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Distance Closed Restrained Domination in Graph

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Abstract: *In a graph G=(V,E), a set* $S \subset V(G)$ *is a distance closed set of G if for each vertex* $u \in S$ *and for each* $w \in V\text{-}S$*, there exists at least one vertex* $v \in S$ *such that* $d_{<S>}(u, v) = d_G(u, w)$*. Also, a vertex subset D of V(G) is a restrained dominating set of G if every vertex in V-D is adjacent to a vertex in D and at least a vertex in V-D. In this paper, we define a new concept of domination called distance closed restrained domination (D.C.R.D) and analyze some structural properties of graphs and extremal problems relating to the above concepts.*

Keywords: *domination number, distance, eccentricity, radius, diameter, self centered graph, neighborhood, induced sub graph, component, unique eccentric point graph, ciliates, distance closed dominating set, distance closed restrained dominating set.*

1. Introduction

Graphs discussed in this paper are connected and simple graphs only. For a graph, let V(G) and E(G) denotes its vertex and edge set respectively. The *degree* of a vertex v in a graph G is denoted by $deg_G(v)$. The length of any shortest path between any two vertices u and v of a connected graph G is called the *distance between u and v* and it is denoted by $d_G(u, v)$. The distance between two vertices in different components of a disconnected graph is defined to be ∞. For a connected graph G, the *eccentricity* $e_G(v) = \max \{d_G(u,v): \forall u \in V(G)\}$. If there is no confusion, we simply use the notion deg (v), d(u, v) and e(v) to denote degree, distance and eccentricity respectively for the connected graph. The minimum and maximum eccentricities are the *radius* and *diameter* of G, denoted by r(G) and diam (G) respectively. If these two are equal in a graph, that graph is called *self-centered* graph with radius r and is called an *r self-centered* graph. Such graphs are 2-connected graphs. A vertex u is said to be an eccentric vertex of v in a graph G, if d (u, v)=e(v) in that graph. In general, u is called an *eccentric vertex*, if it is an eccentric vertex of some vertex. For $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is the set of all vertices adjacent to v in G. The set $N_G[v]= N_G(v) \cup \{v\}$ is called the *closed neighborhood* of v. A set S of edges in a graph is said to be independent if no two of the edges in S are adjacent. An edge e=(u, v) is a *dominating edge* in a graph G if every vertex of G is adjacent to at least one of u and v. For any set S of vertices in G, the induced sub graph <S> is the maximal sub graph with vertex set S. Also, a vertex cover is a set of vertices which covers all the edges of a graph G. The concept of distance and related properties are studied in [2].

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One important aspect of the concept of distance and eccentricity is the existence of polynomial time algorithm to analyze them. The concept of domination in graphs originated from the chess games theory and that paved the way to the development of the study of various domination parameters and its relation to various other graph parameters. The list of survey of domination theory papers are in [11] and [12].

A set $D \subseteq V(G)$ is called dominating set of G if every vertex in V(G)-D is adjacent to some vertex in D and D is said to be a minimal dominating set if D-{v} is not a dominating set for any $v \in D$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set. We call a set of vertices a γ-set if it is a dominating set with cardinality $\gamma(G)$.Different types of dominating sets have been studied by imposing conditions on the dominating sets. A dominating set D is called *connected* (*independent*) dominating set if the induced sub graph <D> is connected (independent) and D is called a *restrained dominating*(*R.D*)*set* if every vertex in V-D is adjacent to a vertex in D and a vertex in V-D and the *restrained domination number* $\gamma_r(G)$ is the minimum cardinality of a R.D set. The restrained domination in various graphs and their structural properties are studied in [5], [6], [7], [8], [9] and [10].

A dominating set D of a graph G without isolated vertices is called a *total dominating set* of G if the sub graph <D> induced by D has no isolated vertices. If D is both total and restrained dominating set then D is called a *total restrained dominating* (*T.R.D*) *set*. Every graph without isolated vertex has a total restrained dominating set. The concept of total restrained domination was introduced by Chen, Ma and Sun in [3] and further studied in [4], [15]and[17] We may note that the concept of total restrained domination was also introduced by Telle and Proskurowski [16], Albeit indirectly, as a vertex partitioning problem. If G is a graph without isolated vertices then every R.D set is a T.R.D set of G.

