International Journal of Engineering Science, Advanced Computing and Bio-Technology
Vol. 2, No. 1, January –March 2011, pp. 25-37

k-edge-graceful labeling and k-global edge-graceful labeling of some graphs

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Abstract: *In 1985, Lo [7] introduced the notion of edge-graceful graphs. The concept of global edge-graceful labeling was introduced in 2009 by Gayathri and Duraisamy [2]. In 2004, Sin - Min Lee, Kuo - Jye Chen and Yung - Chin Wang [8] introduced the k-edge-graceful graphs. In this paper, we extend the concept of global edge-graceful graphs to k-global edge-graceful graphs. Here, we investigate the k-edge gracefulness and k-global edge-gracefulness of some families of graphs.*

Keywords: *k-edge-graceful, global edge-graceful, k-global edge-graceful.*

AMS (MOS) Subject Classification: 05C78

1. Introduction

All graphs in this paper are finite, simple and undirected. Terms not defined here are used in the sense of Harary [6]. Let the symbols $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G denoted by p. The cardinality of the edge set is called the size of G denoted by q. A graph with p vertices and q edges is called a (p, q) graph.

In 1985 Lo [7] introduced the k-edge-graceful graphs. The concept of global edge-graceful labeling was introduced in 2009 by Gayathri and Duraisamy [2]. In 2004, Sn-Min Lee, Kuo - Jye Chen and Yung-Chin Wang [8] introduced the k-edge-graceful graphs. In this paper, we extend the concept of global edge-graceful graphs to k-global edge-graceful graphs. Here, we investigate the k-edge gracefulness and k-global edge-gracefulness of some families of graphs. Throughout this paper, we assume that k is a positive integer greater than or equal to 0.

2. Edge – graceful and k – edge – graceful

Definition 2.1 [7]: A graph $G(V, E)$ is said to be **edge-graceful** if there exists a bijection f from E to $\{1, 2, ..., |E|\}$ such that the induced mapping f^+ from V to $\{0, 1, 2, ..., |V| - 1\}$ given by $f^+(x) = (\Sigma f(xy)) \pmod{|V|}$ taken over all edges xy incident at x is a bijection.

Necessary Condition 2.2 [7]: A necessary condition for a graph G with p vertices and q edges to be edge-graceful is

Received: 22 July, 2010; Revised: 18 September, 2010; Accepted: 29 September, 2010

$$q(q + 1) \equiv \frac{p(p+1)}{2} \pmod{p}$$

Conjecture 2.3 [7]: All trees of odd order are edge-graceful.

Definition 2.4 [8]: Given an integer $k \geq 0$, a graph $G = (V, E)$ with p vertices and q edges is said to be ***k*-edge-graceful** if there is a bijection $f : E \rightarrow \{k, k + 1, ..., k + q - 1\}$ such that the induced mapping $f^+ : V \rightarrow Z_p$, given by $f^+(u) = \Sigma\{f(u, v) : (u, v) \text{ in } E\} \pmod{p}$ is a bijection.

Necessary Condition 2.5 [8]: If a (p, q) graph G is k-edge-graceful, then

$$q(q + 2k - 1) \equiv \frac{p(p+1)}{2} \pmod{p} \qquad \text{... (1)}$$

Corollary 2.6 [8]: If p is odd, (1) is equivalent to $q(q + 2k - 1) \equiv 0 \pmod{p}$.

Corollary 2.7 [8]: If p is even, (1) is equivalent to $q(q + 2k - 1) \equiv \frac{p}{2} \pmod{p}$.

Corollary 2.8 [8]:

If a (p, q)-graph G has a k-edge-graceful labeling, then $p \equiv 0, 1$ or $3 \pmod{4}$.

By the conjecture 2.3, we now present another conjecture on k-edge-graceful labeling.

Conjecture 2.9: Whether all odd order trees are k-edge-graceful?

Towards attempting the conjecture 2.9, we have some observation and necessary conditions

Observation 2.10: If a gr.aph G is 1-edge-graceful, then G need not be k-edge-graceful for all k, as the following example illustrates.

Example 2.11: Consider the graph $G(p, q)$ given below.

