totally disconnected, V = D and hence, $|V - D| = k = 0$, a contradiction. Hence, G has a non trivial component and then $\langle D \rangle$ contains at least one edge. Then, by a similar argument as in case i, contradiction arises.

In both cases, we arrive at a contradiction. Thus, k = 0. Then, $|V - D| = \sum_{u \in D} \deg(u) = 0$.

Therefore, V = D and hence, for each $u \in V$, deg(u) = 0. Thus, G is a totally disconnected graph and hence $G = kK_1$.

Corollary 4.1.10. For any non trivial connected graph with a dc-set D,

$$\sum_{u \in D} \deg(u) \geq |V - D| + 2.$$

Proof: If G is vertex-color-critical, then V = D and $\sum_{u \in D} \deg(u) = 2q \geq 2 = |V - D| + 2$.

Suppose G is not vertex-color-critical. As G is non trivial, $\chi(G) \geq 2$. By similar argument as in Case (i) of Proposition 4.1.9, $\sum_{u \in D} \deg(u) \geq |V - D| + 2$.

Proposition 4.1.11. For any graph G, $\left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor \leq \gamma_{ch}(G)$ and equality holds if and only if $G = pK_1$, $p \geq 1$.

Proof: From Theorem 3.2, lower bound is trivial. If $G = pK_1$, then the result follows. Suppose $\left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor = \gamma_{ch}(G) = k$ and D is a γ_{ch}-set of G.

Case i: G is connected.

If $k \geq 2$, then G is a non trivial connected graph. Then by Corollary 4.1.11, $|V - D| < \sum_{u \in D} \deg(u)$. Thus, $p - k < \sum_{u \in D} \deg(u) \leq k\Delta(G)$ and hence, $\frac{p}{\Delta(G)+1} < k$.

Hence, $k > \frac{p}{\Delta(G)+1} \geq \left\lfloor \frac{p}{\Delta(G)+1} \right\rfloor = k$, a contradiction. Thus, k = 1 and hence, $\gamma_{ch}(G)=1$.

Therefore, $G = K_1$.

Case ii: G is disconnected.

Suppose G is not totally disconnected. Then G has at east one non trivial component. By similar argument as in Case i, contradiction arises. Therefore, $G = pK_1$.

Theorem 4.1.12. If G is a connected split graph and $\Delta(G) < p - 1$, then

i. $\gamma_{ch}(G) = \omega(G) = cpn(G)$.

ii. $\gamma_{ch}(\overline{G}) = \omega(\overline{G}) = cpn(\overline{G})$.

Proof:

i) As G is a split graph, its vertex set can be partitioned into two sets X and Y such that <X> is complete and <Y> is a null graph. It may be assumed that X is maximal. As discussed in proposition 3.6, each $y \in Y$ is not adjacent to at least one vertex of X and $\chi(G) = \chi(<X>)$ and $cpn(G) = |X| = \omega(G)$. As G is connected, each $y \in Y$ is adjacent to at least one vertex of X. Therefore, X dominates Y and hence, a γ_{ch}-set of G.

ii) From Proposition 3.3, $\overline{G}$ is a connected split graph and hence, the result follows.

Proposition 4.1.13. If G is a perfect graph, then

i. $\gamma_{ch}(G) \leq \gamma(G) + \omega(G)$.

ii. $\gamma_{ch}(\overline{G}) \leq \beta_o(G) + \omega(G) - 1$.

Proof:

i) Let S be a maximal clique in G and D a γ-set in G. Since G is perfect, $\omega(G)= \chi(G)$, and hence, $\chi(<S>) = \chi(G)$. Therefore, $\chi(<S \cup D>) = \chi(G)$. Thus, $S \cup D$ is a dc-set of G and the result follows.

cs) Since G is perfect, $\chi(G) = \omega(G)$. Let $\omega(G) = k$. Let I be a maximum independent set in G. Then I is a maximum clique in $\overline{G}$. Hence, $\beta_o(G) = | I | = \omega(\overline{G})$. Further in $\overline{G}$ <I> is a complete graph on $\omega(\overline{G})$ vertices and therefore, $\chi(<I>) = \chi(\overline{G})$. Let $\{V_1, V_2,\dots,V_k\}$ be a χ-partition of V(G) and S be a maximum clique in G. Then $|S| = \omega(G) = k$.

