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On Eccentric domination in Trees

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Abstract: *A subset D of the vertex set V(G) of a graph G is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an eccentric dominating set if for every $v \in V-D$, there exists at least one eccentric point of v in D. The minimum of the cardinalities of the eccentric dominating sets of G is called the eccentric domination number $\gamma_{ed}(G)$ of G. In this paper, eccentric domination parameter of trees is studied. Characterization of trees with $\gamma_{ed}(T) = \gamma(T)+2$, $\gamma_{ed}(T) = \gamma(T)+1$ and $\gamma_{ed}(T) = \gamma(T)$ are also studied and bounds for $\gamma_{ed}(T)$, its exact value for some particular classes of trees are found.*

Key words: *Eccentric dominating set, Eccentric domination number.*

1.Introduction

Let G be a finite, simple, undirected graph on n vertices with vertex set V(G) and edge set E(G). For graph theoretic terminology refer to Harary [4], Buckley and Harary [1].

Definition 1.1 Let G be a connected graph and u be a vertex of G. The **eccentricity** e(v) of v is the distance to a vertex farthest from v. Thus, e(v) = max {d(u, v) : $u \in V$}. The **radius** r(G) is the minimum eccentricity of the vertices, whereas the **diameter** diam(G) is the maximum eccentricity. For any connected graph G, r(G) $\leq$ diam(G) $\leq$ 2r(G). v is a central vertex if e(v) = r(G). The **center C(G)** is the set of all central vertices. The central subgraph < C(G) > of a graph G is the subgraph induced by the center. v is a peripheral vertex if e(v) = d(G). The periphery P(G) is the set of all peripheral vertices.

For a vertex v, each vertex at a distance e(v) from v is an **eccentric vertex**. **Eccentric set of a vertex v** is defined as E(v) = {$u \in V(G)$ / d(u, v) = e(v)}.

Definition 1.2 Caterpillar is a tree in which the removal of pendent vertices results in a path. This path is known as the underlying path of the caterpillar.

Definition 1.3 A lobster is a tree in which removal of pendent vertices results in a caterpillar.

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Definition 1.4 [2, 8] A set S $\subseteq$ V is said to be a **dominating set** in G, if every vertex in V$-$S is adjacent to some vertex in S. A dominating set D is an **independent dominating set**, if no two vertices in D are adjacent that is D is an independent set. A dominating set D is a **connected dominating set**, if < D > is a connected subgraph of G. A set D $\subseteq$ V(G) is a **global dominating set**, if D is a dominating set in G and $\overline{G}$.

Definition: 1.5 [6] A set D $\subseteq$ V(G) is an **eccentric dominating set** if D is a dominating set of G and for every $v \in V - D$, there exists at least one eccentric point of v in D.

An eccentric dominating set D is a **minimal eccentric dominating set** if no proper subset $D'' \subseteq D$ is an eccentric dominating set.

Definition: 1.6 [6] The **eccentric domination number** γ_{ed}**(G)** of a graph G equals the minimum cardinality of an eccentric dominating set.

Definition: 1.7 [6] **Eccentric point set of G:**

Let S $\subseteq$ V(G). Then S is known as an **eccentric point set of G** if for every $v \in V{-}S$, S has at least one vertex u such that $u \in E(v)$. An eccentric point set S of G is a **minimal eccentric point set** if no proper subset S' of S is an eccentric point set of G. S is known as a **minimum eccentric point set** if S is an eccentric point set with minimum cardinality. The minimum cardinality of an eccentric point set of G denoted as **e(G)** is known as **eccentric number of G**.

Let D be a minimum dominating set of a graph G and S be a minimum eccentric point set of G. Clearly, D $\cup$ S is an eccentric dominating set of G. Hence, $\gamma_{ed}(G) \leq \gamma(G)+e(G)$.

Theorem: 1.1 For a connected graph G, $\gamma(G) \geq \lceil (\gamma_c(G)+2)/3 \rceil$.