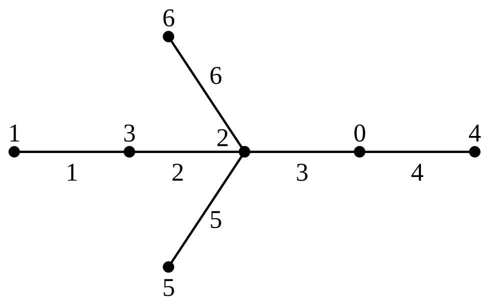

Figure 1: 1-*EGL* of *G*(7, 6)

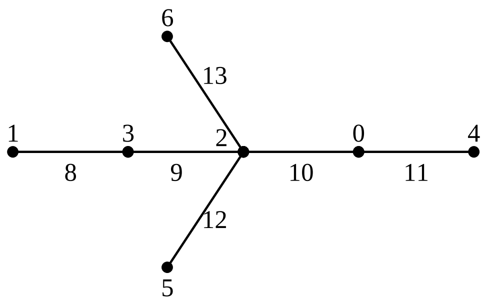

Figure 2: 8-*EGL* of *G*(7, 6)

Then G is 1-edge-graceful and not 2-edge-graceful. But G is 8-edge-graceful (see Figures).

By the observation, we have the following necessary condition.

Theorem 2.12: If a (p, q) graph G is a k-edge-graceful tree of odd order then k is of the form $\frac{p}{2}(l-1)+1$ where l is any odd positive integer and hence $k \equiv 1 \pmod{p}$.

Proof: By hypothesis and corollary 2.6, we have

$$q(q + 2k - 1) \equiv 0 \pmod{p} \qquad \text{... (2)}$$

As G is a tree, (2) reduces to $p + 2k \equiv 2 \pmod{p}$ and hence $p + 2k = lp + 2$ for some l, which implies $k = \frac{p}{2}(l-1)+1$. Since k is a positive integer and p is odd, it follows that l is an odd positive integer. Thus $k \equiv 1 \pmod{p}$.

Theorem 2.13: If a (p, q) graph G is 1-edge-graceful then G is k-edge-graceful for all $k \equiv 1 \pmod{p}$.

Proof: Let G be a 1-edge-graceful graph. Then there exists $f : E \longrightarrow \{1, 2, 3, ..., q\}$ $\ni : f^+ : V \longrightarrow \{0, 1, 2, ..., p - 1\}$ is a bijection. Let $k \equiv 1 \pmod{p}$.

Define $g : E \longrightarrow \{k, k + 1, ..., k + q - 1\}$ by

$g(e) = f(e) + k - 1$

Then for any vertex v, the induced vertex label

$$\begin{aligned} g^+(v) &= f^+(v) + \deg(v)(k-1) \\ &= f^+(v) + \deg(v)\, p \qquad (\ \textit{since}\ \ k \equiv 1 (\textit{mod}\ p)) \\ &= f^+(v) \pmod{p} \end{aligned}$$

Clearly g^+ is a bijection as f^+ is bijective. Hence, G is k-edge-graceful.

For an attempt, we check the edge-gracefulness of a specific family of tree given in theorem below.

Theorem 2.14: If G is an odd order tree consisting of only one vertex of degree 2 and all other vertices of degree either 1 or 3 then G is edge-graceful.

Proof: First we observe that G is a rooted tree with root vertex v as degree 2. By the definition of G, the vertices which are adjacent to v has either degree 1 or degree 3. Also given a tree $G(p, p - 1)$, the construction $G(p + 2, p + 1)$ requires addition of two more edges to $G(p, p - 1)$. Again by the definition of G, these two edges have to be attached together with a pendant vertex of G.

Consider the Diophantine equation $x + y = p$. The pair of solution of the equation is of the form $(t, p - t)$ where t is a number in $1 \leq t \leq \frac{p-1}{2}$. We now label the edges of G as follows.

Let v be a vertex of degree 2. Let $e_1 = (vv_1)$ and $e_2 = (vv_2)$ be the edges incident with v. Define $f : E \longrightarrow \{1, 2, 3, ..., q\}$ by

$f(e_1) = 1$ and $f(e_2) = p - 1$

Now for each of the vertex w of degree 3, there will be three edges incident with w. For any two of the edges incident with w, label them by any pair of solutions to the diophantine equation in any order. Then the induced vertex labels are as follows.

$f^+(v) = 0$ and the vertex labels of degree 3 vertices will have the edge label of the edge other than the pair. The pendant vertices will have the labels of the edges with which they are incident. Clearly they are distinct. Hence, G is an edge-graceful graph.

Edge-graceful labeling of $G(9, 8)$ and $G(13, 12)$ are given respectively in Figures 3 and 4.