Let $S = \{v_1, v_2,\dots, v_k\}$. Hence, each v_i is in distinct V_j's. Without loss of generality, let $v_i \in V_i$, $1 \leq i \leq k$. In $\overline{G}$, each V_i induces a complete graph. Hence, in $\overline{G}$ each v_i dominates V_i and hence, S is a dominating set of $\overline{G}$. Let $x \in I$. Then $x \in V_i$ for some i. Then x dominates V_i. Therefore, $(S - v_i) \cup \{x\}$ is a dominating set of $\overline{G}$. Then $I \cup (S - v_i)$ is a dc-set of $\overline{G}$. Therefore, $\gamma_{ch}(\overline{G}) \leq |I \cup (S - u)|$ and hence, $\gamma_{ch}(\overline{G}) \leq |I| + |S| - 1 = \beta_o(G) + \omega(G) - 1$.

Proposition 4.1.14. If G is a graph with diam(G) = 2, $\chi(G) = 3$ and $\gamma(G) = 1$, then $\gamma_{ch}(G) = 3$.

Proof: Let {u} be a γ-set of G. Since $\chi(G) = 3$, there exists a pair of adjacent vertices $x, y \neq u$. Then $<\{x, y, u\}> = K_3$ and then $\gamma_{ch}(G) = 3$.

Proposition 4.1.15. If G is a graph with diam(G) = 2, $\chi(G) = 3$ and $\gamma(G) = 2$, then $3 \le \gamma_{ch}(G) \le 5$.

Proof: Lower bound is trivial. Let S = {a, b} be a γ-set of G. Since diam(G) = 2, $g_o(G) = 3$ or 5.

Case i: $g_o(G) = 3$.

Let C be a 3-cycle xyzx. If a, b $\notin$ C, then 2 vertices of C are adjacent to a and one vertex is adjacent to b or vice versa, otherwise K_4 is induced, a contradiction. Let x and y be adjacent to a, and z be adjacent to b. Then axya is a 3-cycle. Hence, {a, x, y, b} is a dc-set of G. If a or b is in the 3-cycle, then the 3-cycle together with the remaining vertex of S is a dc-set of G.

Case ii: $g_o(G) = 5$.

Let C be a 5-cycle uvwxyu. If a, b $\notin$ C, then as S is dominating, vertices of C are adjacent to a or b and not to both, otherwise a 3-cycle is induced. Also no two consecutive vertices of C can be both adjacent to a or b, otherwise a 3-cycle is induced. Then S can dominate at most 4 vertices of C, a contradiction. Hence, a or b $\in$ C. Let a $\in$ C and b $\notin$ C. Let u = a. Then x and w are adjacent to b and hence, a 3-cycle is induced, a contradiction. Therefore, both a, b $\in$ C and hence, C is a γ_{ch}-set of G.

From Case (i) and (ii), the upper bound is proved.

Proposition 4.1.16. If G is a vertex-color-critical graph, then $\alpha_o(G) < \gamma_{ch}(G)$.

Proof : Let u be any vertex of G. The result follows from the fact that V- u is a vertex cover of G.

Proposition 4.1.17. If G is vertex-color-critical graph with diam(G) ≥ 2, then $\alpha_o(G) + 2 \le \gamma_{ch}(G)$.

Proof : Let u and v be two non adjacent vertices of G. Then V- {u, v} is a vertex cover of G.

Proposition 4.1.18. If G is triangle free with $\chi(G) \ge 3$, then $\gamma_{ch}(G) \ge 5$.

Proof : Since $\chi(G) \ge 3$, any dc-set of G contains an odd cycle. Since G is triangle free, $\gamma_{ch}(G) \ge 5$.

Proposition 4.1.19. If H is a spanning sub graph of G such that $\chi(H) = \chi(G)$, then $\gamma_{ch}(G) \le \gamma_{ch}(H)$.