The new concepts such as distance closed sets, distance preserving sub graphs, eccentricity preserving sub graphs, Super eccentric graph of a graph, pseudo geodetic graphs are introduced and structural properties of those graphs are studied in [13]. Janakiraman and Alphonse [1] introduced and studied the concept of weak convex dominating sets, which mixes the concept of dominating set and distance preserving set. Using these, structural properties of various dominating parameters are studied.

In this paper, we introduce a new dominating set called distance closed restrained dominating set of a graph through which we studied the properties of the graph. We find upper and lower bounds for the new domination number in terms of various already known parameters. Also, the Nordhaus-Gaddum type results for restrained domination number are established.

2. Prior Results

The concept of distance closed set is defined and studied in the doctoral thesis of Janakiraman [9] and the concept of distance closed sets in graph theory is due to the related concept of ideals in ring theory in algebra. The ideals in a ring are defined with respect to the multiplicative closure property with the elements of that ring. Similarly, the distance closed dominating set is defined with respect to the distance closed property and the dominating set of the graph. Thus, the distance closed dominating set of a graph G is defined as follows:

A subset $S \subseteq V(G)$ is said to be a distance closed dominating (D.C.D) set, if

(i) <S> is distance closed;

(ii) S is a dominating set.

The cardinality of a minimum D.C.D set of G is called the distance closed domination number of G and is denoted by γ_{dcl}.

Clearly from the definition, $1 \leq \gamma_{dcl} \leq p$ and graph with $\gamma_{dcl}=p$ is called a 0-distance closed dominating graph. Also, if S is a D.C.D set of G then the complement V-S need not be a D.C.D set of G. The definition and the extensive study of the above said distance closed dominating set in Graphs are studied in [14].

Following are some of the results related to the distance closed domination number of a graph presented in [14].

Theorem 2.1[14]: If T is a tree with number of vertices $p \geq 2$, then $\gamma_{dcl}(T) = p-k+2$, where k is the number of pendant vertices in T.

Theorem 2.2 [14]: Let G be a self centered graph of diameter 2.Then $\gamma_{dcl}(G) \leq \delta+2$.

Theorem 2.3 [14]: Let G be a graph of order p. Then

(i) $\gamma_{dcl}(G)=2$ if and only if G has at least two vertices of degree p-1.

(ii) If G has exactly a vertex of degree p-1, then $\gamma_{dcl}(G)=3$.

Theorem 2.4 [14]: Let G be a graph of order p. If G has exactly a vertex of degree p-1, then $\gamma_{dcl}(G)=3$.

Theorem 2.5 [14]: If a graph G is connected and diam(G) ≥3, then $\gamma_{dcl}(\overline{G})=4$.

Theorem2.6 [14]: For any connected graph G such that $\overline{G}$ is also connected $\gamma_{dcl}(G)+\gamma_{dcl}(\overline{G}) \leq p+4$, where $\gamma_{dcl}(G)$ and $\gamma_{dcl}(\overline{G})$are the cardinality of minimal distance closed dominating set of G and $\overline{G}$ respectively.

3. Main Results

Continuing the above study, we define a new domination parameter namely, distance closed restrained domination as follows.

3.1 Distance Closed restrained Dominating Sets in Graphs

Definition 3.1

A restrained dominating set S said to be a distance closed restrained dominating set (D.C.R.D), if S is a distance closed set.

The cardinality of the minimum distance closed restrained dominating set is called the distance closed restrained domination number and it is denoted by $\gamma_{rdcl}(G)$.