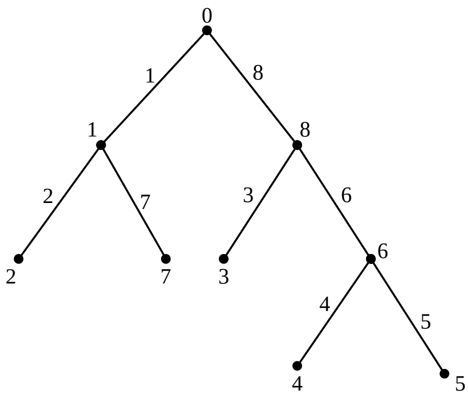

Figure 3: *EGL* of $G(9, 8)$

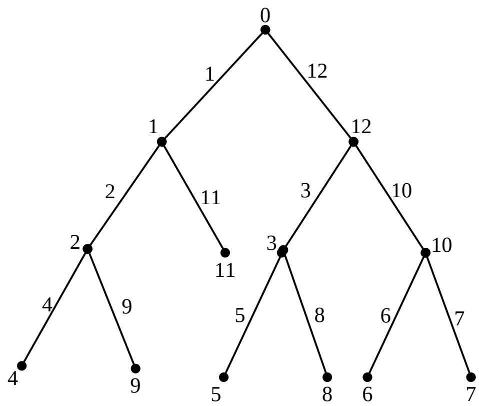

Figure 4: *EGL* of $G(13, 12)$

Corollary 2.15: If G is an odd order tree consisting of only one vertex of degree 2 and all other vertices of degree either 1 or 3 then G is k-edge-graceful for all $k \equiv 1 \pmod{p}$.

Proof: Follows from Theorem 2.13 and 2.14.

We now consider the even order analogue of conjecture 2.9.

Conjecture 2.16: Whether all even order trees are k-edge-graceful?

First and foremost, we see that no even order tree is edge-graceful (by Lo's condition). So the theory of 1-edge-graceful is completely different from that of k-edge-graceful, when the tree is of even order. For example, tree of order 4 is 2-edge-graceful but not 1-edge-graceful (see Figure).

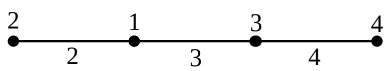

Figure 5: 2-*EGL* of p_4

We now present below a necessary condition for a k-edge-graceful graph of even order.

Theorem 2.17:

If a (p, q) graph G is a k-edge-graceful tree of even order with $p \equiv 0 (\text{mod } 4)$ then $k = \frac{p}{4}(2l-1)+1$ where l is any positive integer.

$$\text{Further } k \equiv \begin{cases} \frac{p+4}{4} (\text{mod } p) & \text{if } l \text{ is odd} \\ \frac{3p+4}{4} (\text{mod } p) & \text{if } l \text{ is even} \end{cases}$$

Proof: By hypothesis and corollary 2.7, we have

$$q(q + 2k - 1) \equiv \frac{p}{2} (\text{mod } p) \quad \text{... (3)}$$

As G is a tree, (3) reduces to $2k - 2 \equiv -\frac{p}{2}(\text{mod } p)$ and hence $2(k - 1) = lp - \frac{p}{2}$ which implies $k = \frac{p}{4}(2l-1)+1$. Since $p \equiv 0$ (mod 4), l can be any positive integer.

$$\text{Thus } k \equiv \begin{cases} \frac{p+4}{4} (\text{mod } p) & \text{if } l \text{ is odd} \\ \frac{3p+4}{4} (\text{mod } p) & \text{if } l \text{ is even} \end{cases}$$

In the case of even order tree, we observe that if G is $\frac{p+4}{4}$ or $\frac{3p+4}{4}$ - edge-graceful then G is k-edge-graceful for all $k \equiv \frac{p+4}{4}$ or $\frac{3p+4}{4}$ (mod p) if $p \equiv 0$ (mod 4).

Theorem 2.18: If a (p, q) graph G with $p \equiv 0$ (mod 4) is $\frac{p+4}{4}$ or $\frac{3p+4}{4}$ - edge-graceful then G is k-edge-graceful for all $k \equiv \frac{p+4}{4}$ or $\frac{3p+4}{4}$ (mod p).

Proof: Let $p \equiv 0$ (mod 4)

Case 1: G is $\frac{p+4}{4}$ - edge-graceful

Then there exists $f : E \longrightarrow \left\{\frac{p+4}{4}, \frac{p+4}{4}+1, \ldots \frac{p+4}{4}+q-1\right\} \ni$: the induced vertex labeling is a bijection when taken modulo p.