4.2. Dom-chromatic sets of some classes of graphs

Theorem 4.2.1. If G is graph with $\beta_o(G) = 2$, then the bounds for independence graph I(G) is given by $\frac{p}{2} + 1 \leq \gamma_{ch}(I(G)) \leq p$ and lower bound is attained if and only if G is a complete $\frac{p}{2}$ - partite graph $K_{2,2,\ldots,2}$ and upper bound is attained if and only if G $=K_p - e$.

Proof: Let k be the number of doubleton sets in G.

Claim : $1 \leq k \leq \frac{p}{2}$.

By property 3.3(ii), G can have at most $\left\lfloor \frac{p}{2} \right\rfloor$ doubleton sets. Since $\beta_o(G) = 2$, G has at least one doubleton set, and hence, both inequalities hold.

From property 3.4, each doubleton set intersects exactly 2 singleton sets, and no two doubleton set intersect and there are p singleton sets, $I(G) = kK_{1,2} \cup (p - 2k) K_1$. Therefore,

$$\gamma_{ch}(I(G)) = p - k + 1 \quad \text{--------------} \ (1)$$

Thus, from claim, $\frac{p}{2} + 1 \leq \gamma_{ch}(I(G)) \leq p$.

Suppose G is a complete $\frac{p}{2}$-partite graph $K_{2,2,\ldots,2}$. Then G has $\frac{p}{2}$ doubleton sets and p singleton sets. Also each vertex of G is in a doubleton set of G. Hence, $I(G) = \frac{p}{2} K_{1,2}$ and then $\gamma_{ch}(I(G)) = \frac{p}{2} + 1$. Thus, the lower bound is obtained.

Conversely, suppose $\gamma_{ch}(I(G)) = \frac{p}{2} + 1$. Hence, G is of even order and from (1), $k = \frac{p}{2}$. Thus, the number of doubleton sets in G is $\frac{p}{2}$. Further no two doubleton sets intersect, and each vertex of G is not adjacent to exactly one vertex of G. Hence, G is a complete $\frac{p}{2}$-partite graph $K_{2,\,2\,\ldots\,2}$. Now let $G = K_p - e$. Then G has exactly one doubleton set. Then $I(G) = K_{1,2} \cup (p - 2)K_1$ and hence, $\gamma_{ch}(I(G)) = p$.

Conversely, suppose $\gamma_{ch}(I(G)) = p$. From (1), $k = 1$. Hence, G has exactly one doubleton set. Then exactly one pair of vertices are non adjacent and every other pair of vertices are adjacent. Hence, $G = K_p - e$.

In all results pertaining to the splitting graph S(G) of a graph G, u' denotes the new vertex in S(G) corresponding to $u \in V$ and V' denotes the set of all such new vertices u'.

Observation 4.2.2. For a graph G,

i. $\chi(G) = \chi(S(G))$.
ii. Any total dominating set of G dominates S(G).
iii. A dominating set of G with isolated vertices cannot dominate S(G).
iv. A total dc-set of G is a dc-set of S(G).
v. A dc-set of G with isolated vertices is a not a dc-set of S(G).
vi. If S is a dc-set of S(G), then $S \not\subset V'$.

Proof: Without loss of generality let G be connected.

i. Trivial.

ii. Let S be total dominating set of G. Suppose S does not dominate S(G) and let u' be not dominated by S. Since S is total, each $v \in V$ is dominated by a vertex in S. Hence, u is dominated by some $v \in S$. Then v dominates u', a contradiction.

iii. Let S be a dominating set of G with isolated vertices and u be an isolated vertex in S. Since u' is adjacent only to vertices of N(u), S cannot dominate u'.

iv. Follows from (i) and (ii).

v. Follows from (iii).

vi. Follows from the fact that V' induces a null graph in S(G).

Theorem 4.2.3. For any graph G, $\gamma_{ch}(G) \leq \gamma_{ch}(S(G) \leq p$. Further lower equality holds if and only if there is a γ_{ch}-set of G with no isolates and upper equality holds if and only if G a is vertex-color-critical graph.