The following theorems give the bounds for the distance closed restrained domination number of some special class of graphs:

Theorem 3.1: For any graph G with n vertices, we have the following

(i) $\gamma_{dcl}(G) \leq \gamma_{rdcl}(G)$;

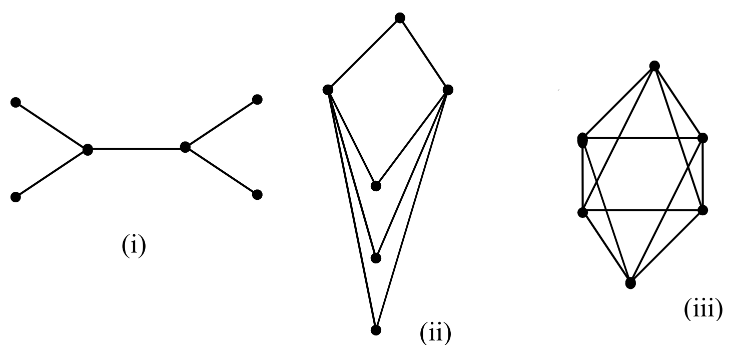
(ii) If v is a vertex of degree 1 in G, then v belongs to all D.C.R.D sets of G.

Proof: Let G be a graph on n vertices.

(i) Proof is trivial.

(ii) If D is a D.C.R.D set of G, then every vertex in V-D is adjacent to at least one other vertex of V-D. Thus $d(v) \geq 2$, for every $v \in V\text{-}D$. Hence a vertex of degree 1 must be in every D.C.R.D set of G.

Example 3.1: Example of graphs with $\gamma_{rdcl}(G) > \gamma_{dcl}(G)$.



For the above graphs, $\gamma_{dcl}(G)=4$ and $\gamma_{rdcl}(G)=6$ and hence $\gamma_{rdcl}(G) > \gamma_{dcl}(G)$.

Proposition 3.1: There exists no graph G on p vertices such that $\gamma_{rdcl}(G) = p-1$.

Proof: If suppose that G has a D.C.R.D set D with $|D|$ $\gamma_{rdcl}= p-1$, then $|V-D|=1$.

=> There exists a vertex $v \in V(G)$ such that V-$D=\{v\}$.
=> D cannot be a restrained dominating set.

Proposition 3.2: For any tree T, $\gamma_{rdcl}(T)=p$ the number of vertices in G. (figure(i) of example 3.1).

Proof: We have, $\gamma_{dcl}(T)=p-k+2$, where k is the number of pendant vertices in T. That is, if D is a D.C.D set of T then every vertex in V-D is a pendant vertex. Hence, by result (ii) of Theorem 3.1, $\gamma_{rdcl}(T)=p$.

Remark 3.1: Trees, cycles, paths and ciliates are some graphs with $\gamma_{rdcl}(G)=p$. Those graphs are called 0-distance closed restrained dominating (0-D.C.R.D) graphs.

Theorem 3.2: Let G be a graph with p vertices. Then $\gamma_{rdcl}(G)=2 \Leftrightarrow$ there exist two vertices of degree p-1 and all other vertices have degree greater than or equal to 3 in G.

Proof: Let $\gamma_{rdcl}(G)=2$. Then there exists a D.C.R.D set D with $|D|=2$. If $D=\{u,v\}$, then $deg(u)=deg(v)=p-1$.Since D is a D.C.R.D set, each vertex in V-D is adjacent to at least one vertex in V-D and already they are adjacent with both u and v. Hence degree of any vertex in V-D must be greater than or equal to 3.

Conversely, suppose that G contains two vertices of degree p-1 then that two vertices will form a distance closed dominating set D, and if all other vertices are of degree greater than or equal to 3 then every vertex in V-D must be adjacent to at least one vertex of V-D. Hence D forms a D.C.R.D set of G and hence $\gamma_{rdcl}(G)=2$.

Note 3.1: In this case $\gamma_{dcl}(G) = \gamma_{rdcl}(G)$.

Corollary 3.1: If G has a vertex of degree less than or equal to 1, then $\gamma_{rdcl}(G)\geq 3$.

Theorem 3.3: Let G be a graph with p vertices. Then $\gamma_{rdcl}(G)=3 \Leftrightarrow$ there exist exactly one vertex of degree p-1 and all other vertices have degree greater than or equal to 2 in G.
Proof: Proof is similar as that of Theorem 3.2.

Note 3.2: In this case also $\gamma_{dcl}(G) = \gamma_{rdcl}(G)$.