Let $l = \frac{p+4}{4}$, then by adding the value $l - 1$ to each edge of the graph and applying the argument used in Theorem 2.13, we can prove that the graph G is a k-edge-graceful for $k \equiv \frac{p+4}{4} (\text{mod } p)$.

Case 2: G is $\frac{3p+4}{4}$ - edge-graceful.

Proof: Follows on similar lines of case 1.

Theorem 2.19: If a (p, q) graph G with $p \equiv 2 \pmod 4$ then G is not k-edge-graceful for all k.

Proof: Follows from corollary 2.8.

By theorems 2.17, 2.18 and 2.19 conjecture 2.16 boils down to check whether all even order trees are $\frac{p+4}{4}$ or $\frac{3p+4}{4}$ (mod p)-edge-graceful.

Towards attempting to this conjecture, we take a special class of tree which is considered as in Theorem 2.14.

Theorem 2.20: If G is an even order tree consisting of only vertices of degree 3 and degree 1 with $p \equiv 0 \pmod 4$ then G is $\frac{p+4}{4}$ or $\frac{3p+4}{4}$-edge-graceful.

Proof: First we observe that G is a rooted tree with root vertex v as degree 1. By the definition of G, the vertices which are adjacent to v has degree 3. Also given a tree $G(p, p - 1)$, $p \equiv 0 \pmod 4$ the construction of $G(p + 4, p + 3)$ requires addition of 4 more edges to $G(p, p - 1)$. Again by the definition of G, these 4 edges have to be attached as a pair with any two vertices of G.

Case 1: G is $\frac{p+4}{4}$ - edge-graceful.

Consider the Diophantine equations

$$x + y = p \quad \dots (1)$$

and

$$x + y = 2p \quad \dots (2)$$

The pair of solution of the equations (1) and (2) is of the form $(t, p - t)$ where t is a number in $\frac{p+4}{4} \le t \le \frac{p-2}{2}$ and $(t_1, 2p - t_1)$ where t_1 is a number in $\frac{3p+4}{4} \le t_1 \le p-1$.

We now label the edges of G as follows:

Let v be any vertex of degree 1. Let $e = (vv_1)$ be an edge incident with v. Define $f(e) = p$. Now label the edges incident with v_1 as $\frac{p}{2}$ and $\frac{3p}{4}$.

For each of the vertex w of degree 3, there will be three edges incident with w. For any two of the edges incident with w, label them by any pair of solutions to the Diophantine equations considered in (1) and (2) in any order.

Then the induced vertex labels are as follows $f^+(v) = 0$, $f^+(v_1) = \frac{p}{4}$ and the vertex label of degree 3 vertex will have the edge label (mod p) of the edge other than the pair. The pendant vertex will have label of the edge (mod p) with which they are incident. Clearly they are distinct. Hence G is a $\frac{p+4}{4}$-edge-graceful graph.

Edge-graceful labeling of $G(16, 15)$ and $G(20, 19)$ are given respectively in Figures 6 and 7.

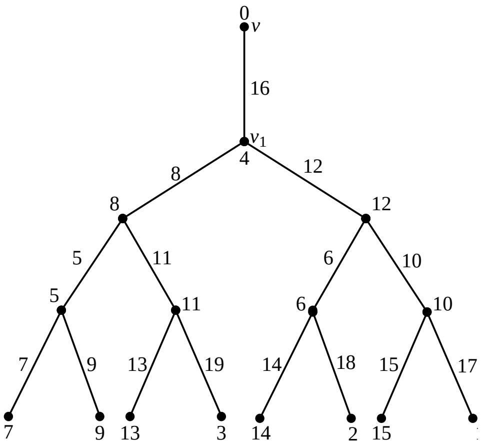

Figure 6: 5-*EGL* of *G*(16, 15)

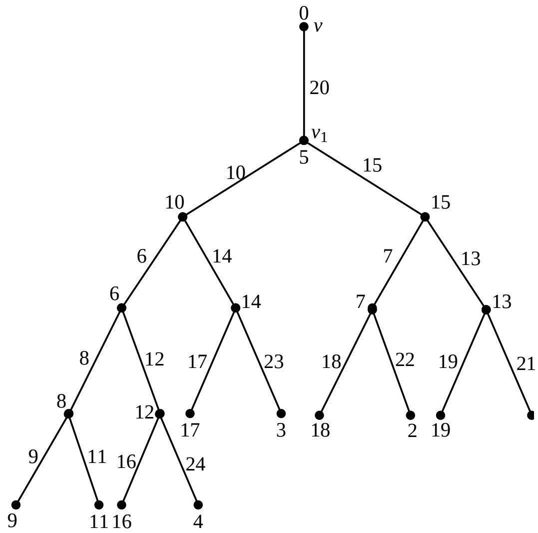

Figure 7: 6-*EGL* of *G*(20, 19)

Case 2: G is $\frac{3p+4}{4}$ - edge-graceful.