Theorem: 1.2 [8] For a connected graph G on n vertices $\gamma(G) \leq n/2$.

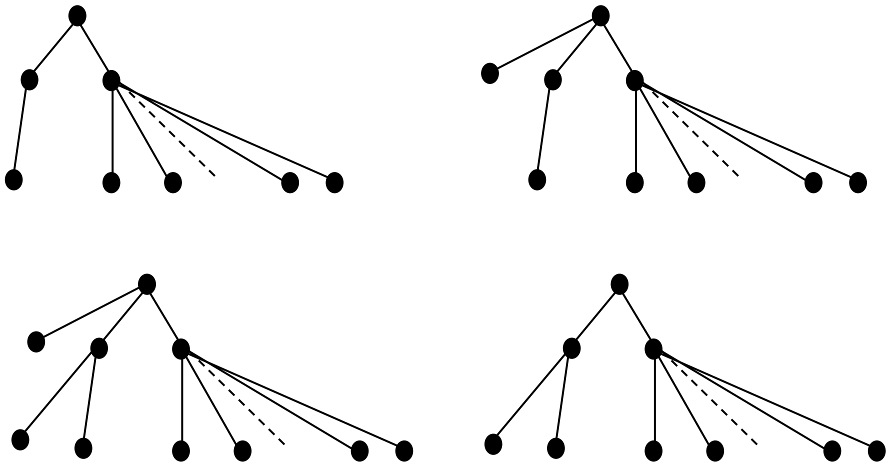
Theorem: 1.3 [6] $\lceil n/(1+\Delta(G)) \rceil \leq \gamma_{ed}(G) \leq \lfloor (n+\gamma(G))/2 \rfloor$.

Theorem: 1.4 [6] $\gamma_{ed}(P_n) = \lceil n/3 \rceil$, if n = 3k+1,

$$\gamma_{ed}(P_n) = \lceil n/3 \rceil + 1, \text{ if } n = 3k \text{ or } 3k+2.$$

Theorem: 1.5 [6] For a tree T, $\gamma(T) \leq \gamma_{ed}(T) \leq \gamma(T)+2$.

Theorem: 1.6 [6] Let T be a tree with radius 2 and diameter 4. $\gamma_{ed}(T) = n - \Delta(T)$ if and only if any one of the following is true: (i) $T = P_5$. (ii) T is a wounded spider having at least two non wounded legs. (iii) T is any one of the following four types of trees.



2.Eccentric domination in Trees

In [6] we have proved that for any tree T, $\gamma(T) \leq \gamma_{ed}(T) \leq \gamma(T)+2$. In this paper, we characterize trees for which $\gamma_{ed}(T) = \gamma(T)+2$, $\gamma(T)+1$ or $\gamma(T)$ and give bounds for $\gamma_{ed}(T)$ in terms of number of vertices, pendent vertices and support vertices of T.

First let us characterize trees for which $\gamma_{ed}(T) = \gamma(T)+2$.

Theorem: 2.1 For a tree T, $\gamma_{ed}(T) = \gamma(T)+2$ if and only if supports of all the peripheral vertices are in every γ-set.

Proof: This theorem follows from the fact that, an eccentric dominating set of T must contain atleast two peripheral vertices at distance d = diam(T) to each other.

Next Theorem follows from the definition of eccentric dominating set.

Theorem: 2.2 For a tree T, if $\gamma_{ed}(T) = \gamma(T)$ there exists atleast two peripheral vertices at distance d = diam(T) to each other such that support of these vertices are of exactly degree two.

Proof: From the definition of eccentric dominating set, any γ_{ed}-set of T must contain atleast two peripheral vertices at distance d = diam(T) to each other. If $\gamma_{ed}(T) = \gamma(T)$, there exists a γ-set D of T such that D contains two peripheral vertices x, y at distance d =

diam(T) to each other. Thus D is minimum implies that supports of x and y are not in D. This implies that supports of x and y are of exactly degree two.