Proposition 3.3: If G is an n-self centered graph with (2n+1) vertices, then G is a 0-D.C.R.D graph.
Proof: Let G be an n-self centered graph with (2n+1) vertices. If G has a D.C.R.D set then the distance closed restrained domination number must be less than or equal to 2n-1(by Proposition 3.1), which is not possible as $\gamma_{dcl}(G) \geq 2n$. Hence G is a 0-D.C.R.D graph.

Corollary 3.2: If G is a graph with n vertices and $\gamma_{dcl}(G)$ = n-1, then G is a 0-D.C.R.D graph.

Proposition 3.4: If every D.C.D set of a graph G is a vertex cover for G, then G is a 0-D.C.R.D graph.

Proof: Let D be a D.C.D set of G which covers all the edges of G. Then, clearly every edge has its both ends in D. Hence, $\gamma_{rdcl}(G)=p$ and hence G is a 0-D.C.R.D graph.

Proposition 3.5: Let G be a self centered graph of diameter 2 and $\delta \geq 3$. If for some $u \in V(G)$, $\langle N_1(u) \rangle$ and $\langle N_2(u) \rangle$ are independent, then $\gamma_{rdcl}(G)=4$.

Proof: If there exists a vertex $u \in V(G), d(u) \geq 3$ such that $\langle N_1(u) \rangle$ and $\langle N_2(u) \rangle$ are independent then each vertex in $N_2(u)$ must be adjacent to all the vertices of $N_1(u)$. Then clearly $\{u, u', u'', w\}$ will form a D.C.R.D set, where $u', u'' \in N_1(u)$ and $w \in N_2(u)$. Hence $\gamma_{rdcl}(G)=4$.

Corollary 3.3: Let G be a self centered graph of diameter 2 and $\delta=2$. If for some $u \in V(G)$, $\langle N_1(u) \rangle$ and $\langle N_2(u) \rangle$ are independent, then $\gamma_{rdcl}(G)=p$.(fig (ii) of example 3.1)

Proposition 3.6: Let G be a self centered graph of diameter 2. If u is a vertex of degree $\delta \geq 3$ such that $\langle N_2(u) \rangle$ is independent, then $\gamma_{rdcl}(G) = 4$.

Proof: Suppose for a vertex u of degree $\delta \geq 3$, $\langle N_2(u) \rangle$ is an independent set. Then each vertex of $N_2(u)$ is adjacent with all of $N_1(u)$ (since deg(u)= δ and deg(v) $\geq \delta$, for all $v \in N_2(u)$). Also, any vertex in $N_1(u)$ has its eccentric node in $N_1(u)$ itself. Hence u and any two non adjacent vertices of $N_1(u)$ together with a vertex in $N_2(u)$ will form a restrained distance closed dominating set (D.C.R.D) for G. That is $\gamma_{rdcl}(G) = 4$.

Proposition 3.7: If G is a (p-2) regular graph with $p \geq 7$, then $\gamma_{rdcl}(G)=4$.

Proof: Let G be a (p-2) regular graph with $p \geq 7$. Then clearly G is 2-self centered and $\gamma_{dcl}(G)=4$. Let D be the D.C.D set of G. Since $|D|=4$, $|V-D| \geq 3$ and also every vertex in V-D is of degree p-2. That is, every vertex is non adjacent exactly one vertex of G. Hence every vertex v in V-D is not adjacent to at most one vertex of V-D and hence every vertex in V-D is adjacent to at least one vertex of V-D as $|V-D| \geq 3$. Therefore, $\gamma_{rdcl}(G)=4$.

Corollary 3.4: If G is a (p-2) regular graph with p=6, then $\gamma_{rdcl}(G)=p$.(figure (iii) of example 3.1)

Proposition 3.8: Let G be a self centered graph of diameter 2 and let u be a vertex of degree $\delta \geq 3$. If every component of <N_2(u)> has cardinality at least 3, then $\gamma_{rdcl}(G) \leq \delta+2$.

Proof: Let u be a vertex of degree $\delta \geq 3$ of a 2-self centered graph G and let C be a component of <N_2(u)>. Since $|C| \geq 3$, any vertex in C must be adjacent to at least one vertex of C in N_2(u). Hence the set $\{u\} \cup N_1(u) \cup \{u'\}$ where $u' \in N_2(u)$ will form a D.C.R.D set and hence $\gamma_{rdcl}(G) \leq \delta+2$.