Consider the Diophantine equations

$$x + y = 2p \qquad \text{... (3)}$$

and

$$x + y = 4p \qquad \text{... (4)}$$

The pair of solution of the equations (3) and (4) is of the form $(t, 2p - t)$ where t is a number in $\left(\frac{3p+4}{4} \le t \le p-1\right)$ and $(t_1, 4p - t_1)$ where t_1 is a number in $\left(\frac{5p+4}{4} \le t_1 \le \frac{3p-2}{2}\right)$.

We now label the edges of G as follows. Let v be any vertex of degree 1. Let $e = (vv_1)$ be an edge incident with v.

Define $f(e) = p$. Now label the edges incident with v_1 as $\frac{5}{4}p$ and $\frac{3}{2}p$. For each of the vertex w of degree 3, there will be three edges incident with w, label them by any pair of solutions to the Diophantine equations considered in (3) and (4) in any order.

Then the induced vertex labels are as follows. $f^+(v) = 0$, $f^+(v_1) = \frac{3}{4}p$ and the vertex label of degree 3 vertex will have the edge label (mod p) to the edge other than the pair. The pendant vertex will have label of the edge (mod p) with which they are incident. Clearly they are distinct. Hence G is a $\frac{3p+4}{4}$ - edge-graceful graph.

Edge-graceful labeling of $G(8, 7)$ and $G(12, 11)$ are given respectively in Figure 8 and 9.

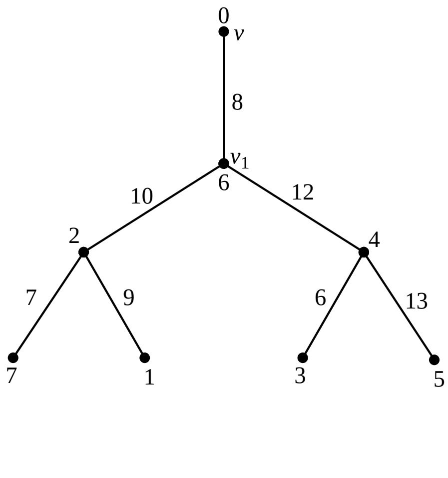

Figure 8: 7-*EGL* of *G*(8, 7)

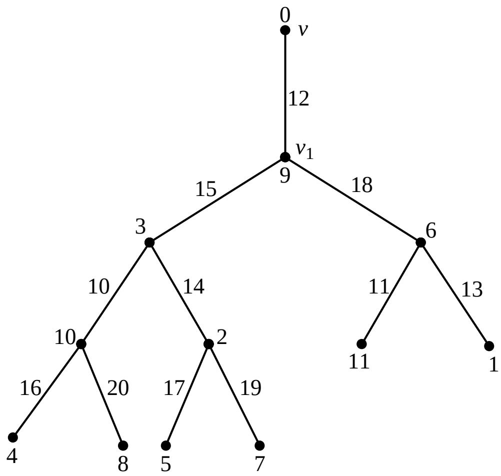

Figure 9: 10-*EGL* of *G*(12, 11)

Corollary 2.21: If G is an even order tree consisting of only vertices of degree 3 and degree 1 with $p \equiv 0 \pmod 4$ then G is k-edge-graceful for all $k \equiv \frac{p+4}{4}$ or $\frac{3p+4}{4} \pmod p$.

Proof: Follows from Theorem 2.18 and 2.20.

3. *k*-global edge graceful

The study of global-edge-graceful graphs was initiated by Gayathri and Duraisamy [2]. In this section, we extend the concept of global edge-graceful graphs to k-global edge-graceful graphs.

Definition 3.1 [2]: A graph G is called **global edge-graceful** if both G and its complement G^c are edge-graceful.