Proof: Without loss of generality let G be connected and S' be a γ_{ch}-set of S(G). Let $S = (S' \cap V(G)) \cup \{u \in V | u' \in S'\}$. Clearly, $|S| \leq |S'|$ and S is a dc-set of G. Hence, the lower inequality is obtained. V is a dc-set of S(G) and hence, the upper inequality is obtained. Suppose S is γ_{ch}-set of G with no isolated vertex. By (iv) of Observation 4.2.2, S is a dc-set of S(G). Hence, $\gamma_{ch}(S(G)) \leq \gamma_{ch}(G)$ and the equality holds. Conversely, suppose $\gamma_{ch}(S(G)) = \gamma_{ch}(G)$ and each γ_{ch}-set of G has isolated vertices. Let S a γ_{ch}-set of S(G).

Claim 1: $S \not\subset V$

Suppose $S \subseteq V$. Then S is a γ_{ch}-set of G. Hence, from (v) of Observation 4.2.2, S cannot have isolated vertices, a contradiction.

Let $D = S \cap V$ and $D' = S \cap V'$. Clearly, $D \cap D' = \phi$.

Claim 2: If $x \in D$ is an isolated vertex in S, then $x' \in D'$ and is an isolated vertex in S and vice versa.

Suppose x is an isolated vertex is S. Then $N(x) \subseteq V - S$ and henc,e $x' \in S$. Consequently $x' \in D'$ is an isolated vertex in D'.

Let M be the set of isolated vertices of S in D'.

Case i: $M = \phi$.

Let $u' \in D'$. Then there exists $x \in N(u) \cap S$. Clearly, $u \notin S$. For, if $u \in S$, then $S - u'$ is a dc-set of S(G), a contradiction. Now $S' = (S - u') \cup \{u\}$ is a γ_{ch}-set of S(G) and u is not an isolated vertex in S'. Repeat this for each $u' \in D'$. Then the resultant set S_o is a γ_{ch}-set of S(G) whose vertices are from V and without isolated vertices. Hence, S_o is a γ_{ch}-set of G without isolated vertices, a contradiction.

Case ii: $M \neq \phi$.

Let $x' \in M$. Then $x' \in D'$. By claim 2, $x \in D$ and x is an isolated vertex in S. Let $y \in N(x)$. Now $S' = (S - x') \cup \{y\}$ is a γ_{ch}-set of S(G) and x is not an isolated vertex in S'. As in case i, repeat the above for each $x' \in M$ and resultant set is a γ_{ch}-set of S(G) whose vertices are from V and without isolated vertices, a contradiction.

Thus, there exists a γ_{ch}-set of G with no isolates.

Now let G be a vertex-color-critical graph. Then $\gamma_{ch}(G) = p$ and hence, $\gamma_{ch}(S(G)) = p$. Suppose $\gamma_{ch}(S(G)) = p$ and G is not vertex-color-critical. Then $\gamma_{ch}(G) \leq p - 1$. Let S be a dc-set of a G of cardinality p - 1. Let $V - S = \{u\}$. Clearly, S cannot be a dc-set of S(G). Since $\chi(<S>) = \chi(S(G))$, domination property is not satisfied. Also S dominates V, there exists $x' \in V'$ not dominated by S.

Claim 3: $x' \neq u'$.

Suppose $x' = u'$. Now u is adjacent to some $y \in S$. Then y dominates x', a contradiction.

Hence, $x \neq u$ and $x \in S$. If x is adjacent to some $y \in S$, then again x' is dominated by S, a contradiction. Hence, x is a pendant vertex in S. Since G is connected, x is adjacent to u. If p = 2, then $G = K_2$, a vertex-color-critical graph, a contradiction. Thus, $p \geq 3$ and hence, u is adjacent to some $y \in S$. Let $S' = V - x$. Then $\chi(<S'>) = \chi(G)$. Clearly, S' dominates V.

Claim 4: S' dominates V'.

Let $v' \in V'$. Three cases arise.

Case i: $v = u$.

u adjacent to $y \in S'$ and y dominates v'.

Case ii: $v = x$.

Since x adjacent to u, u dominates v'.

<u>Case iii:</u> $v \neq u, x$.

Since $v \neq x$, $v \in S$. As G is connected and x is adjacent to u only, v adjacent to some $z \in S$ (z and u need not be distinct). Thus, z dominates v' and hence, S' is a dc-set of S(G) of cardinality p - 1, a contradiction. Therefore, G is a vertex-color-critical graph.