Theorem 3.4: Let G be a self centered graph of diameter 2 with exactly one vertex of degree 2 and the remaining vertices are of degree greater than or equal to 5, then $\gamma_{rdcl}(G) \leq 5$.

Proof: Let u be a vertex of degree 2 in G and let u', u'' be the vertices adjacent to u.

Case 1: suppose u', u'' are non adjacent.

If u', u'' are non adjacent then every vertex in N_2(u) is adjacent to any one of u' or u'' or both as G is 2- self centered. That is, every vertex of N_2(u) is adjacent to at most two vertices of N_1(u). Let $w \in N_2(u)$.Then the set {u, u', u'', w} forms D.C.D set for G. Also degree of any vertex in the induced sub graph <N_2(u)- {u, u', u'', w}> is greater than or equal to 2.Hence the set {u, u', u'', w} will form a D.C.R.D set for G and hence $\gamma_{rdcl}(G)=4$.

Case 2: suppose u', u'' are adjacent.

If u' and u'' are adjacent then the eccentric vertices of u' and u'' must be in N_2(u).Let v, w be the eccentric vertices of u' and u'' respectively in N_2(u). Since G is 2-self centered, u' is adjacent with w and u'' is adjacent with u. Therefore, the set {u, u', u'', v, w} forms D.C.D set for G. Also, if all the vertices other than u has degree greater than or equal to 5 then all the vertices in the induced sub graph <N_2(u)- { u, u', u'', v, w }> must have degree greater than or equal to one. Hence the set {u, u', u'', v, w} will form a D.C.R.D set for G and hence $\gamma_{rdcl}(G) \leq 5$.

Theorem 3.5: Let G and $\overline{G}$ be both self centered graphs of diameter 2. If for some vertex u, any vertex in N_1(u) is not adjacent to at least two vertices of N_1(u) and any vertex in N_2(u) is adjacent to at least two vertices in N_2(u), then $\gamma_{rdcl}(G)+\gamma_{rdcl}(\overline{G}) \leq p+3$.

Proof: Let u be a vertex of G of degree k. If the cardinality of all the components G_i (i=1 to k) of <N_2(u)> is greater than or equal to 3, then {u}U { N_1(u)} U a vertex in N_2(u) will form a R.D.C.D set for G. If every vertex in N_1(u) is not adjacent to at least two vertices in N_1(u) then {u} U {N_2(u)} U a vertex in N_1(u) forms a R.D.C.D set for $\overline{G}$.

Hence $\gamma_{rdcl}(G) + \gamma_{rdcl}(\overline{G}) \leq k+2+[p-(k+1)+2]$

$\Rightarrow \gamma_{rdcl}(G) + \gamma_{rdcl}(\overline{G}) \leq p+3$.

Theorem 3.6: Let G be a graph. If there exist two pairs of vertices (u_1,u_2) and (v_1,v_2) such that $d(u_1,u_2)= d(v_1,v_2)=3$, then $\gamma_{rdcl}(G)+ \gamma_{rdcl}(\overline{G}) \leq p+4$.

Proof: If there exist two pairs of vertices (u_1,u_2) and (v_1,v_2) such that $d(u_1,u_2)= d(v_1,v_2)=3$ then clearly u_1u_2 is a dominating edge in $\overline{G}$ and all the vertices are adjacent with v_1 or v_2 in $\overline{G}$. Since $d(u_1,u_2)=3$, there exists at least one vertex u_1' which is non adjacent to u_1 in $\overline{G}$ and similarly there exists at least one vertex u_2' which is non adjacent to u_2 in $\overline{G}$. Therefore, the set of vertices $\{u_1, u_2', u_1', u_2\}$ will form a D.C.R.D set in $\overline{G}$. Hence, $\gamma_{rdcl}(\overline{G})= 4$ and hence $\gamma_{rdcl}(G) + \gamma_{rdcl}(\overline{G}) \leq p+4$.

Corollary 3.5: Let G be a graph. If there exists two pair of vertices (u_1,u_2) and (v_1,v_2) such that $d(u_1,u_2)= d(v_1,v_2)>3$, then $\gamma_{rdcl}(G) + \gamma_{rdcl}(\overline{G}) \leq p+4$.