Example 3.2:

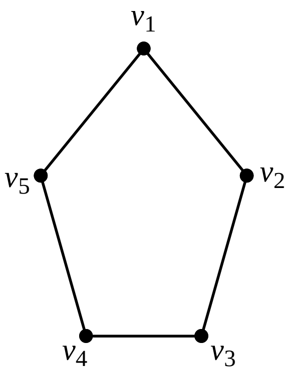

Figure 10: C_5 with ordinary labeling

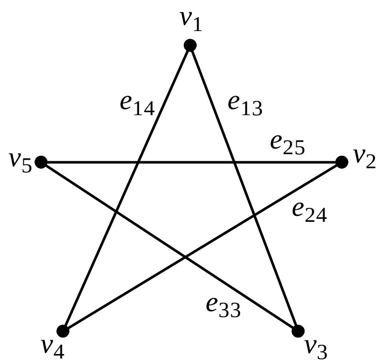

Figure 11: C_5^c with ordinary labeling

Definition 3.3: A graph G is called ***k*-global edge-graceful** if both G and its complement G^c are k-edge-graceful.

Theorem 3.4: The graph $C_n(n > 3)$ is k-global edge-graceful if n is prime and for all $k \equiv z \pmod{n}$.

Proof:

Case 1: C_n is k-edge-graceful

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of C_n, edges $e_i = (v_i, v_{i+1})$ for $i = 1, 2, \ldots, n - 1$ and $e_n = (v_n, v_1)$ are given in Figure 10.

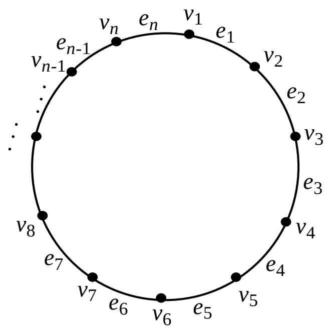

Figure 10: C_n with ordinary labeling

We first label the edges as follows:

For $1 \leq i \leq n$, $f(e_i) = k + i - 1$ for all $k \geq 0$

Then the induced vertex labels are

Case (i): For $k \equiv 0 \pmod{n}$, $f^+(v_1) = n - 1$

$$f^+(v_i) = \begin{cases} 2i - 3 & \text{for } 2 \leq i < \frac{n+3}{2} \\ 2i - n - 3 & \frac{n+3}{2} \leq i \leq n \end{cases}$$

Case (ii): $k \equiv z \pmod{n}$, $1 \leq z \leq n - 1$

Let a_{ik} denote the k-edge-graceful labeling of the i^{th} vertex v_i.

For $k \equiv 1 \pmod{n}$ to $(n - 1) \pmod{n}$ and $i = 1$ to n,

$$a_{ik} = \begin{cases} a_{(k+i)0} & \text{if } k + i \leq n \\ a_{(k+i-n)0} & \text{if } k + i > n \end{cases}$$

Clearly all the vertex labels are distinct. Hence C_n is k-edge-graceful if n is odd and for all $k \equiv z \pmod{n}$.

k-EGL of C_9 and C_{13} are given respectively in Figures 11 and 12.