The converse of the previous theorem is not true. For example, for T = P_9, path on nine vertices $\gamma_{ed}(T) = \gamma(T)+1$.

Theorem: 2.3 For a tree T, $\gamma_{ed}(T) = \gamma(T)$ if and only if there exists a γ-set D of T such that D contains atleast two peripheral vertices at distance d = diam(T) to each other.

Proof: From the definition of eccentric dominating set, any γ_{ed}-set of T must contain atleast two peripheral vertices at distance d = diam(T) to each other. Therefore $\gamma_{ed}(T) = \gamma(T)$ if and only if there exists a γ-set D of T such that D contains two peripheral vertices x, y at distance d = diam(T) to each other.

Remark: If there exists a γ-set D of T such that D contains two peripheral vertices x, y at distance d = diam(T) to each other, then their supports u, v are not in D. Therefore, deg u = deg v = 2. Then S = (D − {x,y})$\cup${u, v} is also a γ-set of T. But S must not be efficient.

Theorem: 2.4 If the degree of support of each peripheral vertex is greater than two in a tree T then $\gamma_{ed}(T) = \gamma(T)+2$.

Proof: If the degree of support of each peripheral vertex is greater than two, then that supports must be in every γ-set. Hence by theorem 2.1 $\gamma_{ed}(T) = \gamma(T)+2$.

The converse of the previous theorem is not true. For example, for T given in figure 2.1, $\gamma_{ed}(T) = \gamma(T)+2$, but the degree of support of each peripheral vertex is two.

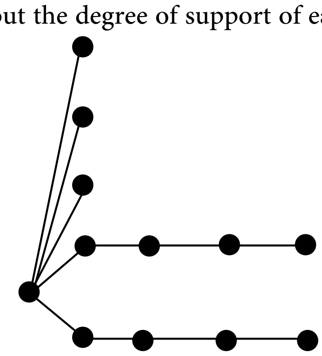


Figure 2.1

Theorem: 2.5 For a tree T, if there exists an efficient dominating set containing all the supports of peripheral vertices, then $\gamma_{ed}(T) = \gamma(T)+2$.

Proof: Let D be an efficient dominating set of T. Efficient dominating set is always a γ-set and distance between any two elements of D is $\geq$ 3. Hence by theorem 2.1, $\gamma_{ed}(T) = \gamma(T)+2$.

Theorem: 2.6 For a tree T with radius two which is not a path, (i) $\gamma_{ed}(T) = \gamma(T)$ if there exists atleast two peripheral vertices with degree of their supports = 2. (ii) $\gamma_{ed}(T) = \gamma(T)+1$ if there exists only one peripheral vertex with degree of its support = 2. (iii) $\gamma_{ed}(T) = \gamma(T)+2$ if degree of all the support vertices are greater than two.

Proof: Let T be a bi-central tree. In this case, diam(T) = 3 and T = $\overline{K}_n+K_1+K_1+\overline{K}_m$ for n, m $\geq$ 1. When n = m = 1,$\gamma_{ed}(T) = \gamma(T) = 2$. When n, m $>$ 1,$\gamma_{ed}(T) = \gamma(T)+2 = 4$, otherwise $\gamma_{ed}(T) = \gamma(T)+1 = 3$.

Next, let us consider a unicentral tree T. In this case, diam(T) = 4 and T $\neq P_5$. Let S be the set of all support vertices of T.

(i) if there exists two peripheral vertices with degree of their supports = 2, then degree of the central vertex is greater than two. Let x, y be two peripheral vertices and u, v be their supports such that deg u = deg v = 2. Then S is γ-set of T and $(S-\{u, v\})\cup\{x, y\}$is a γ_{ed}-set of T. Hence $\gamma_{ed}(T) = \gamma(T)$.

(ii) if there exists only one peripheral vertex with degree of its support = 2, let x be a peripheral vertex and u be its support such that deg u = 2. Let y be another peripheral vertex which is at distance 4 from x. Then S is γ-set of T and $(S-\{u\})\cup\{x, y\}$is a γ_{ed}-set of T. Hence $\gamma_{ed}(T) = \gamma(T)+1$.