Theorem 3.7: If G is a graph with diameter greater than or equal to 5, then
$\gamma_{rdcl}(G)+ \gamma_{rdcl}(\overline{G}) \leq p+4$.

Proof: Since G is of diameter greater than or equal to 5, there exists a path of length 5 say uxyzvw in G. Clearly, $d(u,z)= d(x,v)=d(y,w)=3$. Therefore $\{u,z\}$ will form a dominating set for $\overline{G}$. Also, $\{x,z,u,v\}$ form a R.D.C.D set for $\overline{G}$, since all the vertices in $V-\{x,z,u,v\}$ must be adjacent to $\{y,w\}$ in $\overline{G}$. Therefore, $\gamma_{rdcl}(\overline{G})=4$.

Hence, $\gamma_{rdcl}(G) + \gamma_{rdcl}(\overline{G}) \leq p+4$.

Remark 3.2: The above bound is sharp and attained for the following graphs.

(i) Ciliates

(ii) Cycles, paths and trees with number of vertices $p \geq 6$.

References

[1] P.J.A. Alphonse, "On distance, domination and related concepts in graphs and their Applications", Ph.D thesis submitted to Bharathidasan University, 2003.

[2] F. Buckley and Harary – Distance in graphs, Addison- Wesley, Redwood City, CA (1990).

[3] X.G. Chen, D.X. Ma and L. sun, "On total restrained domination in graphs",Czechoslovak Math. J.55(130) (2005) 165-173.

[4] J. Cyman and J. Raczek, "On the total restrained domination number of a graph", Australas. J. Combin.36(2006), 91-100.

[5] P. Dankelmenn, D. Day, J.H. Hattingh, M.A. Henning, L.R. Markus and H.C. Swart, "On equality in an upper bound for the restrained and total domination numbers of a graph", Discrete Math. (307)(2007) 2845-2852.

[6] P. Dankelmenn, J.H. Hattingh, M.A. Henning, and H.C. Swart, “Trees with equal domination and restrained domination numbers”, J. Global. Optim. 34(2006), 597-607.

[7] G.S. Domke, J.H. Hattingh, M.A. Henning and L.R. Markus, “ Restrained domination in graphs with minimum degree two”, J. Combin. Math. Combin. Comput. 35(2000), 239-254.

[8] G.S. Domke, J.H. Hattingh, S.T. Hedetiemi, R.C. Laskar and L.R. Markus, “ Restrained domination in graphs”, Discrete Math.203(1999), 61-69.

[9] G.S. Domke, J.H. Hattingh, S.T. Hedetiemi, and L.R. Markus, “ Restrained domination in Trees”, Discrete Math. 211(2000), 1-9.

[10] M.A. Henning, “Graphs with large restrained domination number”, Discrete Math.197|198 (1999), 415-429.

[11]Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater: Fundamentals of domination in graphs. Marcel Dekker, New York 1998.

[12] Teresa W. Haynes, Stephen T. Hedetniemi, Peter.J. Slater, Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998.

[13] T.N. Janakiraman, “On some eccentricity properties of the graphs”, Ph.D thesis submitted to Madras University, 1991.

[14] T.N. Janakiraman, P.J.A. Alphonse and V.Sangeetha, “Distance closed domination in graph”, International Journal of Engineering Science, Advanced computing and Bio-Technology, Vol.1, 2010, 109-117.

[16] T.N. Janakiraman and P.J.A. Alphonse “Weak convex restrained domination in graphs”, International Journal of Engineering Science, Advanced computing and Bio-Technology, Vol.2, 2011, 1-10.

[15] Ernst. Joubert, “Aspects of total restrained domination in graphs”- Ph.D thesis submitted to University of Johannesburg, 2007.

[16] J.A. Telle and A. Proskurowski, Algorithms for vertex partitioning problems on partial k-trees. SIAM J. Discrete Math. 10(1997) 529-550.

[17] B. Zelinka, “Remarks on restrained and total restrained domination in graphs”, Czchoslovak Math. J. 55(130)(2005), 393-396.