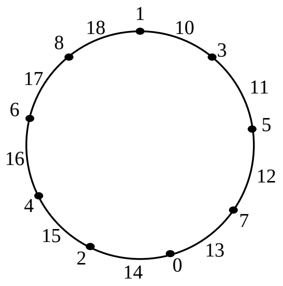

Figure 11: 10-*EGL* of C_9

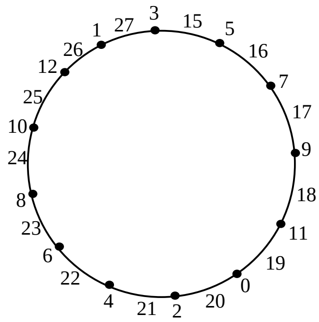

Figure 12: 15-*EGL* of C_{13}

Case 2: C_n^c is k – edge – graceful

Let the vertices are defined as in case 1 and edges are

$e_{1j} = (v_1, v_j),\ 3 \leq j \leq n - 1$

For $2 \leq i \leq n - 2$, $e_{ij} = (v_i, v_j)$, $i + 2 \leq j \leq n$

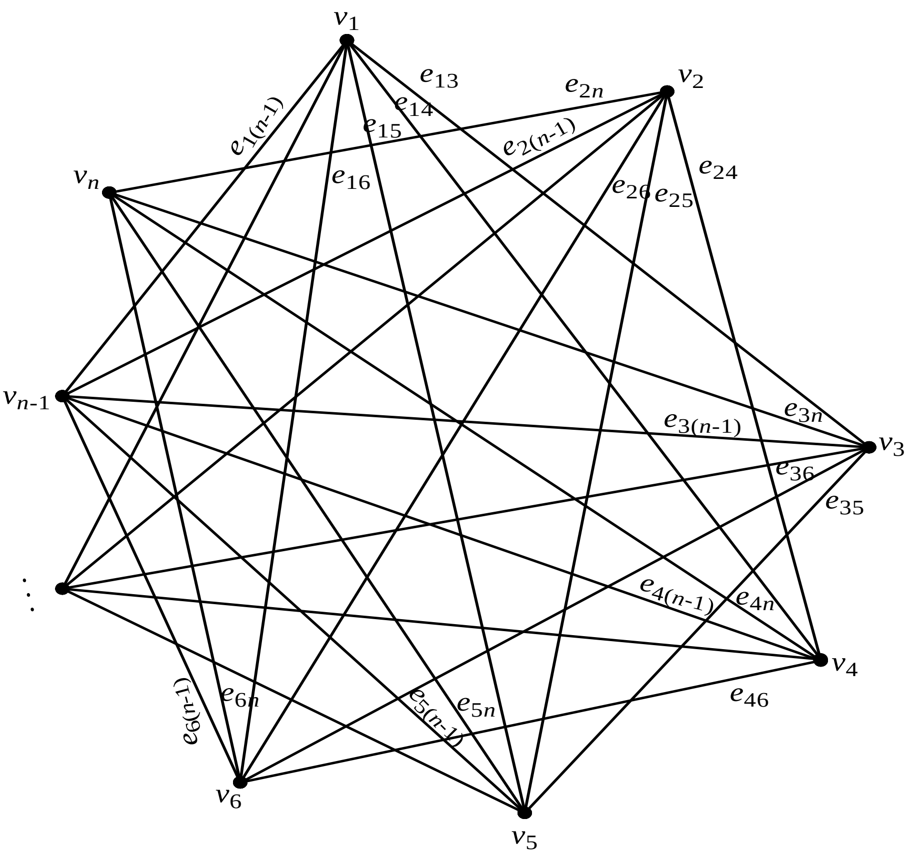

Figure 13: C_n^c with ordinary labeling

We first label the edges as follows:

For $i = 1$, $i + 2 \leq j \leq n - 1$ and $k \geq 0$

$$f(e_{ij}) = k + \frac{(j-3)(2n-j)}{2}$$

For $2 \leq i \leq n - 2$, $i + 2 \leq j \leq n$ and $k \geq 0$

$$f(e_{ij}) = k + i + \frac{(j-i-2)(2n-j+i-1)}{2} - 1$$

Then the induced vertex labels are

Case (i): $k \equiv 0 (\text{mod } n)$

For $n = 6l_1 - 1$, $l_1 \geq 1$,

$$f^+(v_i) = \begin{cases} n - 3i + 1 & \text{for } 1 \leq i < \frac{n+4}{3} \\ 2n - 3i + 1 & \text{for } \frac{n+4}{3} \leq i < \frac{2n+2}{3} \\ 3n - 3i + 1 & \frac{2n+2}{3} \leq i \leq n \end{cases}$$

For $n = 6l_1 + 1$, $l_1 \geq 1$

$$f^+(v_i) = \begin{cases} n - 3i + 1 & \text{for } 1 \leq i < \frac{n+2}{3} \\ 2n - 3i + 1 & \text{for } \frac{n+2}{3} \leq i < \frac{2n+4}{3} \\ 3n - 3i + 1 & \frac{2n+4}{3} \leq i \leq n \end{cases}$$

Case (ii): $k \equiv z (\text{mod } n)$, $1 \leq z \leq n - 1$

Let a_{ik} denote the *k*-edge-graceful labeling of the i^{th} vertex v_i

For $k \equiv 1 (\text{mod } n)$ to $n - 1 (\text{mod } n)$,

$$a_{ik} = \begin{cases} a_{(k+i)0} & \text{if } k + i \leq n \\ a_{(k+i-n)0} & \text{if } k + i > n \end{cases}$$

Clearly, all the vertex labels are distinct. Hence, the graph C_n^c is *k*-edge-graceful if *n* is prime and for all $k \equiv z$ (mod *n*). Then by case 1 and case 2, C_n is a *k*-global edge-graceful graph. *k-EGL* of C_{13}^c is given in Figure 14.