(iii) If degree of all the support vertices are greater than two, then S is γ-set of T and $S\cup\{x, y\}$ where x, y are any two peripheral vertices at distance 4 to each other is a γ_{ed}-set of T. Hence $\gamma_{ed}(T) = \gamma(T)+2$.

Bounds of $\gamma_{ed}(T)$ interms of number of vertices, pendent vertices and support vertices of T.

Let us assume that T be a tree on n vertices, with s support vertices and p pendent vertices. Let T' be a tree on n+2 vertices obtained from T as follows:

Let u, v be any two pendent vertices of T at distance d = diam(T) to each other. Attach a new vertex x to u by an edge and y to v by an edge. Denote the new tree obtained as T'. Then $|V(T')| = n+2 = |V(T)|+2$ and $|E(T')| = n+1 = |E(T)|+2$.

Example:

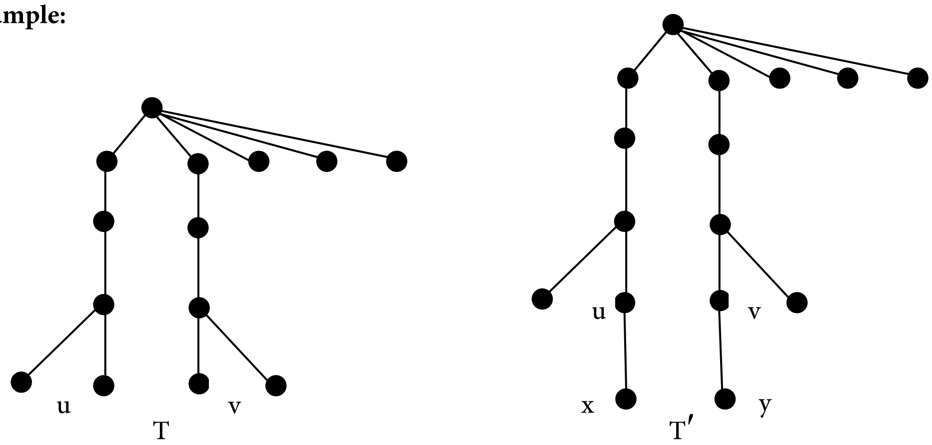


Figure 2.2

Clearly $\gamma_{ed}(T) = \gamma(T')$. deg u = deg v = 2 in T' implies that there is a minimum dominating set of T' containing u and v; that dominating set is a minimum eccentric dominating set for T. Number of pendent vertices in T' is also equal to p. For example, if T is a path on n vertices then $\gamma_{ed}(T) = \gamma(T') = \lceil (n+2)/3 \rceil$ for all $n \geq 3$.

It is well known that, for any graph G, $\gamma(G) \geq \lceil (\gamma_c(G)+2)/3 \rceil$ where $\gamma_c(G)$ denotes the connected domination number of G. Also for a tree T on n vertices having p pendent vertices $\gamma_c(T) = n-p$.

Thus, we have, $\gamma_c(T') = n+2-p$ and $\gamma(T') \geq \lceil (\gamma_c(T')+2)/3 = \lceil (n+4-p)/3 \rceil$.
Also, since T' is connected with n+2 vertices, we have $\gamma(T') \leq (n+2)/2$. But $\gamma_{ed}(T) = \gamma(T')$. Therefore, $(n/2)+1 \geq \gamma_{ed}(T) \geq \lceil (n+4-p)/3 \rceil$. Hence we have the following important Theorem.

Theorem: 2.7 For any tree T on n vertices having p pendent vertices, $(n/2)+1 \geq \gamma_{ed}(T) \geq \lceil (n+4-p)/3 \rceil$. If $T = P_n$, then $\gamma_{ed}(P_n) = \gamma(P_{n+2}) = \lceil (n+2)/3 \rceil$ for all $n \geq 3$.