Proposition 4.2.4. If G is a graph with $p \geq 2$ vertices, then $\gamma_{ch}(G^+) = p$.

Proof: Since each vertex of G is adjacent to a pendant vertex in G^+, V is a γ-set of G^+. Since $\chi(G) = \chi(G^+)$, V is a dc-set of G^+. Hence, $\gamma_{ch}(G^+) = p$.

Proposition 4.2.5. If G is a (p, q)- graph, then $\gamma_{ch}(s(G)) \leq \frac{p+q}{2}$.

Proof: Since s(G) is a bipartite (p + q, 2q)-graph, $\gamma_{ch}(s(G)) \leq \frac{p+q}{2} + 1$. From Theorem 3.9, equality holds if and only if $s(G) = mK_2$, and for no graph G, s(G) can be mK_2. Hence, $\gamma_{ch}(s(G)) \leq \frac{p+q}{2}$.

Theorem 4.2.6. If G has a perfect matching and $\chi'(G) = \Delta(G)$, then

i. $\gamma_{ch}(L(G)) \leq \Delta + \frac{p}{2} - 1$.

ii. $\gamma_{ch}(L(G)) \leq \Delta + \gamma(L(G) - |N[M]|)$ where M is a Δ-clique in L(G).

Proof :

i) Let $V = \{ x_1, x_2, \ldots x_{p/2}, y_1, y_2 .. y_{p/2} \}$ and M be a perfect matching given by $M = \{ x_1y_1, x_2y_2 \ldots, x_{p/2}y_{p/2} \}$. Since $\chi'(<G>) = \Delta(G)$,

$$\chi'(G) = \Delta(G) = \chi(L(G)) \qquad \text{---------} \quad (1)$$

Let $u \in V$ such that $\deg(u) = \Delta(G)$ and $N(u) = \{u_1, u_2, \ldots, u_\Delta\}$. Then $uu_i \in V(L(G))$ for each i and the set $S = \{uu_1, uu_2, \ldots, uu_\Delta\}$ induces a maximal clique isomorphic to $K_{\Delta(G)}$ in L(G). Now $S \cup M$ is a vertex subset of V(L(G)) and S induces $K_{\Delta(G)}$ in L(G) together imply that $\chi(<S \cup M>) \geq \chi(<S>)$. Therefore, from (1), $\chi(<S \cup M>) = \chi(L(G))$. Let $e \in E - (S \cup M)$. Then $e = x_ix_j$, x_iy_j or y_iy_j, $i \neq j$ and hence, in L(G), the vertex corresponding to e adjacent to at least two vertices of $S \cup M$. Thus, $S \cup M$ is a dominating set of L(G).Thus, $S \cup M$ is a dc-set of G. Since M is a perfect matching, $uu_i \in M$ for some i and hence, $|M \cap S| = 1$. Therefore, $\gamma_{ch}(L(G)) \leq \frac{p}{2} + \Delta(G) - 1$.

ii) Since line graph of a graph is perfect, L(G) is a perfect graph.

4.3 Dom-chromatic partition of graphs

Imitating the definitions of domatic partition and domatic partition number of a graph dom-chromatic partition and subsequently the partition number are defined below.

Definition 4.3.1. A ***dom-chromatic- partition*** (or ***dc-partition***)of graph G is a partition of V into dom-chromatic sets. The maximum cardinality of a partition of V into dom-chromatic sets is the ***dom-chromatic-partition number*** (***dc- partition number***) and is dented by $d_{ch}(G)$.

Observation 4.3.2.

i. If a graph G has pendant vertices, then $d_{ch}(G) \leq 2$.
ii. For any graph G, $d_{ch}(G) \leq d(G)$.
iii. If G is a bipartite graph with no isolates, then $d_t(G) \leq d_{ch}(G)$.

Proposition 4.3.3. For any graph G, $d_{ch}(G) \leq \left\lfloor \frac{p}{\gamma_{ch}(G)} \right\rfloor$.

Proposition 4.3.4. For any graph G, $\gamma_{ch}(G)d_{ch}(G) \leq p$.