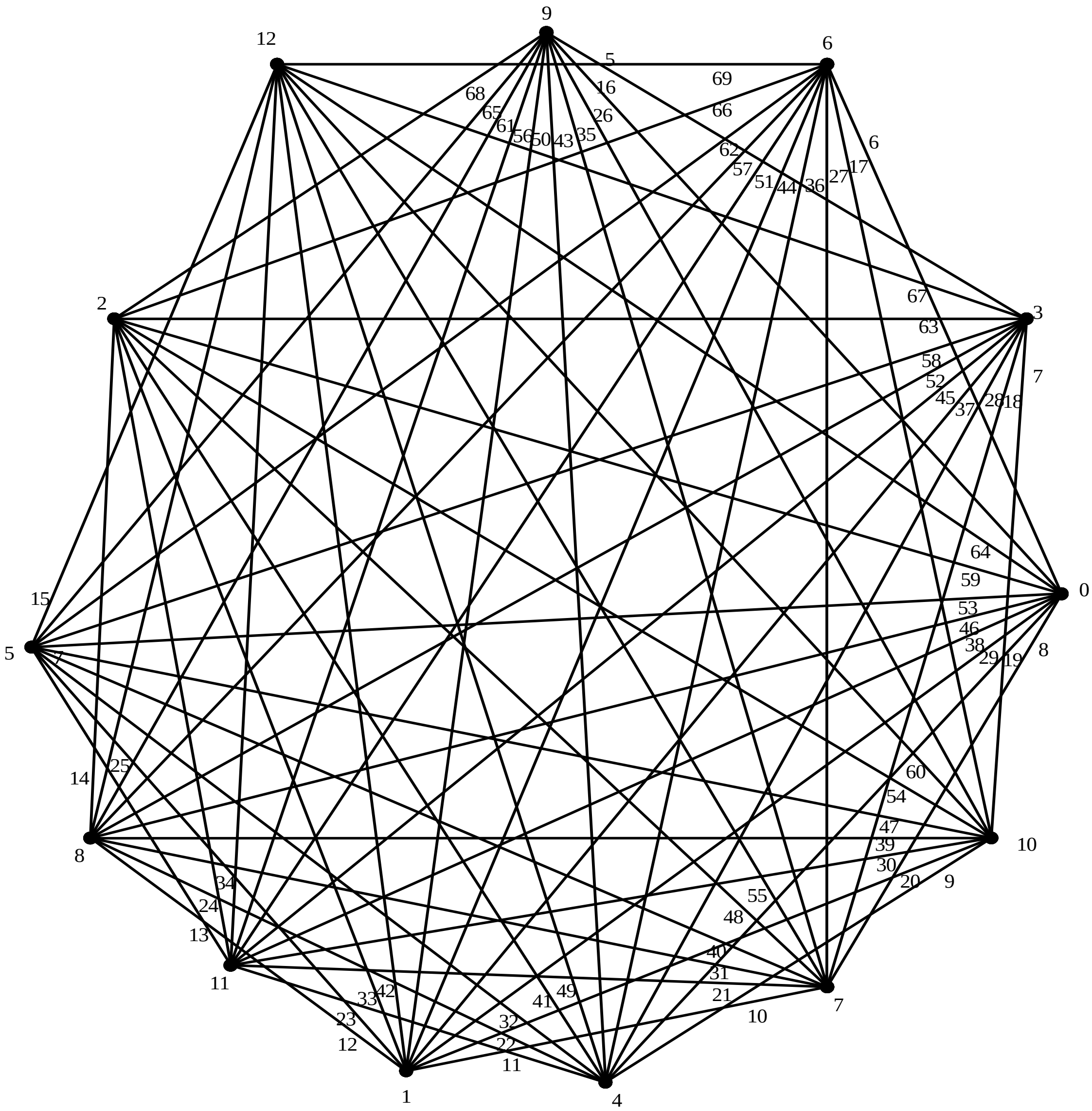

Figure 14: 5-*EGL* of C_{13}^{c}

Conclusion

In this paper, we have established a specific family of tree to be edge-graceful and *k*-edge-graceful. Further, we have obtained some necessary conditions and characterizations for *k*-edge-gracefulness of any tree. Then, we have extended the concept of global edge-graceful graphs to *k*-global edge-graceful graphs. More results on *k*-global edge-graceful graphs are discussed in [5]. Also, *k*-even edge-graceful graphs are completely studied in [3] and [4].

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