Theorem: 2.8 For a tree T, $\gamma_{ed}(T) = (n/2)+1$ if and only if T can be obtained as follows:
Let H be a tree on $(n-2)/2$ vertices with exactly two peripheral vertices x and y, and let $G = H \cdot K_1$. With G, add two vertices u, v and edges joining x to u and y to v. Name the new graph as T. Clearly T is a tree with n vertices.

Proof: If T is a tree as given in the theorem, it is easy to verify that $\gamma_{ed}(T) = (n/2)+1$. On the other hand, assume that $\gamma_{ed}(T) = (n/2)+1$. Therefore, $\gamma_{ed}(T) = (n/2)+1 = (n-2)/2+2$. Since T is a tree, two peripheral vertices at distance d = diameter of T is necessary to dominate T eccentrically. So among the remaining n—2 vertices, half of them are in a minimum eccentric dominating set implies that (n—2)/2 vertices are pendent vertices with distinct support vertices. So deleting all the (n—2)/2+2 pendent vertices from T, we get a tree on (n—2)/2 vertices. Hence the theorem follows.

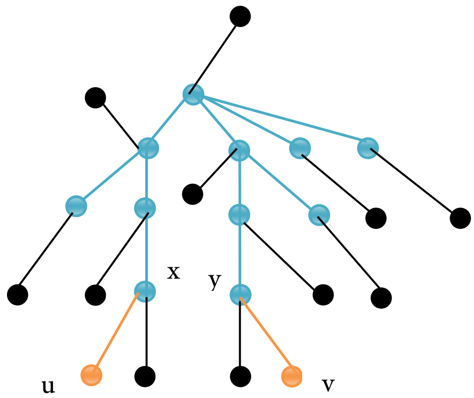


Figure 2.3

Theorem: 2.9 If T is a tree with p pendent vertices and s support vertices, then $\gamma_{ed}(T) \leq \lfloor (n-p-s)/2 \rfloor + s + 2$.

Proof: Let T be a tree with p pendent vertices and s support vertices. Delete all the pendent vertices of T from T. Then we get a tree T_1 on n—p vertices with s pendent vertices. Again deleting these s pendent vertices from T_1 we obtain a tree T_2 on n—p—s vertices. Now, a γ-set of T can be formed by taking atmost (n—p—s)/2 vertices from $V(T_2)$ and s support vertices of T.

Therefore, $\gamma(T) \leq \lfloor (n-p-s)/2 \rfloor + s$; $\gamma_{ed}(T) \leq \lfloor (n-p-s)/2 \rfloor + s + 2$. This proves the Theorem.

Theorem: 2.10 For a tree T, $\lceil (n+4-p)/3 \rceil \leq \gamma_{ed}(T) \leq \lfloor (n-p-s)/2 \rfloor + s + 2$.

Proof: Follows from Theorem 2.7 and 2.9.

Both of these bounds are sharp, since for T = P_n, $\gamma_{ed}(T) = \lceil (n+4-p)/3 \rceil$ and for T in figure 2.4, $\gamma_{ed}(T) = \lfloor (n-p-s)/2 \rfloor + s + 2 = \lfloor (22-12-10)/2 \rfloor + 10 + 2 = 12$.

Theorem: 2.11 If T is a caterpillar, $\lceil (n-p)/3 \rceil + 1 \leq \gamma_{ed}(T) \leq n-p+2$.

Proof: Let T be a caterpillar on n vertices. then its underlying path is P = P_m, m = n—p, where p is the number of pendent vertices of T. Domination parameter of this path is $\lceil (n-p)/3 \rceil$ and $\gamma_{ed}(P) = \lceil (n-p)/3 \rceil$ or $\lceil (n-p)/3 \rceil + 1$. But diameter of T = diameter of

P+2. Therefore, $\lceil (n-p)/3 \rceil +1 \leq \gamma_{ed}(T)$. Also, $(V-U)\cup\{x, y\}$, where U is the set of all pendent vertices and x, y are any two peripheral vertices at distance d is an eccentric dominating set. Hence, $\gamma_{ed}(T) \leq n-p+2$.