Proof: Let $d_{ch}(G) = k$ and $\{V_1, V_2, \ldots, V_k\}$ be a dc-partition of G. Then $\sum_{i=1}^{k} |V_i| = p$ and for each i, $|V_i| \geq \gamma_{ch}(G)$. Therefore, $p \geq k\gamma_{ch}(G) = d_{ch}(G)\gamma_{ch}(G)$.

Dc-partition numbers of some standard graphs are given below.

Proposition 4.3.5.

i. $d_{ch}(K_{m,n}) = \min\{m, n\}$.
ii. $d_{ch}(W_n) = 1$.
iii. $d_{ch}(P_n) \leq 2$.
iv. $d_{ch}(C_n) \leq 2$.

4.4. Nordhaus Gaddum type of results

It is customary to obtain Nordhaus-Gaddum type of results for any graph parameter and we have obtained few results.

Proposition 4.4.1. For a graph G, $\gamma_{ch}(G) + \gamma_{ch}(\overline{G}) = 2p$ if and only if f G and $\overline{G}$ are one of the following types

i) G and $\overline{G}$ are vertex-color-critical.

ii) G and $\overline{G}$ are K_p and pK_1.

iii) G is vertex-color-critical and G has exactly one non trivial component which is vertex-color-critical.

Proposition 4.4.2. If G is a connected split graph & $\Delta(G) < p - 1$, then $\gamma_{ch}(G) + \gamma_{ch}(\overline{G}) = p$.

Proof : By similar discussion as in Theorem 4.1.12, if X is a maximal clique of G, then X is a γ_{ch}-set of G and V - X is a γ_{ch}-set of G and V - X is a γ_{ch}-set of $\overline{G}$ and the equality follows.

Theorem 4.4.3. If G is a tree of diameter 3, then $\gamma_{ch}(G) + \gamma_{ch}(\overline{G}) = p$.

Proof :

i) Since G is a tree of diameter 3, G has a dominating edge. Therefore, $\gamma_{ch}(G) = 2$. Let uv be the dominating edge of G and V_1, V_2 be the set of pendant vertices adjacent to u and v respectively. Then in $\overline{G}$, $<V_1 \cup V_2>$ induces a complete graph K_{p-2} and, u and v are non adjacent. Further u is adjacent to each vertex of V_1 and v is adjacent to each vertex of V_2 in $\overline{G}$. Thus, $V_1 \cup V_2$ is a γ_{ch}-set of $\overline{G}$. Therefore, $\gamma_{ch}(\overline{G}) = p - 2$ and (i) follows.

Proposition 4.4.4. If G is a perfect graph, then

$\gamma_{ch}(G) + \gamma_{ch}(\overline{G}) \leq \gamma(G) + \beta_o(G) - 2\omega(G) - 1$.

Proof: From Proposition 4.1.12, $\gamma_{ch}(G) \leq \gamma(G) + \omega(G)$ and $\gamma_{ch}(\overline{G}) \leq \beta_o(G) + \omega(G) - 1$ and hence, the upper bound is obtained.

Theorem 4.4.5. If G is an incomplete bipartite graph with no isolated vertices, then

$\gamma_{ch}(G) + \gamma_{ch}(\overline{G}) \leq 3\frac{p}{2} + 1$.

Proof : Let $\{V_1, V_2\}$ be a vertex partition of V. Clearly, $|V_1|$, $|V_2| > 1$. From Theorem 3.9, $\gamma_{ch}(G) \leq \frac{p}{2} + 1$. Two cases arise.

<u>**Case i :**</u> $\overline{G}$ is vertex-color-critical.

Then $\gamma_{ch}(\overline{G}) = p$ and then $\gamma_{ch}(G) + \gamma_{ch}(\overline{G}) \leq 3\frac{p}{2} + 1$.

<u>**Case ii**</u> **:** $\overline{G}$ is not vertex-color-critical.

If $\overline{G}$ is totally disconnected, then $G = K_p$. Since G is bipartite, $G = K_2$, a contradiction as G is incomplete. Hence, $\gamma_{ch}(\overline{G}) \leq p - 1$ and the result follows.

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