Both of these bounds are sharp, since for $T = P_7$, $\gamma_{ed}(T) = \lceil (7-2)/3 \rceil +1 = 4$ and for T in figure 2.4, $\gamma_{ed}(T) = n-p+2 = 22-12+2 = 12$.

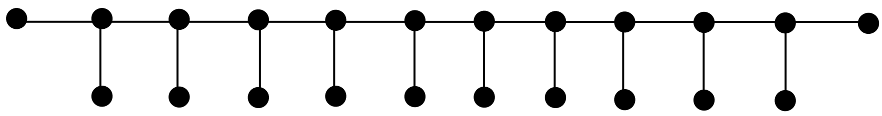


Figure 2.4

Theorem: 2.12 If T is a lobster with p pendent vertices and s support vertices then $\lceil (n-p-s_1)/3 \rceil + s_1 \leq \gamma_{ed}(T) \leq \lfloor (n-p-s_1)/3 \rfloor +s+2$, where s_1 denotes the number of support vertices of T which are not in the underlying path P.

Proof: Let T be a given lobster. Then its underlying path P contains $n-p-s_1$ vertices. To dominate the end vertices of T which are at distance two from P we need s_1 support vertices. So if S is the set of support vertices which are not in P and if D is a γ-set of P, then $D\cup S$ dominate vertices of T except those which are at distance one from P and not adjacent to elements of D. Hence, $\gamma_{ed}(T) \geq |D\cup S|$.

Therefore, $\lceil (n-p-s_1)/3 \rceil + s_1 \leq \gamma_{ed}(T)$.

Suppose no support vertices is on P. Then $\lceil (n-p-s)/3 \rceil$ vertices are necessary to dominate P. If D_1 is such a dominating set, then $D_1\cup S_1$, where S_1 is the set of all support vertices of T, form a dominating set of T. Thus $\gamma_{ed}(T) \leq \lfloor (n-p-s)/3 \rfloor +s+2$.

Suppose some support vertices of T are in P. Then $\lfloor (n-p-s_1)/3 \rfloor +s$ vertices dominate the caterpillar obtained from T after deleting pendent vertices and is also a dominating set for T. Hence to dominate T eccentrically atmost $\lfloor (n-p-s_1)/3 \rfloor +s+2$ vertices are needed. Thus $\gamma_{ed}(T) \leq \lfloor (n-p-s_1)/3 \rfloor +s+2$.

Both of these bounds in Theorem 2.12 are sharp, since for T in figure 2.5, $\gamma_{ed}(T) = 6 = \lceil (n-p-s_1)/3 \rceil + s_1 = \lceil (19-9-4)/3 \rceil +4$, and for T in figure 2.6, $\gamma_{ed}(T) = 14 = \lfloor (n-p-s_1)/3 \rfloor +s+2 = \lfloor (31-10-7)/3 \rfloor +8+2$.

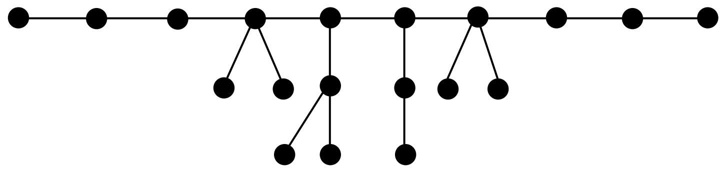


Figure 2.5

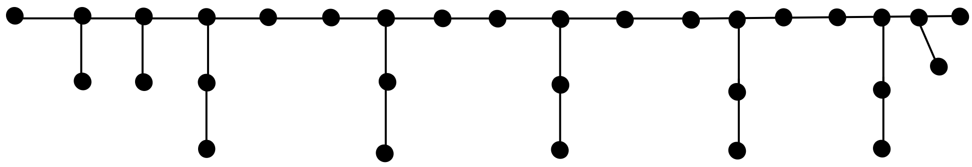


Figure 2.6

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